Homework 8

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Problem 1. Consider the heat equation $u_t = \kappa u_{xx}$ with $u(x,0) = \eta(x)$, $-\infty < x < \infty$.

(a) Calculate the Fourier transform of $\eta(x) = \exp(-\beta x^2)$.

(b) Find the Fourier transform $\hat{u}(\xi, t)$.

(c) Use inverse Fourier transform to calculate $u(x,t)$.

(d) Use the above results to show that the Green's function of heat equation is

$$G(x,t;\tilde{x}) = \frac{1}{\sqrt{4\pi\kappa t}} \exp(-(x-\tilde{x})^2/(4\kappa t))$$

Problem 2. Crank-Nicolson method.

(a) Find the truncation error of Crank-Nicolson method.

(b) Implement Crank-Nicolson method numerically, and justify the order of convergence.

Problem 3. (leapfrog for heat equation) Consider the following method for solving the heat equation $u_t = u_{xx}$,

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}). \tag{1}$$

(a) Determine the order of accuracy of this method (in both space and time).

(b) Suppose we take $k = \alpha h^2$ for some fixed $\alpha > 0$ and refine the grid. For what values of α (if any) will this method be Lax-Richtmyer stable and hence convergent? Hint: Consider the MOL interpretation and the stability region of the time-discretization being used.

(c) Is this a useful method?