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A uniform first order method for the discrete ordinate transport equation with interfaces in X,Y geometry*

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Abstract

A uniformly first order convergent numerical method for the discrete-ordinate transport equation in the rectangle geometry is proposed in this paper. Firstly we approximate the scattering coefficients and source term by piecewise constants determined by their cell averages. Then for each cell, following the work of De Barros and Larsen [19, 1], the solution at the cell edge is approximated by its average along the edge, the solution of the system of equations for the cell edge averages in each cell can be obtained analytically. Finally piece together the numerical solution with the neighboring cells using the interface conditions. When there is no interface or boundary layer, this method is asymptotic-preserving, which implies coarse meshes (meshes that do not resolve the mean free path) can be used to obtain good numerical approximations. Moreover, the uniform first order convergence with respect to the mean free path is shown numerically and the rigorous proof is discussed.

Mathematics subject classification: 41A30, 41A60,65D25 .
Key words: transport equation; interface; diffusion limit; asymptotic preserving; uniform numerical convergence; X,Y geometry

1. Introduction

The transport equation plays an important role in many physical applications, such as neutron transport, radiative transfer, high frequency waves in heterogeneous and random media, semiconductor device simulation and so on. One difficulty about solving this equation numerically is when its mean free path (the average distance a particle travels between two successive interactions with the background media) is small, which requires numerical resolution of the small scale. Historically people use the diffusion limit to approximate the solution when the cost is too much to solve the transport equation directly. This small scale is embodied by the introduction of a dimensionless parameter ϵ into the transport equation and the diffusion limit can be obtain when $\epsilon \to 0$.

Particularly in this paper we consider the steady state isotropic neutron transport equation with interfaces in the X,Y geometry. The interface condition we consider here is that the density of all directions is continuous at the interface, which often arises in neutron transport equation. There is another kind of interface condition which always arises in radiative transfer equations as an approximation of high frequency waves in random and heterogeneous media, where the energy flux is continuous [2, 14, 16]. As the interface condition is local, for the density continuous case, we only need to consider the one dimensional interface analysis which is given in [15].

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The diffusion limit for two dimensional neutron transport equation with interfaces is derived in this paper. For simplicity, we assume that the incoming particle density at the boundaries is isotropic, in which case no boundary layer exists when $\epsilon \to 0$. A first order numerical method uniform with respect to ϵ is proposed and investigated here. The most used determined method for this problem is the discrete ordinate method which is a semi-discretization of the velocity field, our method is based on the discrete ordinate form and the full discretization is considered. The idea is firstly as in [15], to approximate the coefficients by piecewise constants determined by their cell averages. Then for each cell, following the work of De Barros and Larsen [19, 1], the solution at the cell edge is approximated by its average along the edge, the obtained system of equations for the cell edge averages can be solved analytically in each cell and we can get the solution of the whole domain by using the interface conditions to piece together the numerical solution with the neighboring cells. This method is a direct extension of the one dimensional scheme that is proposed in [15].

Asymptotic-Preserving (AP) schemes have been proved rather successful for problems with small scales. That a scheme is AP means it is a good scheme for the original equation, and in the limit as the small parameter goes to zero, becomes a good scheme for the limit equation [10]. It was proved in [9] for the linear transport equation with boundary conditions that an AP scheme converges *uniformly* with respect to ϵ. This implies we can solve the transport equation without resolving the small scale. For more applications of AP schemes we refer to [5, 7, 6] for plasmas and fluids and [3, 11, 12] for hyperbolic systems with stiff relaxations. When there is no interface or boundary layer, the method we propose here is proved to be AP and its uniform first order accuracy is given numerically. The rigorous convergence proof relies heavily on the eigenfunction expansion of the constant coefficient, one-dimensional discrete-ordinate equations, but the idea is quite similar to the proof of uniform second order convergence for one dimensional case in [15]. AP means that at the interior of the materials where the solution varies slowly (no matter whether ϵ is big or small), we do not need to resolve ϵ to get good approximations, though the equation itself requires under-resolving. When using coarse meshes, most methods can not get good results for problems with boundary layers even if they are AP inside. The method proposed in [15] is the first method that is uniformly convergent valid up to the boundary. We'll show numerically that our two dimensional method is not valid at the layers, but this can be improved by resolving ϵ locally at the boundaries or interfaces.

Similar ideas can be found in [17, 19, 1] but with auxiliary equations and in an iterative way. They only discussed piecewise constant case and the required storage is much more than our approach. Problems with coefficients depending on space are investigated in this paper and we also present its AP property and uniform accuracy with respect to ϵ.

The arrangement of this paper is as follows. In section 2, we introduce the two dimensional neutron transport equation and its discrete ordinate form, derive their diffusion limit with interfaces. In section 3, the scheme is given and its AP property is proved in section 4. Several numerical examples are displayed in section 5 to test the AP property and the uniform accuracy is discussed. Finally we conclude in section 6.

2. Neutron transport equation in two dimensional

The steady state, isotropic, neutron transport equation in the X,Y geometry reads as: for $\mathbf{z} \in \Omega \subset \mathbb{R}^2$, $\mu \in S = \{\mathbf{u} \in \mathbb{R}^2 : |\mathbf{u}| = 1\}$,

$$\mathbf{u} \cdot \nabla \Psi(\mathbf{z}, \mathbf{u}) + \frac{\sigma_T(\mathbf{z})}{\epsilon} \Psi(\mathbf{z}, \mu) = \frac{1}{2\pi} \Big(\frac{\sigma_T(\mathbf{z})}{\epsilon} - \epsilon \sigma_a(\mathbf{z}) \Big) \int_S \Psi(\mathbf{z}, \mathbf{u}) d\mathbf{u} + \epsilon q(\mathbf{z}) \tag{2.1}$$

with the boundary conditions

$$\Psi(\mathbf{z}, \mathbf{u}) = \Psi_\Gamma^-, \quad \text{for } (\mathbf{z}, \mathbf{u}) \in \Gamma_u^- = \{\mathbf{z} \in \Gamma = \partial\Omega, \mathbf{u} \cdot n(\mathbf{z}) < 0\}. \tag{2.2}$$

Here σ_T, σ_a, q are the total cross section, absorption cross section and source respectively. $\Psi(\mathbf{z}, \mathbf{u})$ is the function we want, which represent the density of the particles moving in direction $\mathbf{u}$ at position $\mathbf{z}$. For isotropic boundary condition, Ψ_Γ^- only depends on Γ but is independent of $\mathbf{u}$. The diffusion limit can be obtained by introducing

$$\Psi = \sum_{n=0}^{\infty} \epsilon^n \Psi^{(n)} \tag{2.3}$$

into (2.1) and equating the coefficients of different powers of ϵ. The $O(1/\epsilon)$ equation is

$$\Psi^{(0)} = \frac{1}{2\pi} \int_S \Psi^{(0)} d\mathbf{u}. \tag{2.4a}$$

The $O(1)$ and $O(\epsilon)$ equations are

$$\mathbf{u} \cdot \nabla \Psi^{(0)} + \sigma_T \Psi^{(1)} = \frac{\sigma_T}{2\pi} \int_S \Psi^{(1)} d\mathbf{u}, \tag{2.4b}$$

$$\mathbf{u} \cdot \nabla \Psi^{(1)} + \sigma_T \Psi^{(2)} = \sigma_T \frac{1}{2\pi} \int_S \Psi^{(2)} d\mathbf{u} - \sigma_a \frac{1}{2\pi} \int_S \Psi^{(0)} d\mathbf{u} + q. \tag{2.4c}$$

Dividing both sides of (2.4b) by σ_T gives

$$\Psi^{(1)} = -\frac{1}{\sigma_T} \mathbf{u} \cdot \nabla \Psi^{(0)} + \frac{1}{2\pi} \int_S \Psi^{(1)} \, d\mathbf{u}.$$

Then take the gradient of both sides of the above equation, left dot it by $\mathbf{u}$ and integrate over S, from $\int_S \mathbf{u} \, d\mathbf{u} = (0,0)^T$, we have

$$\int_S \mathbf{u} \cdot \nabla \Psi^{(1)} \, d\mathbf{u} = -\int_S \mathbf{u} \cdot \nabla \big(\frac{1}{\sigma_T} \mathbf{u} \cdot \nabla \Psi^{(0)} \big) \, d\mathbf{u}.$$

Finally integrating both sides of (2.4c) and noting $\int_S \, d\mathbf{u} = 2\pi$, $\int_S \mathbf{u} \cdot \mathbf{u} d\mathbf{u} = 2\pi$ gives

$$-\nabla \cdot \big(\frac{1}{2\sigma_T} \nabla \Phi \big) + \sigma_a \Phi = q, \tag{2.5}$$

where $\Phi = \frac{1}{2\pi} \int_S \Psi^{(0)} d\mathbf{u}$. This is the diffusion limit of (2.1) which means that when $\epsilon \to 0$, the solution of (2.1) becomes isotropic and can be approximated by the solution of (2.5).

