Homework 6

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Problem 1. Let A be the tridiagonal matrix generated by either central difference or piecewise linear finite element in 1D. Namely, A is a $d \times d$ Toeplitz matrix, with $\tau_0 = 2$, $\tau_1 = \tau_{-1} = -1$, $b = \mathbf{1} \in \mathbb{R}^d$.

Implement the following methods to solve the linear equation $Ax = b$,

(a) steepest descent,

(b) conjugate gradient,

For the iterative methods (b)-(c), run each method for 1000 steps, record the error ($\|Ax^k - b\|$) at each step, and plot the error with respect to the number of steps. Explain your observation.

For the algorithm of conjugate gradient method, read Iserles 316-317.

Note: You can use Matlab commmand *toeplitz* to generate the matrix A,

Problem 2. Consider the boundary value problem

$$-\Delta u = f(x), \quad x \in \Omega. \tag{1}$$

with boundary condition $u(x) = 0$ for $x = (x_1, x_2) \in \partial\Omega$. Let $\Omega = [0,1] \times [0,1]$, and $f(x) = -2\pi^2 \sin(\pi x_1)\sin(\pi x_2)$, we can impose the exact solution as

$$u(x) = \sin(\pi x_1)\sin(\pi x_2). \tag{2}$$

(a) Solve this problem with 5-point finite difference method for $m = 4, 8, 16, 32, 64$ with conjugate gradient method. You can use natural ordering for the unknowns. Evaluate the error with respect to the exact solution u in terms of the norm

$$\|v_h\|^2 := h^2 \sum_{i=1}^{m} \sum_{j=1}^{m} v_h^2(ih, jh) \tag{3}$$

where $h = 1/(m+1)$. Plot the convergence history.

(b) Solve this problem with finite element method using piecewise linear basis for $m = 4, 8, 16, 32, 64$, the triangulation is the same as last homework.

We have constructed the stiffness matrix A in last homework. The difference with 5-point finite difference method is, now you need to evaluate $\int f\phi_i \,\mathrm{d}x$ for each i instead using the value of f directly.

It requires some numerical quadrature technique to evaluate the H^1 error, but you can still measure the error with the norm $\|\cdot\|$ as above. In fact, this norm is equivalent to L^2 norm. Plot the convergence history.

Problem 3. Think about your project topic, some possible topics can be

1. Convergence of G-S method and SOR method for 2D Poisson equation,
2. Multigrid method with application on Poisson equation (convection-diffusion equation, etc),
3. Numerical methods for conservation laws,
4. Solve Poisson's equation in higher dimension ($d > 3$).
5. Numerical methods for Allen-Cahn equation,
6. Numerical methods for wave equation in 2D,
7. Multidimensional methods,
8. Numerical methods for nonlinear equations,
9. ...

You can look into the reference books of this class, or talk to me. And please send me and TA your topic with a short description by next Thursday (Apr 17).