Homework 2

March 12, 2014, due March 20

Problem 1. Find a, b, c, and d, such that the multistep scheme (1) has the highest possible order,

$$u_{n+1} + au_n + bu_{n-1} = h(cf(t_n, u_n) + df(t_{n-1}, u_{n-1})) \tag{1}$$

What is the leading order term for the truncation error?

Problem 2. (a) What does it mean to say that a linear multistep method is zero-stable? Formulate an equivalent characterisation of zero-stability of a linear multistep method in terms of the roots of its first characteristic polynomial.

(b) Show that there is a value of the parameter b such that the linear multistep method defined by the formula

$$U_{n+3} + (2b - 3)(U_{n+2} - U_{n+1}) - U_n = b\Delta t(F_{n+2} + F_{n+1}) \tag{2}$$

is fourth order accurate (with $F_n = f(t_n, U_n)$). Show further that the method is not zero-stable for this value of b.

Problem 3. (a) Impletment Runge's method (RK2),

$$\begin{cases} y_{n+1} = y_n + hk_2, \\ k_1 = f(t_n, y_n), \\ k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1). \end{cases} \tag{3}$$

Heun's method

$$\begin{cases} y_{n+1} = y_n + h(\frac{1}{4}k_1 + \frac{3}{4}k_3), \\ k_1 = f(t_n, y_n), \\ k_2 = f(t_n + \frac{1}{3}h, y_n + \frac{1}{3}hk_1), \\ k_3 = f(t_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_2). \end{cases} \tag{4}$$

and classical 4th order Runge-Kutta method (RK4),

$$
\begin{cases}
y_{n+1} = y_n + h(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4), \\
\quad k_1 = f(t_n, y_n), \\
\quad k_2 = f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1), \\
\quad k_3 = f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2), \\
\quad k_4 = f(t_n + h, y_n + k_3).
\end{cases} \tag{5}
$$

Justify the rate of convergence numerically.

(b) Choose appropriate Runge-Kutta method to initialize Adams-Bashforth method of order 3,

$$
y_{n+3} = y_{n+2} + h[\frac{23}{12}f(t_{n+2}, y_{n+2}) - \frac{4}{3}f(t_{n+1}, y_{n+1}) + \frac{5}{12}f(t_n, y_n)]. \tag{6}
$$

Justify the rate of convergence numerically.

Problem 4. Consider the Runge-Kutta method

$$
\frac{U_{n+1} - U_n}{\Delta t} = (c_1 k_1 + c_2 k_2), \tag{7}
$$

where

$$
k_1 = f(tn, Un),
$$

and

$$
k_2 = f(t_n + b_{2,1}\Delta t, U_n + b_{2,1}\Delta t k_1),
$$

and where c_1, c_2, $b_{2,1}$ are real parameters.

(a) Show that there is a choice of these parameters such that the truncation error of the method,

$$
\frac{T_n}{\Delta t} = \frac{u_{n+1} - u_n}{\Delta t} - [c_1 f(t_n, u_n) + c_2 f(t_n + b_{2,1}\Delta t, u_n + b_{2,1}\Delta t f(t_n, u_n))], \tag{8}
$$

is order 2 as $\Delta t \to 0$.

(b) Suppose that a second-order method of the above form is applied to the initial value problem $u' = -\lambda u$, $u(0) = 1$, where λ is a positive real number. Show that the sequence $\{U_n\}_{n \geq 0}$ is bounded if and only if $\Delta t \leq \frac{2}{\lambda}$ (*interval of absolute stability*). *(Hard; not compulsory!)*: Show further that, for such λ,

$$
|u_n - U_n| \leq \frac{1}{6}\lambda^3 \Delta t^2 t_n, \quad n \geq 0. \tag{9}
$$