In the Cartesian coordinate system, assume

$$\mathbf{z} = (x, y); \qquad \mathbf{u} = (\mu, \nu).$$

Let

$$V = \{-2M, \cdots, -1, 1, \cdots, 2M\}.$$

The discrete-ordinates form of (2.1) is

$$\mu_m \frac{\partial}{\partial x}\psi_m + \nu_m \frac{\partial}{\partial y}\psi_m + \frac{\sigma_T}{\epsilon}\psi_m = \left(\frac{\sigma_T}{\epsilon} - \epsilon\sigma_a\right)\sum_{n\in V}\psi_n w_n + \epsilon q, \quad m \in V, \tag{2.6}$$

where $\mu_m^2 + \nu_m^2 = 1$. For simplicity, the computational domain we consider here is a rectangle:

$$\Omega = \{(x,y)\big| x \in [0,a],\ y \in [0,b]\}.$$

Other more complex domains can be approximated by rectangles, thus are direct extensions. When Ω is a rectangle, the boundary conditions become

$$\begin{aligned} &\psi_m(0,y) = \psi_L(y), \quad \mu_m > 0; \quad \psi_m(a,y) = \psi_R(y), \quad \mu_m < 0, \\ &\psi_m(x,0) = \psi_B(x), \quad \nu_m > 0; \quad \psi_m(x,b) = \psi_T(x), \quad \nu_m < 0. \end{aligned} \tag{2.7}$$

The corresponding diffusion limit of (2.6) can be obtained similarly as for (2.1) by introducing

$$\psi_m = \sum_{n=0}^{\infty} \epsilon^n \psi_m^{(n)}.$$

The equations corresponding to (2.4) are

$$\psi_m^{(0)} = \sum_{n\in V} w_n \psi_n^{(0)}, \tag{2.8a}$$

$$\mu_m \partial_x \psi_m^{(0)} + \nu_m \partial_y \psi_m^{(0)} + \sigma_T \psi_m^{(1)} = \sigma_T \sum_{n\in V} w_n \psi_n^{(1)}, \tag{2.8b}$$

$$\mu_m \partial_x \psi_m^{(1)} + \nu_m \partial_y \psi_m^{(1)} + \sigma_T \psi_m^{(2)} = \sigma_T \sum_{n\in V} w_n \psi_n^{(2)} - \sigma_a \sum_{n\in V} w_n \psi_n^{(0)} + q. \tag{2.8c}$$

In order to get the same form of diffusion equation as (2.5), the quadrature set $\{\mu_n, \nu_n, w_n\}$ should satisfy

$$\begin{aligned} &\textstyle\sum_{n\in V} w_n = 1, \quad \sum_{n\in V} w_n\mu_n = 0, \quad \sum_{n\in V} w_n\nu_n = 0, \\ &\textstyle\sum_{n\in V} w_n\mu_n\nu_n = 0, \quad \sum_{n\in V} w_n\mu_n^2 = \frac{1}{2}, \quad \sum_{n\in V} w_n\nu_n^2 = \frac{1}{2}, \end{aligned} \tag{2.9}$$

and diffusion limit now is

$$-\partial_x\big(\frac{1}{2\sigma_T}\partial_x\phi\big) - \partial_y\big(\frac{1}{2\sigma_T}\partial_y\phi\big) + \sigma_a\phi = q, \tag{2.10}$$

where $\phi = \sum_{n\in V} w_n \psi_n^{(0)}$. Because of the isotropic boundary condition, we have

$$\phi\big|_{x=0} = \psi_L, \quad \phi\big|_{x=a} = \psi_R, \quad \phi\big|_{y=0} = \psi_B, \quad \phi\big|_{y=b} = \psi_T. \tag{2.11}$$

The commonly used $\{\mu_n, \nu_n, w_n\}$ are symmetric, we can assume for $\theta_m \in (0, \frac{\pi}{2})$, $m = 1, \cdots, M$,

$$\begin{aligned} &\theta_m = \theta_{m+M} - \tfrac{\pi}{2} = \theta_{-m} - \pi = \theta_{-m-M} - \tfrac{3}{2}\pi, \quad m = 1, \cdots, M \\ &\mu_m = \cos\theta_m, \quad \nu_m = \sin\theta_m, \qquad m \in V \\ &w_m = w_{-m} = w_{-m-M} = w_{m+M} > 0, \qquad m = 1, \cdots, M. \end{aligned} \tag{2.12}$$

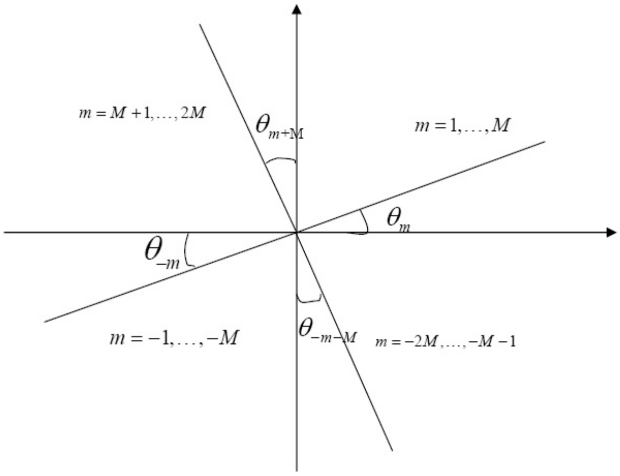

Fig. 2.1. The distribution of the quadrature set.

It is easy to check that when $\{\mu_n, \nu_n, w_n\}$ satisfy (2.12), (2.9) holds automatically. The distribution of the quadrature set can be seen more clearly from Figure 2.1.

The problem we consider here is when some heterogeneous media are put together and the particles do not change their directions when passing through the interfaces. As for the equation (2.6) that controls the movement of the particles, that means at the interfaces the coefficients σ_T, σ_a and q have discontinuities, but ψ_m are continuous. For any interface lines α, assuming the two different media are denoted by $+$ and $-$, we have

$$\psi_m^+\big|_\alpha = \psi_m^-\big|_\alpha. \tag{2.13}$$

In this paper we only consider the case when the interfaces consist of pieces of lines parallel to the x or y coordinates. Because the discontinuities only occur in one direction for each piece of interface line, we only need to use the one dimensional interface analysis. As proved in [15], in one dimension where the space coordinate is z, the interface conditions for the diffusion limit are

$$\phi^+ = \phi^-, \qquad \frac{1}{\sigma_T^+}\partial_z\phi^+ = \frac{1}{\sigma_T^-}\partial_z\phi^-.$$

Thus the interface conditions for our two dimensional case are

$$\begin{aligned} \phi^+\big|_\alpha &= \phi^-\big|_\alpha, \\ \tfrac{1}{\sigma_T^+}\partial_z\phi^+\big|_\alpha &= \tfrac{1}{\sigma_T^-}\partial_z\phi^-\big|_\alpha, \end{aligned} \tag{2.14}$$

where z is $x(y)$ when α is parallel to $y(x)$.

3. Derivation of the scheme

The scheme is focused on the space discretization of the discrete ordinate form (2.6). The basic idea of this method includes two approximations:

i) to approximate the coefficients by piecewise constants;

ii) to approximate the solution on each cell edge by its cell edge average as in (3.6).

