Homework 4

March 27, 2014

Problem 1. Please implement the trapezoidal rule

$$y_{n+1} = y_n + \frac{1}{2}h[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \tag{1}$$

Justify numerically the convergence rate for the trapezoidal rule is 2nd order.

Problem 2. To solve the boundary value problem

$$-u_{xx}(x) = f(x), \quad x \in (0,1). \tag{2}$$

with boundary condition $u(0) = u(1) = 0$, $f \in C^0$, we subdivide interval $[0,1]$ into n equal subintervals with $h = 1/n$. Let $x_j = jh$, $j = 0, \ldots, n$, we are looking for u_j, the approximations to the exact solution $u(x_j)$ at x_j.

(a) (Formulation) If we use central differences to approximate u_{xx},

$$u_{xx}(x_i) \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{3}$$

write down the resulting finite difference scheme (including boundary condition), and the associated linear system $Au = F$ for the unknowns, specify A, u and F.

(b) (Existence and Uniqueness) Prove that A is nonsingular, therefore the finite difference scheme has a unique solution. *hint: there are many ways to do this, one way is to show that $v^T Av > 0$ for any $v \neq 0$, namely, A is symmetrically positive definite.*

(c) (Matlab) A is a tri-diagonal matrix, $Au = F$ can be efficiently solved by Gaussian elimination method which will be introduced later. In this homework, suppose that $f = (3x + x^2)e^x$, implement the numerical scheme in Matlab. Take $n = 4$, plot the solution you obtain.

(d) (Maximum Principle) For $v \in \mathbb{R}^m$, we say that $v \geq 0$ if $v_i \geq 0$ for $1 \leq i \leq m$. Show that if $Aw = v$ and $v \geq 0$, then $w \geq 0$. Furthermore, this implies that $\alpha_{ij} \geq 0$, where α_{ij} are the entries of A^{-1}. Use this property to show that if $f \geq 0$, then $u_j \geq 0$, for $j = 0, \ldots, n$.

(e) (Discrete Stability) The function $v(x) = \frac{x(1-x)}{2}$ satisfies

$$-\frac{v(x_{j+1}) - 2v(x_j) + v(x_{j-1})}{h^2} = 1. \tag{4}$$

Use this to show that the entries α_{ij} of A^{-1} satisfies

$$0 \leq \sum_{j=1}^{n-1} \alpha_{ij} \leq \frac{1}{8}. \tag{5}$$

Prove that

$$\max_{1\leq i\leq n-1} |u_i| \leq \frac{1}{8} \max_{1\leq i\leq n-1} |f(x_i)| \tag{6}$$

(f) (Truncation Error) Like ODE, we can define truncation error,

$$T_j = -\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} - f(x_j) \tag{7}$$

Calculate the leading order term of T_j.

(g) (Error Equation) Let $e_j = u(x_j) - u_j$ be the discretization error. Show that e_j satisfies the equation $Ae = T$, where $e = (e_1, \ldots, e_{n-1})^T$ and $T = (T_1, \ldots, T_{n-1})^T$. Using (**??**) to prove the convergence result

$$\max_{1\leq i\leq n-1} |u(x_i) - u_i| \leq \frac{h^2}{96} \max_{0\leq x\leq 1} |u^{(4)}(x)|. \tag{8}$$

(h) (Justification) When $f = (3x + x^2)e^x$, the exact solution is $u(x) = x(1-x)e^x$. Take $n = 4, 8, 16, 32, 64, 128, 256$, and compute numerical solutions u^n with Matlab. Calculate $\|u - u^n\|_\infty := \max_{1\leq i\leq n-1} |u(x_i) - u_i^n|$, plot (log-log) the convergence with resepct to n. Numeically estimate the prefactor in the estimate $\|u - u^n\|_\infty \simeq Ch^\alpha$, compare it with the result in (**??**)