Homework 10

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Problem 1. Let $U = [U_0,\ U_1,\ \ldots,\ U_m]^T$ be a vector of function values at equally spaced points on the interval $0 \le x \le 1$, and suppose the underlying function is periodic and smooth. Then we can approximate the first derivative u_x at all of these points by DU, where D is circulant matrix such as

$$D_- = \frac{1}{h}\begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}, \qquad D_+ = \frac{1}{h}\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{bmatrix} \tag{1}$$

for first-order accurate one-sided approximations or

$$D_0 = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix} \tag{2}$$

for a second-order accurate centered approximation. (These are illustrated for a grid with $m + 1 = 5$ unknowns and $h = 1/5$.)

The advection equation $u_t + au_x = 0$ on the interval $0 \le x \le 1$ with periodic boundary conditions gives rise to the MOL discretization $U'(t) = -aDU(t)$ where D is one of the matrices above.

(a) Discretizing $U' = -aD_-U$ by forward Euler gives the first order upwind method

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n), \tag{3}$$

where the index i runs from 0 to m with addition of indices performed mod $m + 1$ to incorporate the periodic boundary conditions.

Suppose instead we discretize the MOL equation by the second-order Taylor series method,

$$U^{n+1} = U^n - akD_-U^n + \frac{1}{2}(ak)^2 D_-^2 U^n. \tag{4}$$

Compute D_-^2 and also write out the formula for U_j^n that results from this method.

(b) What are the orders of the methods (3) and (4) derived in part (a)?

(c) Suppose we make the method (3) more symmetric:

$$U^{n+1} = U^n - \frac{ak}{2}(D_+ + D_-)U^n + \frac{1}{2}(ak)^2 D_+ D_- U^n. \tag{5}$$

Write out the formula for U_j^n that results from this method.

(d) Use von Neumann analysis to study the stability condition of the methods (4) and (5).

(e) Use the numerical methods in (3), (4) and (5) to solve the equation $u_t + u_x = 0$ on $[0, 1] \times [0, 1]$, with periodic boundary condition and initial value $u(x, 0) = \sin(2\pi x)$. Draw the L^2 error with respect to exact solution for $h = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256$, and $k = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256$ (in case the stability condition is satisfied). Verify the order of convergence for each method.