The details are as follows:

Generate a set of grid points

$$G = \{(x_i, y_j)\big|0 = x_0 < x_1 < \cdots < x_N = a,\, 0 = y_0 < y_1 < \cdots < y_{N'} = b\}.$$

Let

$$\triangle = \max_{i=0,\cdots,N-1;j=0,\cdots,N'-1} \Big\{|x_{i+1} - x_i|, |y_{j+1} - y_j|\Big\}$$

and for $i = 0, \cdots, N-1, j = 0, \cdots, N'-1$,

$$\sigma_{aij} = \frac{1}{(x_{i+1} - x_i)(y_{j+1} - y_j)} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} \sigma_a dxdy,$$

$$\sigma_{Tij} = \frac{1}{(x_{i+1} - x_i)(y_{j+1} - y_j)} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} \sigma_T dxdy,$$

$$q_{ij} = \frac{1}{(x_{i+1} - x_i)(y_{j+1} - y_j)} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} q dxdy.$$

The first approximation of (2.6) is

$$\mu_m \frac{\partial}{\partial x}\tilde{\psi}_m + \nu_m \frac{\partial}{\partial y}\tilde{\psi}_m + \frac{\tilde{\sigma}_T}{\epsilon}\tilde{\psi}_m = \left(\frac{\tilde{\sigma}_T}{\epsilon} - \epsilon\tilde{\sigma}_a\right) \sum_{n\in V} \tilde{\psi}_n w_n + \epsilon\tilde{q}, \quad m \in V, \tag{3.1}$$

where $\tilde{\sigma}_a$, $\tilde{\sigma}_T$, $\tilde{q}$ are piecewise constants:

$$\tilde{\sigma}_a(x, y) = \sigma_{aij}, \quad \tilde{\sigma}_T(x, y) = \sigma_{Tij}, \quad \tilde{q}(x, y) = q_{ij}, \quad (x, y) \in (x_i, x_{i+1}] \times (y_j, y_{j+1}]. \tag{3.2}$$

Now (3.1) can be solved by similar ideas as in [17, 19, 1], but the approach is different. On each rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we seek the solution of

$$\mu_m \partial\tilde{\psi}_m + \nu_m \partial y\tilde{\psi}_m + \frac{\sigma_{Tij}}{\epsilon}\tilde{\psi}_m = \left(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\right) \sum_{n\in V} \tilde{\psi}_n w_n + \epsilon q_{ij}, \quad m \in V. \tag{3.3}$$

Let

$$\hat{\psi}_{mj}(x) = \frac{1}{y_{j+1} - y_j} \int_{y_j}^{y_{j+1}} \tilde{\psi}_m dy, \qquad \check{\psi}_{mi}(y) = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \tilde{\psi}_m dx. \tag{3.4}$$

Integrating both sides of (3.3) from y_j to y_{j+1} gives

$$\begin{aligned} \mu_m \partial_x \hat{\psi}_{mj}(x) + \frac{\sigma_{Tij}}{\epsilon}\hat{\psi}_{mj}(x) \quad &= \quad \left(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\right) \sum_{n\in V} w_n \hat{\psi}_{nj}(x) + \epsilon q_{ij} \\ &\quad -\nu_m \frac{\tilde{\psi}_m(x, y_{j+1}) - \tilde{\psi}_m(x, y_j)}{y_{j+1} - y_j}, \end{aligned} \tag{3.5a}$$

and from x_i to x_{i+1},

$$\begin{aligned} \nu_m \partial_y \check{\psi}_{mi}(y) + \frac{\sigma_{Tij}}{\epsilon}\check{\psi}_{mi}(y) \quad &= \quad \left(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\right) \sum_{n\in V} w_n \check{\psi}_{ni}(y) + \epsilon q_{ij} \\ &\quad -\mu_m \frac{\tilde{\psi}_m(x_{i+1}, y) - \tilde{\psi}_m(x_i, y)}{x_{i+1} - x_i}. \end{aligned} \tag{3.5b}$$

Introduce the second approximation

$$\tilde{\psi}_m(x, y_{j+1}) \approx \check{\psi}_{mi}(y_{j+1}), \qquad \tilde{\psi}_m(x, y_j) \approx \check{\psi}_{mi}(y_j), \tag{3.6a}$$

in (3.5a) and

$$\tilde{\psi}_m(x_i, y) \approx \hat{\psi}_{mj}(x_i), \qquad \tilde{\psi}_m(x_{i+1}, y) \approx \hat{\psi}_{mj}(x_{i+1}), \tag{3.6b}$$

in (3.5b), we have

$$\begin{aligned} \mu_m \partial_x \hat{\psi}_{mj}(x) + \frac{\sigma_{Tij}}{\epsilon} \hat{\psi}_{mj}(x) &= \left(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\right) \sum_{n\in V} w_n \hat{\psi}_{nj}(x) + \epsilon q_{ij} \\ &\quad - \nu_m \frac{\check{\psi}_{mi}(y_{j+1}) - \check{\psi}_{mi}(y_j)}{y_{j+1} - y_j}, \quad m \in V, \qquad (3.7a) \\ \nu_m \partial_y \check{\psi}_{mi}(y) + \frac{\sigma_{Tij}}{\epsilon} \check{\psi}_{mi}(y) &= \left(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\right) \sum_{n\in V} w_n \check{\psi}_{ni}(y) + \epsilon q_{ij} \\ &\quad - \mu_m \frac{\hat{\psi}_{mj}(x_{i+1}) - \hat{\psi}_{mj}(x_i)}{x_{i+1} - x_i}, \quad m \in V. \qquad (3.7b) \end{aligned}$$

The boundary conditions for $\hat{\psi}_{mj}$ and $\check{\psi}_{mi}$ corresponding to (2.7) become

$$\begin{aligned} \hat{\psi}_{mj}\big|_{x=0} &= \tfrac{1}{y_{j+1}-y_j} \int_{y_j}^{y_{j+1}} \psi_L dy = \psi_{Lj}, \quad \mu_m < 0; \\ \hat{\psi}_{mj}\big|_{x=a} &= \tfrac{1}{y_{j+1}-y_j} \int_{y_j}^{y_{j+1}} \psi_R dy = \psi_{Rj}, \quad \mu_m > 0; \\ \check{\psi}_{Bi}\big|_{y=0} &= \tfrac{1}{x_{i+1}-x_i} \int_{x_i}^{x_{i+1}} \psi_B dx = \psi_{Bj}, \quad \nu_m < 0; \\ \check{\psi}_{Ti}\big|_{y=b} &= \tfrac{1}{x_{i+1}-x_i} \int_{x_i}^{x_{i+1}} \psi_T dx = \psi_{Tj}, \quad \nu_m > 0. \end{aligned} \tag{3.8}$$

Clearly there are $4MN + 4MN'$ boundary conditions in all.

The system of equations (3.7) can be solved analytically using the same idea as in [15] and the following part is trying to find the general solution of (3.7). We should emphasize here that in (3.7a), μ_m could be the same for different m. From (2.12), the same μ_m occurs in

$$S_\mu = \{\mu_n, | n \in V\}$$

twice at most. Define a new set

$$\hat{S} = \{\hat{\mu}_n, \hat{w}_n | n \in \hat{V}\},$$

where $\hat{V} = \{-\hat{M}, \cdots, -1, 1, \cdots, \hat{M}\}$ in the following way: if μ_m occurs once in S_μ, keep μ_m, w_m unchanged in $\hat{S}$, otherwise for $\mu_m = \mu_{m_1}$, $m \neq m_1$, let μ_m occur once as $\hat{\mu}_{m'}$ in $\hat{S}$ and the corresponding $\hat{w}_{m'} = w_m + w_{m_1}$. $2\hat{M}$ is the order of this new set $\hat{S}$, in which $\hat{\mu}_n$ are different from each other. Obviously, in S_μ there are $4M - 2\hat{M}$ μ_m occur twice and $4\hat{M} - 4M$ occur once.

Introduce some new variables

$$\begin{aligned} \hat{\varphi}_{m'j} &= \hat{\psi}_{mj}, & \mu_m &= \hat{\mu}_{m'} & &\mu_m \text{ occurs once in } \hat{S} \\ \hat{\varphi}_{m'j} &= \tfrac{w_m \hat{\psi}_{mj} + w_{m_1} \hat{\psi}_{mj}}{w_m + w_{m_1}}, & \mu_m &= \mu_{m_1} = \hat{\mu}_{m'} & &\mu_m \text{ occurs twice in } \hat{S} \end{aligned} \tag{3.9}$$

and from (3.7a) $\hat{\varphi}_{m'j}$, $m' \in \hat{V}$ satisfy

$$\hat{\mu}_{m'} \partial_x \hat{\varphi}_{m'j}(x) + \frac{\sigma_{Tij}}{\epsilon} \hat{\varphi}_{m'j}(x) = \left(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\right) \sum_{n\in V} \hat{w}_n \hat{\varphi}_{nj}(x) + \epsilon q_{ij} - \hat{r}_{m'ij} \tag{3.10}$$

where

$$\hat{r}_{m'ij} = \begin{cases} \nu_m \frac{\check{\psi}_{mi}(y_{j+1}) - \check{\psi}_{mi}(y_j)}{y_{j+1}-y_j}, & \mu_m = \hat{\mu}_{m'}, \quad \mu_m \text{occurs once in } S_\mu \\ \frac{1}{w_m + w_{m_1}} \Big(w_m \nu_m \frac{\check{\psi}_{mi}(y_{j+1}) - \check{\psi}_{mi}(y_j)}{y_{j+1}-y_j} + w_{m_1} \nu_{m_1} \frac{\check{\psi}_{m_1 i}(y_{j+1}) - \check{\psi}_{m_1 i}(y_j)}{y_{j+1}-y_j} \Big), & \\ \qquad \mu_m = \mu_{m_1} = \hat{\mu}_{m'}, \quad \mu_m \text{occurs twice in } S_\mu \end{cases} \tag{3.11}$$

$\hat{r}_{m'ij}$ are undetermined constants on $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. For this new system of equations (3.10), $\hat{\mu}_{m'}$ are different from each other. Assume

$$1 > \hat{\mu}_{\hat{M}} > \cdots > \hat{\mu}_1 > 0 > \hat{\mu}_{-1} > \cdots > \hat{\mu}_{-\hat{M}} > -1$$

and the symmetry: $\hat{\mu}_n = -\hat{\mu}_{-n}$, $\hat{w}_n = \hat{w}_{-n}$ can be obtained from (2.12).

