Homework 5

Lei Zhang

April 3, 2014

Problem 1. Given 1-d finite element mesh in Figure 1 $x_i - x_{i-1} = h_i$, h_i's are not

Figure 1: Discretization of $(0,1)$

necessarily equal, formulate the Galerkin finite element method for

$$\begin{cases} -u_{xx} = f, & x \in (0,1) \\ u = 0, & x = 0,1 \end{cases} \tag{1}$$

Calculate $A_h = \{a(\phi_i, \phi_j)\}$ in terms of h_i.

Problem 2. Consider Poisson's equation

$$\begin{cases} -\Delta u = f, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{2}$$

We have the following regular triangulation in Figure 2, if we index the nodes as shown in the Figure 2 (a), compute the entries $a(\phi_1, \phi_i)$ for $i = 1, 2, \ldots, 7$, and the entries $m(\phi_1, \phi_i) = \int \phi_1 \phi_i \mathrm{dx}$. ϕ_i is the piecewise linear nodal basis at node x_i. Compare with 5-point finite difference metohd. Furthermore, calculate $a(\phi_1, \phi_i)$ in (b).

Problem 3. For a banded sparse $n \times n$ matrix A with band width s, prove that the cost for LU decomposition of A is $O(s^2 n)$.

Problem 4. Let A be the tridiagonal matrix generated by either central difference or piecewise linear finite element in 1D. Namely, A is a $d \times d$ Toeplitz matrix, with $\tau_0 = 2$, $\tau_1 = \tau_{-1} = -1$, $b = \mathbf{1} \in \mathbb{R}^d$.

Implement the following methods to solve the linear equation $Ax = b$,

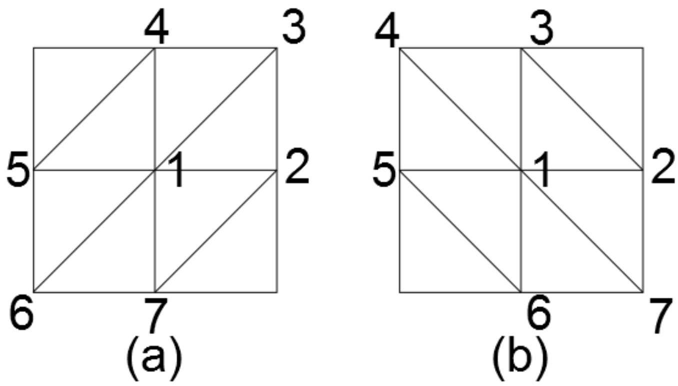

Figure 2: Triangulation in 2D

(a) LU factorization,

(b) Jacobi iteration,

(c) Gauss-Seidel iteration,

For the iterative methods (b)-(c), run each method for 1000 steps, record the error ($\|Ax^k - b\|$) at each step, and plot the error with respect to the number of steps. Explain your observation.

For convergence rate of Jacobi method and Gauss-Seidel method, read Iserles 266-267.

Note: You can use Matlab commmand *toeplitz* to generate the matrix A,