Homework 4

March 20, 2014

Problem 1. Order condition and absolute stability region for 3rd order Runge-Kutta method.

(a) Derive the order condition for 3rd order explicit Runge-Kutta method.

(b) Derive the absolute stability region for 3rd order explicit Runge-Kutta method, and draw the cooresponding region in the complex plane.

Problem 2. Play with the example in `http://w3.bretagne.ens-cachan.fr/math/people/gilles.vilmart/java.html`. For the quartic Hamiltonian system (double well), try 4th order explicit Runge-Kutta and see how long it takes for the particle to move to another basin. In fact, all the explicit Runge-Kutta methods are not symplectic, could you show this by calculating the diagonal entry of the *famous matrix* M? ($m_{k,l} = b_k a_{k,l} + b_l a_{l,k} - b_k b_l$.)

Problem 3. The *symplectic Euler method* for the Hamiltonian system reads

$$\mathbf{p}_{n+1} = \mathbf{p}_n - h\frac{\partial H(\mathbf{p}_{n+1}, \mathbf{q}_n)}{\partial \mathbf{q}}, \quad \mathbf{q}_{n+1} = \mathbf{q}_n + h\frac{\partial H(\mathbf{p}_{n+1}, \mathbf{q}_n)}{\partial \mathbf{p}} \tag{1}$$

(a) Show that this is a first order method.

(b) Prove from basic principles that, as implied by its name, the method is indeed symplectic.
Hint: let $G = \nabla^2 H$*, and write* $G = [G_{11}, G_{12}; G_{21}, G_{22}]$.

(c) Assuming that the Hamiltonian is separable, $H(\mathbf{p}, \mathbf{q}) = T(\mathbf{p}) + V(\mathbf{q})$, where T and V correspond to the kinetic and potential energy respectively, show that the method can be implemented explicitly.

(d) Use symplectic Euler method and explicit Euler method to solve the problem of nonlinear pendulum in MATLAB. The Hamiltonian is $H(p, q) = \frac{1}{2}p^2 - \cos(q)$, and the Hamiltonian equations are

$$\dot{p} = -\sin(q), \quad \dot{q} = p. \tag{2}$$

with initial condition $p(0) = 0, q(0) = 1$. Plot the error of the numerical methods in the Hamiltonian H.