This new system of equations (3.10) is exactly the same as the one dimensional system of equations solved in [15] but the $\hat{r}_{mij}$ terms. We can find the $4\hat{M}$ relations between $\{\hat{\varphi_{m}}_j(x_i), m \in \hat{V}\}$ and $\{\hat{\varphi_{m}}_j(x_{i+1}), m \in \hat{V}\}$ through the general solution of (3.10) as in [15]. As proved in the Appendix of [15], we have the following theorem:

Theorem 3.1. *Consider the equation*

$$\sum_{n \in \hat{V}} \frac{\hat{w}_n}{1 - \hat{\mu}_n \xi} = \frac{1}{1 - \epsilon^2 \frac{\sigma_{aij}}{\sigma_{Tij}}}. \tag{3.12}$$

i) When $\sigma_{aij} \neq 0$, (3.12) has $2\hat{M}$ simple roots that occur in positive/negative pairs, let them be ξ_n $(1 \leq |n| \leq \hat{M})$. Assume

$$\hat{l}_m^{(n)} = \frac{1}{1 - \hat{\mu}_m \xi_n}, \tag{3.13}$$

we have

$$\sum_{m \in \hat{V}} \hat{w}_m \hat{\mu}_m \hat{l}_m^{(k)} \hat{l}_m^{(n)} = \begin{cases} 0 & n \neq k \\ c^{(k)} & n = k \end{cases}, \tag{3.14a}$$

where $\hat{c}^{(k)}$ satisfy

$$\sum_{k \in \hat{V}} \frac{1}{c^{(k)}} \hat{l}_m^{(k)} l_n^{(k)} = \begin{cases} 0 & m \neq n \\ \frac{1}{\hat{w}_n \hat{\mu}_n} & m = n \end{cases}. \tag{3.14b}$$

ii) When $\sigma_{aij} = 0$, (3.12) has $2\hat{M} - 2$ simple roots appear in positive/negative pairs while 0 is a double root. Assume ξ_n $(1 \leq n \leq \hat{M} - 1)$ is the unique (positive, simple) root in $(1/\hat{\mu}_{n+1}, 1/\hat{\mu}_n)$, $\xi_{-n} = -\xi_n$ and $\hat{l}_m^{(n)} = \frac{1}{1 - \hat{\mu}_m \xi_n}$. (3.14a) still holds for $k, n \in \{-\hat{M} + 1, \cdots, \hat{M} - 1\}$. Moreover, defining $c^{(-\hat{M})} = c^{(\hat{M})} = \sum_{m \in \hat{V}} \hat{w}_m \hat{\mu}_m^2 = 1/2$, we have

$$\sum_{1 \leq |k| \leq \hat{M}-1} \frac{1}{c^{(k)}} l_n^{(k)} \hat{l}_m^{(k)} + \frac{\hat{\mu}_n}{\hat{c}^{(-\hat{M})}} + \frac{\hat{\mu}_m}{c^{(\hat{M})}} = \begin{cases} 0 & m \neq n \\ \frac{1}{\hat{w}_n \hat{\mu}_n} & m = n \end{cases}, \tag{3.15a}$$

and

$$\sum_{m \in \hat{V}} \hat{\mu}_m \hat{w}_m \hat{l}_m^{(k)} = 0, \qquad \sum_{m \in \hat{V}} \hat{\mu}_m^2 \hat{w}_m \hat{l}_m^{(k)} = 0. \tag{3.15b}$$

The derivation of the general solution of (3.10) is put into two different cases:

i) When $\sigma_{aij} \neq 0$, multiplying both sides of (3.10) by $w_m \hat{l}_m^{(k)}$ and summing over $\hat{V}$ gives

$$\begin{aligned} &\sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m \hat{l}_m^{(k)} \partial_x \hat{\varphi}_{mj} + \frac{\sigma_{Tij}}{\epsilon} \sum_{m\in\hat{V}} \hat{w}_m \hat{l}_m^{(k)} \hat{\varphi}_{mj} \\ = &\Big(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\Big) \sum_{m\in\hat{V}} \hat{w}_m \hat{l}_m^{(k)} \Big(\sum_{n\in\hat{V}} \hat{w}_n \hat{\varphi}_{nj} + \epsilon q_{ij}\Big) - \sum_{m\in\hat{V}} \hat{w}_m \hat{l}_m^{(k)} \hat{r}_{mij}. \end{aligned} \tag{3.16}$$

Noting (3.12)(3.13), one gets

$$\sum_{m\in\hat{V}} \hat{w}_m \hat{l}_m^{(k)} = \frac{1}{1-\epsilon^2 \frac{\sigma_{aij}}{\sigma_{Tij}}}$$

and

$$\begin{aligned} &\frac{\sigma_{Tij}}{\epsilon} \sum_{m\in\hat{V}} \hat{w}_m \hat{l}_m^{(k)} \hat{\varphi}_{mj} - \Big(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\Big) \sum_{m\in V} \hat{w}_m \hat{l}_m^{(k)} \sum_{n\in V} \hat{w}_n \hat{\varphi}_{nj} \\ = &\frac{\sigma_{Tij}}{\epsilon} \Big(\sum_{m\in\hat{V}} \hat{w}_m \hat{l}_m^{(k)} \hat{\varphi}_{mj} - \sum_{m\in\hat{V}} \hat{w}_m \hat{\varphi}_{mj}\Big) = \xi_k \frac{\sigma_{Tij}}{\epsilon} \sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m \hat{l}_m^{(k)} \hat{\varphi}_{mj}. \end{aligned}$$

Thus (3.16) is

$$\partial_x \sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m \hat{l}_m^{(k)} \hat{\varphi}_{mj} + \xi_k \frac{\sigma_{Tij}}{\epsilon} \sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m \hat{l}_m^{(k)} \hat{\varphi}_{mj} = \big(\sigma_{Tij} - \epsilon^2 \sigma_{aij}\big) q_{ij} - \sum_{m\in\hat{V}} \hat{w} \hat{l}_m^{(k)} \hat{r}_{mij}. \tag{3.17}$$

Multiplying both sides of (3.17) by $\exp\big(\frac{\sigma_{Tij}}{\epsilon}\xi_k x\big)$ and integrating from x_i to x_{i+1} gives

$$\begin{aligned} &\exp\big(\xi_k \frac{\sigma_{Tij}}{\epsilon} x_{i+1}\big) \sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m \hat{l}_m^{(k)} \hat{\varphi}_{mj}(x_{i+1}) - \exp\big(\xi_k \frac{\sigma_{Tij}}{\epsilon} x_i\big) \sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m \hat{l}_m^{(k)} \hat{\varphi}_{mj}(x_i) \\ = &\frac{\epsilon}{\xi_k \sigma_{Tij}} \Big(\big(\sigma_{Tij} - \epsilon^2 \sigma_{aij}\big) q_{ij} - \sum_{m\in\hat{V}} \hat{w}_m \hat{l}_m^{(k)} \hat{r}_{mij}\Big) \Big(\exp\big(\xi_k \frac{\sigma_{Tij}}{\epsilon} x_{i+1}\big) - \exp\big(\xi_k \frac{\sigma_{Tij}}{\epsilon} x_i\big)\Big). \end{aligned} \tag{3.18}$$

These are $2\hat{M}$ independent relations between $\{\hat{\varphi}_{mj}(x_{i+1}), m \in \hat{V}\}$, $\{\hat{\varphi}_{mj}(x_i), m \in \hat{V}\}$ and $\{\hat{r}_{mij}, m \in \hat{V}\}$.

