Homework 11

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Problem 1. We will look at one example simply to give a flavor of the potential difficulties of IBVP. Suppose we use the leapfrog method to solve the IBVP for the advection equation $u_t + au_x = 0$. At the left (inflow) boundary we can use the given boundary condition, but at the right we will need a numerical boundary condition. Suppose we use the first order upwind method at this point,

$$U_{m+1}^{n+1} = U_{m+1}^{n} - \frac{ak}{h}(U_{m+1}^{n} - U_{m}^{n}),$$

which is also consistent with the advection equation. Compute the solution when the initial data is $u(x,0) = \eta(x) = \exp(-5(x-2)^2)$ and the first two time levels for leapfrog are initialized based on the exact solution $u(x,t) = \eta(x - at)$. We see that as the wave passes out the right boundary, a reflection is generated that moves to the left, back into the domain. Run the simulation until you see the generation and leftward propagation of a sawtooth mode.

Problem 2. Let $\Omega = \{(x,y) : r < 1/2\}$, $u = \ln|\ln r|$, where $r = (x^2 + y^2)^{1/2}$. Prove that $u \in H^1(\Omega)$, but $u \notin C(\Omega)$.

Problem 3. Suppose that $\partial\Omega$ is piecewise smooth. Smooth surface S bisects Ω into two subdomains Ω_1 and Ω_2. $u \in H^1(\Omega_1)$ in Ω_1, and $u \in H^1(\Omega_2)$ in Ω_2 separately. Prove that $u \in H^1(\Omega)$ iff the trace (boundary value) of u on S from Ω_1 equals to the trace of u on S from Ω_2.