ii) When $\sigma_{aij} = 0$, (3.12) has $2\hat{M} - 1$ eigenvalues from (ii) of Theorem 3.1. For $1 \le |k| \le \hat{M} - 1$, we can use the same discussion as for $\sigma_{aij} \neq 0$ to obtain $2\hat{M} - 2$ independent relations between $\{\hat{\varphi}_{mj}(x_{i+1}), m \in \hat{V}\}$, $\{\hat{\varphi}_{mj}(x_i), m \in \hat{V}\}$ and $\{\hat{r}_{mij}, m \in \hat{V}\}$. For the other two relations, we can multiply both sides of (3.10) by $\hat{w}_{m'}$ and $\hat{w}_{m'}\hat{\mu}_{m'}$, sum over $\hat{V}$ respectively and obtain

$$\partial_x \sum_{m\in V} \hat{w}_m \hat{\mu}_m \hat{\varphi}_{mj} = \epsilon q_{ij} - \sum_{m\in\hat{V}} \hat{w}_m \hat{r}_{mij}, \tag{3.19}$$

$$\partial_x \sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m^2 \hat{\varphi}_{mj} + \frac{\sigma_{Tij}}{\epsilon} \sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m \hat{\varphi}_{mj} = - \sum_{m\in\hat{V}} \hat{w}_m \hat{\mu}_m \hat{r}_{mij}. \tag{3.20}$$

Integrating both sides of (3.19) from x_i to $x \in (x_i, x_{i+1}]$ gives

$$\sum_{m\in V} \hat{w}_m \hat{\mu}_m \hat{\varphi}_{mj}(x) - \sum_{m\in V} \hat{w}_m \hat{\mu}_m \hat{\varphi}_{mj}(x_i) = (x - x_i)\big(\epsilon q_{ij} - \sum_{m\in\hat{V}} \hat{w}_m \hat{r}_{mij}\big). \tag{3.21}$$

In particular when $x = x_{i+1}$, we have

$$\sum_{m\in V} \hat{w}_m\hat{\mu}_m\hat{\varphi}_{mj}(x_{i+1}) - \sum_{m\in V} \hat{w}_m\hat{\mu}_m\hat{\varphi}_{mj}(x_i) = (x_{i+1}-x_i)\big(\epsilon q_{ij} - \sum_{m\in\hat{V}} \hat{w}_m\hat{r}_{mij}\big). \quad (3.22)$$

Integrating both sides of (3.20) from x_i to x_{i+1} and using (3.21), one gets

$$\begin{aligned} &\sum_{m\in V} \hat{w}_m\hat{\mu}_m^2\hat{\varphi}_{mj}(x_{i+1}) - \sum_{m\in V} \hat{w}_m\hat{\mu}_m^2\hat{\varphi}_{mj}(x_i) + \frac{\sigma_{Tij}}{\epsilon}(x_{i+1}-x_i)\sum_{m\in\hat{V}} \hat{w}\hat{\mu}_m\hat{\varphi}_{mj}(x_i) \\ = \;& \frac{1}{2}(x_{i+1}-x_i)^2\big(\epsilon q_{ij} - \sum_{m\in\hat{V}} \hat{w}_m\hat{r}_{mij}\big) - (x_{i+1}-x_i)\sum_{m\in\hat{V}} \hat{w}_m\hat{\mu}_m\hat{r}_{mij}. \quad (3.23) \end{aligned}$$

(3.22) and (3.23) are the other two relations.

Up to now, for $\sigma_{aij} \neq 0$ and $\sigma_{aij} = 0$, we've found $2\hat{M}$ independent relations of $\{\hat{\varphi}_{mj}(x_{i+1}), m \in \hat{V}\}$, $\{\hat{\varphi}_{mj}(x_i), m \in \hat{V}\}$ and $\{\hat{r}_{mij}, m \in \hat{V}\}$. When coming back to $\hat{\psi}_{mj}$, from (3.7), for $\mu_m = \mu_{m_1} = \hat{\mu}_{m'}$, one has

$$\begin{aligned} &\hat{\mu}_{m'}\partial_x\big(\hat{\psi}_{mj}(x) - \hat{\psi}_{m_1j}(x)\big) + \frac{\sigma_{Tij}}{\epsilon}\big(\hat{\psi}_{mj} - \hat{\psi}_{m_1j}\big) \\ = \;& \nu_{m_1}\frac{\check{\psi}_{m_1i}(y_{j+1}) - \check{\psi}_{m_1i}(y_j)}{y_{j+1}-y_j} - \nu_m\frac{\check{\psi}_{mi}(y_{j+1}) - \check{\psi}_{mi}(y_j)}{y_{j+1}-y_j}. \quad (3.24) \end{aligned}$$

Multiplying both sides of the above equation by $\exp\big(\frac{\sigma_{Tij}}{\epsilon\hat{\mu}_{m'}}x\big)$ and integrating from x_i to x_{i+1} gives

$$\begin{aligned} &\exp\Big(\frac{\sigma_{Tij}}{\epsilon\hat{\mu}_{m'}}x_{i+1}\Big)\Big(\hat{\psi}_{mj}(x_{i+1}) - \hat{\psi}_{m_1j}(x_{i+1})\Big) - \exp\Big(\frac{\sigma_{Tij}}{\epsilon\hat{\mu}_{m'}}x_i\Big)\Big(\hat{\psi}_{mj}(x_i) - \hat{\psi}_{m_1j}(x_i)\Big) \\ = \;& \frac{\epsilon\hat{\mu}_{m'}}{\sigma_{Tij}}\Big(\exp(\frac{\sigma_{Tij}}{\hat{\mu}_{m'}\epsilon}x_{i+1}) - \exp(\frac{\sigma_{Tij}}{\hat{\mu}_{m'}\epsilon}x_i)\Big) \\ &\times\Big(\nu_{m_1}\frac{\check{\psi}_{m_1i}(y_{j+1}) - \check{\psi}_{m_1i}(y_j)}{y_{j+1}-y_j} - \nu_m\frac{\check{\psi}_{mi}(y_{j+1}) - \check{\psi}_{mi}(y_j)}{y_{j+1}-y_j}\Big). \quad (3.25) \end{aligned}$$

Because there are $4M - 2\hat{M}$ μ_m occur twice as discussed previously, we have $4M - 2\hat{M}$ equations of the form (3.25). From (3.9)(3.11), the $2\hat{M}$ relations between $\{\hat{\varphi}_{mj}(x_{i+1}), m \in \hat{V}\}$, $\{\hat{\varphi}_{mj}(x_i), m \in \hat{V}\}$ and $\{\hat{r}_{mij}, m \in \hat{V}\}$ obtained by solving (3.10) are in fact relations between $\{\hat{\psi}_{mj}(x_{i+1}), m \in V\}$, $\{\hat{\psi}_{mj}(x_i), m \in V\}$, $\{\check{\psi}_{mi}(y_{i+1}), m \in V\}$ and $\{\check{\psi}_{mi}(y_i), m \in V\}$. Combing with (3.25), we have $4M$ relations between $\{\hat{\psi}_{mj}(x_{i+1}), m \in V\}$, $\{\hat{\psi}_{mj}(x_i), m \in V\}$, $\{\check{\psi}_{mi}(y_{i+1}), m \in V\}$ and $\{\check{\psi}_{mi}(y_i), m \in V\}$. Similarly, the other $4M$ relations can be found through (3.7b) by the same process. By now, there are $8M$ equations for these $16M$ variables and these are in fact a special finite difference scheme on the rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$.

In summary, this finite difference scheme is as follows

When $\sigma_{aij} \neq 0$:

$$\begin{aligned} &\exp\big(\xi_k\frac{\sigma_{Tij}}{\epsilon}x_{i+1}\big)\sum_{m\in\hat{V}} \hat{w}_m\hat{\mu}_m\hat{l}_m^{(k)}\hat{\varphi}_{mj}(x_{i+1}) - \exp\big(\xi_k\frac{\sigma_{Tij}}{\epsilon}x_i\big)\sum_{m\in\hat{V}} \hat{w}_m\hat{\mu}_m\hat{l}_m^{(k)}\hat{\varphi}_{mj}(x_i) \\ = \;& \frac{\epsilon}{\xi_k\sigma_{Tij}}\Big((\sigma_{Tij} - \epsilon^2\sigma_{aij})q_{ij} - \sum_{m\in\hat{V}} \hat{w}_m\hat{l}_m^{(k)}\hat{r}_{mij}\Big)\Big(\exp\big(\xi_k\frac{\sigma_{Tij}}{\epsilon}x_{i+1}\big) - \exp\big(\xi_k\frac{\sigma_{Tij}}{\epsilon}x_i\big)\Big), \\ & \qquad 1 \le |k| \le \hat{M}, \quad (3.26a) \end{aligned}$$

where

$$\hat{\varphi}_{mj} = \begin{cases} \hat{\psi}_{m'j}, & \mu_{m'} = \hat{\mu}_m, \quad \mu_{m'} \text{ occurs once in } S_\mu \\ \frac{w_{m'}\hat{\psi}_{m'j} + w_{m_1}\hat{\psi}_{m_1 j}}{w_{m'} + w_{m_1}}, & \mu_{m'} = \mu_{m_1} = \hat{\mu}_m, \quad \mu_{m'} \text{ occurs twice in } S_\mu \end{cases}$$

and

$$\hat{r}_{mij} = \begin{cases} \nu_{m'} \frac{\check{\psi}_{m'i}(y_{j+1}) - \check{\psi}_{m'i}(y_j)}{y_{j+1} - y_j}, & \mu_{m'} = \hat{\mu}_m, \quad \mu_{m'} \text{occurs once in } S_\mu \\ \frac{1}{w_{m'} + w_{m_1}} \Big(w_{m'}\nu_{m'} \frac{\check{\psi}_{m'i}(y_{j+1}) - \check{\psi}_{m'i}(y_j)}{y_{j+1} - y_j} + w_{m_1}\nu_{m_1} \frac{\check{\psi}_{m_1 i}(y_{j+1}) - \check{\psi}_{m_1 i}(y_j)}{y_{j+1} - y_j} \Big), & \\ \qquad \mu_{m'} = \mu_{m_1} = \hat{\mu}_m, \quad \mu_m \text{occurs twice in } S_\mu \end{cases} .$$

When μ_m occurs twice in S_μ, let $\mu_{m'} = \mu_{m_1} = \hat{\mu}_m$,

$$\begin{aligned} & \exp\Big(\frac{\sigma_{Tij}}{\epsilon\hat{\mu}_m} x_{i+1}\Big)\Big(\hat{\psi}_{m'j}(x_{i+1}) - \hat{\psi}_{m_1 j}(x_{i+1})\Big) - \exp\Big(\frac{\sigma_{Tij}}{\epsilon\hat{\mu}_m} x_i\Big)\Big(\hat{\psi}_{m'j}(x_i) - \hat{\psi}_{m_1 j}(x_i)\Big) \\ = \ & \frac{\epsilon\hat{\mu}_m}{\sigma_{Tij}}\Big(\exp(\frac{\sigma_{Tij}}{\hat{\mu}_m\epsilon} x_{i+1}) - \exp(\frac{\sigma_{Tij}}{\hat{\mu}_m\epsilon} x_i)\Big) \\ & \times\Big(\nu_{m_1}\frac{\check{\psi}_{m_1 i}(y_{j+1}) - \check{\psi}_{m_1 i}(y_j)}{y_{j+1} - y_j} - \nu_{m'}\frac{\check{\psi}_{m'i}(y_{j+1}) - \check{\psi}_{m'i}(y_j)}{y_{j+1} - y_j}\Big). \end{aligned} \tag{3.26b}$$

The other $4M$ relations according to (3.7b) are:

$$\begin{aligned} & \exp\big(\eta_k\frac{\sigma_{Tij}}{\epsilon} y_{j+1}\big)\sum_{m\in\check{V}} \check{w}_m\check{\nu}_m\check{l}_m^{(k)}\check{\varphi}_{mi}(y_{j+1}) - \exp\big(\eta_k\frac{\sigma_{Tij}}{\epsilon} y_j\big)\sum_{m\in\check{V}} \check{w}_m\check{\nu}_m\check{l}_m^{(k)}\check{\varphi}_{mi}(y_j) \\ = \ & \frac{\epsilon}{\eta_k\sigma_{Tij}}\Big((\sigma_{Tij} - \epsilon^2\sigma_{aij})q_{ij} - \sum_{m\in\check{V}} \check{w}_m\check{l}_m^{(k)}\check{r}_{mij}\Big)\Big(\exp\big(\eta_k\frac{\sigma_{Tij}}{\epsilon} y_{j+1}\big) - \exp\big(\eta_k\frac{\sigma_{Tij}}{\epsilon} y_j\big)\Big), \\ & \qquad 1 \le |k| \le \check{M}, \end{aligned} \tag{3.26c}$$

and for $\nu_{m'}$ occurs twice in S_ν, let $\nu_{m'} = \nu_{m_1} = \check{\nu}_m$,

$$\begin{aligned} & \exp\Big(\frac{\sigma_{Tij}}{\epsilon\check{\nu}_m} y_{j+1}\Big)\Big(\check{\psi}_{m'i}(y_{j+1}) - \check{\psi}_{m_1 i}(y_{j+1})\Big) - \exp\Big(\frac{\sigma_{Tij}}{\epsilon\check{\nu}_m} y_j\Big)\Big(\check{\psi}_{m'i}(y_j) - \check{\psi}_{m_1 i}(y_j)\Big) \\ = \ & \frac{\epsilon\check{\nu}_m}{\sigma_{Tij}}\Big(\exp(\frac{\sigma_{Tij}}{\check{\nu}_m\epsilon} y_{j+1}) - \exp(\frac{\sigma_{Tij}}{\check{\nu}_m\epsilon} y_j)\Big) \\ & \times\Big(\mu_{m_1}\frac{\hat{\psi}_{m_1 j}(x_{i+1}) - \hat{\psi}_{m_1 j}(x_i)}{x_{i+1} - x_i} - \mu_{m'}\frac{\hat{\psi}_{m'j}(x_{i+1}) - \hat{\psi}_{m'j}(x_i)}{x_{i+1} - x_i}\Big). \end{aligned} \tag{3.26d}$$

Here S_ν, $\check{S} = \{\check{\nu}_n, \check{w}_n | n \in \check{V}\}$, $\check{M}$, $\check{V}$, $\check{\varphi}$, $\check{r}$, $\check{l}_m^{(k)}$ are defined parallel to S_μ, $\hat{S} = \{\hat{\mu}_n, \hat{w}_n | n \in \hat{V}\}$, $\hat{M}$, $\hat{V}$, $\hat{\varphi}$, $\hat{r}$, $\hat{l}_m^{(k)}$ respectively. When $\sigma_{aij} = 0$, the scheme can be easily written down by replacing (3.26a) for $1 \le |k| \le \hat{M}$ by (3.26a) for $1 \le |k| \le \hat{M} - 1$ and (3.22)(3.23). So are similar replacements for (3.26c).

When ϵ small, the exponential terms in (3.26) could be extremely big and overflow always happens. To solve this problem, we can divide both sides of (3.26) by some factors to keep the power of the exponential terms negative. For example, when $\xi_k > 0$ in (3.26a), dividing both sides by $\exp\big(\xi_k\frac{\sigma_{Tij}}{\epsilon} x_{i+1}\big)$ gives

$$\begin{aligned} & \sum_{m\in\hat{V}} \hat{w}_m\hat{\mu}_m\hat{l}_m^{(k)}\hat{\varphi}_{mj}(x_{i+1}) - \exp\big(-\xi_k\frac{\sigma_{Tij}}{\epsilon}(x_{i+1} - x_i)\big)\sum_{m\in\hat{V}} \hat{w}_m\hat{\mu}_m\hat{l}_m^{(k)}\hat{\varphi}_{mj}(x_i) \\ = \ & \frac{\epsilon}{\xi_k\sigma_{Tij}}\Big((\sigma_{Tij} - \epsilon^2\sigma_{aij})q_{ij} - \sum_{m\in\hat{V}} \hat{w}\hat{l}_m^{(k)}\hat{r}_{mij}\Big)\Big(1 - \exp\big(-\xi_k\frac{\sigma_{Tij}}{\epsilon}(x_{i+1} - x_i)\big)\Big), \\ & \qquad 1 \le |k| \le \hat{M}. \end{aligned} \tag{3.27}$$

The exponential terms in (3.27) are bounded by 1. Similar treatments can also bound all the other exponential terms in (3.26).

Now consider the matrix form of this special finite difference scheme. Let

$$\hat{\psi}_j = (\hat{\psi}_{-2Mj}, \cdots, \hat{\psi}_{-1j}, \hat{\psi}_{1j}, \cdots, \hat{\psi}_{2Mj})^T, \qquad \check{\psi}_i = (\hat{\psi}_{-2Mi}, \cdots, \hat{\psi}_{-1i}, \hat{\psi}_{1i}, \cdots, \hat{\psi}_{2Mi})^T,$$

(3.26) can be expressed into the matrix form:

$$\hat{A}_{ij}\begin{pmatrix} \hat{\psi}_j(x_i) \\ \hat{\psi}_j(x_{i+1}) \end{pmatrix} + \check{A}_{ij}\begin{pmatrix} \check{\psi}_i(y_j) \\ \check{\psi}_i(y_{j+1}) \end{pmatrix} = \mathbf{b}_{ij}, \tag{3.28}$$

where $\hat{A}_{ij}$, $\check{A}_{ij}$ are $8M \times 8M$ matrices and $\mathbf{b}_{ij}$ is a $8M \times 1$ vector. We piece together the numerical solution with the neighboring cells using the interface conditions (2.13). That is two neighboring cells sharing a common $\hat{\psi}$ or $\check{\psi}$.

Specifically, we can arrange the unknowns as

$$\psi = (\hat{\psi}, \check{\psi})^T$$

where

$$\hat{\psi} = \big(\hat{\psi}_0(x_0), \hat{\psi}_0(x_1), \cdots, \hat{\psi}_0(x_N), \hat{\psi}_1(x_0), \cdots, \hat{\psi}_1(x_N), \cdots, \hat{\psi}_{N'-1}(x_0), \cdots, \hat{\psi}_{N'-1}(x_N)\big)^T,$$

$$\check{\psi} = \big(\check{\psi}_0(y_0), \check{\psi}_0(y_1), \cdots, \check{\psi}_0(y_{N'}), \check{\psi}_1(y_0), \cdots, \check{\psi}_1(y_{N'}), \cdots, \check{\psi}_{N-1}(y_0), \cdots, \check{\psi}_{N-1}(y_{N'})\big)^T.$$

The coefficient matrix is

$$A = \Big(\hat{A}, \check{A}\Big),$$

where

$$\hat{A} = \left(\begin{array}{cccccc}
\hat{A}_{00} & & 0_{8M\times 4M(N-1)} & & & \\
0_{8M\times 4M} & \hat{A}_{10} & 0_{8M\times 4M(N-2)} & & & \\
 & & & \ddots & & \\
0_{8M\times 4M(N-1)} & & \hat{A}_{N-10} & & & \\
0_{8M\times 4M} & & 0_{8M\times 4MN} & \hat{A}_{01} & & \\
 & & & \ddots & \ddots & \\
 & & & & \hat{A}_{0N'-1} & 0_{8M\times 4M(N-1)} \\
 & & \ddots & \ddots & & \\
 & & & & 0_{8M\times 4M(N-1)} & \hat{A}_{N-1N'-1}
\end{array}\right),$$

which is a $8MNN' \times 4MN'(N+1)$ matrix and

$$\check{A} = \left(\begin{array}{cccccc}
\check{A}_{00} & & 0_{8M\times 4M(N'-1)} & & & \\
 & & 0_{8M\times 4M(N'-1)} & \check{A}_{10} & & \\
 & & & \ddots & & \\
 & & & & \check{A}_{N-10} & 0_{8M\times 4M(N'-1)} \\
0_{4M} & \check{A}_{01} & 0_{8M\times 4M(N'-2)} & & & \\
 & & & \ddots & \ddots & \\
0_{8M\times 4M(N'-1)} & & \check{A}_{0N'-1} & & & \\
 & & \ddots & \ddots & & \\
 & & & & 0_{8M\times 4M(N'-1)} & \check{A}_{N-1N'-1}
\end{array}\right),$$

which is a $8MNN' \times 4MN(N'+1)$ matrix. When

$$\mathbf{b} = \left(\mathbf{b}_{00}, \mathbf{b}_{10}, \cdots, \mathbf{b}_{N-10}, \mathbf{b}_{01}, \cdots, \mathbf{b}_{0N'-1}, \cdots, \mathbf{b}_{N-1N'-1}\right)^T,$$

we have

$$A\psi = \mathbf{b}. \tag{3.29}$$

These are $8MNN'$ equations for ψ, together with the $4MN+4MN'$ boundary conditions (3.8), the $8MNN' + 4MN + 4MN'$ unknowns in ψ are determined.

Remark: By integrating both sides of (3.1) from x_i to x_{i+1} and y_j to y_{j+1} gives

$$\begin{aligned} &\mu_m \frac{\hat{\psi}_{mj}(x_{i+1}) - \hat{\psi}_{mj}(x_i)}{x_{i+1} - x_i} + \nu_m \frac{\check{\psi}_{mi}(y_{j+1}) - \check{\psi}_{mi}(y_j)}{y_{j+1} - y_j} + \frac{\sigma_{Tij}}{\epsilon}\tilde{\varphi}_{mij} \\ = &\left(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\right)\sum_{n\in V} w_n \tilde{\varphi}_{nij} + \epsilon q_{ij}, \qquad m \in V \end{aligned} \tag{3.30}$$

are satisfied with $\hat{\psi}_{mj}$, $\check{\psi}_{mi}$ defined in (3.4) and

$$\tilde{\varphi}_{mij} = \frac{1}{(x_{i+1} - x_i)(y_{j+1} - y_j)} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} \tilde{\psi}_m \, dxdy. \tag{3.31}$$

Consider the approximating system of equations (3.7). By integrating both sides of (3.7a) from x_i to x_{i+1} and dividing them by $x_{i+1} - x_i$, one gets

$$\begin{aligned} &\mu_m \frac{\hat{\psi}_{mj}(x_{i+1}) - \hat{\psi}_{mj}(x_i)}{x_{i+1} - x_i} + \nu_m \frac{\check{\psi}_{mi}(y_{j+1}) - \check{\psi}_{mi}(y_j)}{y_{j+1} - y_j} + \frac{\sigma_{Tij}}{\epsilon}\varphi_{mij} \\ = &\left(\frac{\sigma_{Tij}}{\epsilon} - \epsilon\sigma_{aij}\right)\sum_{n\in V} w_n \varphi_{nij} + \epsilon q_{ij}, \end{aligned} \tag{3.32}$$

with

$$\varphi_{mij} = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \hat{\psi}_{mj} dx.$$

Then we integrate both sides of (3.7b) from y_j to y_{j+1}, divide them by $y_{j+1} - y_j$, compare the obtained equation with (3.32) and get

$$\varphi_{mij} = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \hat{\psi}_{mj} dx = \frac{1}{y_{j+1} - y_j} \int_{y_j}^{y_{j+1}} \check{\psi}_{mi} dy.$$

This consists with the definition of $\hat{\psi}_{mj}$, $\check{\psi}_{mi}$ in (3.4) and φ_{mij} is an approximation of $\tilde{\varphi}_{mij}$ in (3.31). Moreover, that (3.32) is the same as (3.30) implies the second approximation conserves the equation $\tilde{\varphi}_m$ satisfies.

4. Asymptotic preserving property

This method is to solve (3.7) exactly. When $\epsilon \ll 1$, $\Delta/\epsilon \gg 1$, in each rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, substituting

$$\hat{\psi}_{mj} = \sum_{n=0}^{\infty} \epsilon^n \hat{\psi}_{mj}^{(n)}, \qquad \check{\psi}_{mi} = \sum_{n=0}^{\infty} \epsilon^n \check{\psi}_{mi}^{(n)}$$

into (3.7) and equating the same order of ϵ gives

$$\hat{\psi}_{mj}^{(0)} = \sum_{n\in V} w_n \hat{\psi}_{nj}^{(0)}, \qquad \check{\psi}_{mi}^{(0)} = \sum_{n\in V} w_n \check{\psi}_{ni}^{(0)}, \tag{4.1}$$

$$\hat{\psi}_{mj}^{(1)} = -\frac{\mu_m}{\sigma_{Tij}}\partial_x \hat{\psi}_{mj}^{(0)} + \sum_{n\in V} w_n \hat{\psi}_{nj}^{(1)} - \frac{\nu_m}{\sigma_{Tij}} \frac{\check{\psi}_{mi}^{(0)}(y_{j+1}) - \check{\psi}_{mi}^{(0)}(y_j)}{y_{j+1}-y_j}, \tag{4.2a}$$

$$\check{\psi}_{mi}^{(1)} = -\frac{\nu_m}{\sigma_{Tij}}\partial_y \check{\psi}_{mi}^{(0)} + \sum_{n\in V} w_n \check{\psi}_{ni}^{(1)} - \frac{\mu_m}{\sigma_{Tij}} \frac{\hat{\psi}_{mj}^{(0)}(x_{i+1}) - \hat{\psi}_{mj}^{(0)}(x_i)}{x_{i+1}-x_i}, \tag{4.2b}$$

and

$$\begin{aligned} &\mu_m \partial_x \hat{\psi}_{mj}^{(1)} + \sigma_{Tij}\hat{\psi}_{mj}^{(2)} \\ = \;& \sigma_{Tij}\sum_{n\in V} w_n \hat{\psi}_{nj}^{(2)} - \sigma_{aij}\sum_{n\in V} w_n \hat{\psi}_{nj}^{(0)} + q_{ij} - \nu_m \frac{\check{\psi}_{mi}^{(1)}(y_{j+1}) - \check{\psi}_{mi}^{(1)}(y_j)}{y_{j+1}-y_j}, \end{aligned} \tag{4.3a}$$

$$\begin{aligned} &\nu_m \partial_y \check{\psi}_{mi}^{(1)} + \sigma_{Tij}\check{\psi}_{mi}^{(2)} \\ = \;& \sigma_{Tij}\sum_{n\in V} w_n \check{\psi}_{nj}^{(2)} - \sigma_{aij}\sum_{n\in V} w_n \check{\psi}_{nj}^{(0)} + q_{ij} - \nu_m \frac{\hat{\psi}_{mj}^{(1)}(x_{i+1}) - \hat{\psi}_{mj}^{(1)}(x_i)}{x_{i+1}-x_i}. \end{aligned} \tag{4.3b}$$

For both equations in (4.3), multiplying both sides by w_m, summing over V and using (4.1)(4.2)(2.9), one gets

$$-\partial_x\big(\frac{1}{2\sigma_{Tij}}\partial_x\hat{\psi}_j^{(0)}\big) + \sigma_{aij}\hat{\psi}_j^{(0)} = \frac{1}{2\sigma_{Tij}} \frac{\partial_y\check{\psi}_i^{(0)}(y_{j+1}) - \partial_y\check{\psi}_i^{(0)}(y_j)}{y_{j+1}-y_j} + q_{ij}, \tag{4.4a}$$

$$-\partial_y\big(\frac{1}{2\sigma_{Tij}}\partial_y\check{\psi}_i^{(0)}\big) + \sigma_{aij}\check{\psi}_i^{(0)} = \frac{1}{2\sigma_{Tij}} \frac{\partial_x\hat{\psi}_j^{(0)}(x_{i+1}) - \partial_x\hat{\psi}_j^{(0)}(x_i)}{x_{i+1}-x_i} + q_{ij}, \tag{4.4b}$$

where

$$\hat{\psi}_j^{(0)} = \sum_{n\in V} w_n \hat{\psi}_{nj}^{(0)}, \qquad \check{\psi}_i^{(0)} = \sum_{n\in V} w_n \check{\psi}_{ni}^{(0)}. \tag{4.5}$$

Noting the interface condition (2.13), the definitions of $\hat{\psi}_{mj}, \check{\psi}_{mi}$ in (3.4) and $\hat{\psi}_j^{(0)}, \check{\psi}_i^{(0)}$ in (4.5), by using the interface analysis as discussed previously, we can get the connection conditions for $\hat{\psi}_j^{(0)}, \check{\psi}_i^{(0)}$, which are themselves and their first order derivative divided by σ_{Tij} are continuous at the grid lines. Now, for the boundary conditions of $\hat{\psi}_j^{(0)}$ or $\check{\psi}_i^{(0)}$, we have

$$\hat{\psi}_j^{(0)}\big|_{x=0} = \psi_{Lj}, \quad \hat{\psi}_j^{(0)}\big|_{x=a} = \psi_{Rj}, \quad \check{\psi}_i^{(0)}\big|_{y=0} = \psi_{Bi}, \quad \check{\psi}_i^{(0)} = \psi_{Ti}. \tag{4.6}$$

Considering (2.10), firstly approximating the coefficients by piecewise constants like in (3.1)(3.2) and integrating both sides from y_j to y_{j+1} and from x_i to x_{i+1} respectively, we have in the rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$,

$$-\partial_x\big(\frac{1}{2\sigma_{Tij}}\partial_x\hat{\phi}_j\big) - \frac{1}{2\sigma_{Tij}} \frac{\partial_y\phi(x,y_{j+1}) - \partial_y\phi(x,y_j)}{y_{j+1}-y_j} + \sigma_{aij}\hat{\phi}_j = q_{ij}, \tag{4.7a}$$

$$-\partial_y\big(\frac{1}{2\sigma_{Tij}}\partial_y\check{\phi}_i\big) - \frac{1}{2\sigma_{Tij}} \frac{\partial_x\phi(x_{i+1},y) - \partial_x\phi(x_i,y)}{x_{i+1}-x_i} + \sigma_{aij}\check{\phi}_i = q_{ij}, \tag{4.7b}$$

where

$$\hat{\phi}_j = \frac{1}{y_{j+1} - y_j}\int_{y_j}^{y_{j+1}} \phi dy, \qquad \check{\phi}_i = \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} \phi dx.$$

By approximating

$$\begin{aligned} \partial_y\phi(x, y_{j+1}) &\approx \tfrac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \partial_y\phi(x, y_{j+1})dx = \partial_y\check{\phi}_i(y_{j+1}), \\ \partial_y\phi(x, y_j) &\approx \tfrac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \partial_y\phi(x, y_j)dx = \partial_y\check{\phi}_i(y_j), \\ \partial_x\phi(x_{i+1}, y) &\approx \tfrac{1}{y_{j+1}-y_j}\int_{y_j}^{y_{j+1}} \partial_x\phi(x_{i+1}, y)dy = \partial_x\hat{\phi}_j(x_{i+1}), \\ \partial_x\phi(x_i, y) &\approx \tfrac{1}{y_{j+1}-y_j}\int_{y_j}^{y_{j+1}} \partial_x\phi(x_i, y)dy = \partial_x\hat{\phi}_j(x_i), \end{aligned} \tag{4.8}$$

in (4.7), solving (4.7) in each cell analytically and piecing together the neighboring cells using common $\hat{\phi}_j(x_i)$, $\frac{1}{\sigma_{Tij}}\partial_x\hat{\phi}_j(x_i)$ or $\check{\phi}_i(y_j)$, $\frac{1}{\sigma_{Tij}}\check{\phi}_i(y_j)$, we can obtain a discretization of (2.10) with interface conditions (2.14). This is just what $\hat{\psi}_j^{(0)}, \check{\psi}_i^{(0)}$ in (4.4) satisfy. Moreover, from (2.11), $\hat{\phi}$ and $\check{\phi}$ satisfy the same boundary conditions as for $\hat{\psi}_j^{(0)}, \check{\psi}_i^{(0)}$ in (4.6). Thus this scheme is AP.

5. Numerical Example

The performance of the scheme described above is shown in this section. In fact we can find numerically that when there is no boundary layer (the incoming density is homogeneous), this method is first order convergent uniformly with respect to the mean free path ϵ.

The numerical results are presented for several problems in two dimension and the computational domain for all the test problems is $[0, 1] \times [0, 1]$. There is no specified 'best' quadrature set in two dimension and the problem we concern about here is only the space discretization. From (2.12), we use $M = 2$ and $\theta_m = \pi/6, \pi/3$ when $\theta_m \in (0, \pi/2)$, the eight μ_m, ν_m now are

$$\begin{array}{cccc} \mu_1 = 0.866, & \mu_2 = 0.5, & \mu_3 = -0.5, & \mu_4 = -0.866, \\ \mu_{-1} = -0.866, & \mu_{-2} = -0.5, & \mu_{-3} = 0.5, & \mu_{-4} = -0.866 \\ \nu_1 = 0.5, & \nu_2 = 0.866, & \nu_3 = 0.866, & \nu_4 = 0.5, \\ \nu_{-1} = -0.5, & \nu_{-2} = -0.866, & \nu_{-3} = -0.866, & \nu_{-4} = -0.5 \end{array}.$$

for $m = 1, \cdots, 8$, let $w_m = 1/8$. We use $\Delta x = \Delta y = 1/64$ to get the "exact" solutions for all the numerical examples. In the figures and tables, the average density defined by

$$\rho(x, y) = \sum_{m\in V} w_m\psi_m(x, y)$$

are displayed. Assume $\hat{\rho}_{ij}$, $\check{\rho}_{ij}$ are the average of ρ along the vertical and horizonal edges of the cell $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ respectively, that are the quantities obtained by our method. The errors in the tables are the relative error given by

$$\max\Big\{\frac{\max_{i,j}\{|\hat{\rho}_{ij} - \int_{y_j}^{y_{j+1}} \rho(x_i, y)\, dy|\}}{\|\rho\|_{L^\infty(\Omega)}}, \frac{\max_{i,j}\{|\check{\rho}_{ij} - \int_{x_i}^{x_{i+1}} \rho(x, y_j)\, dx|\}}{\|\rho\|_{L^\infty(\Omega)}}\Big\},$$

where $\rho(x, y)$ is the exact average density.

Example 1

$$\begin{aligned} &\psi_L = 1, \quad \psi_R = 1, \quad \psi_B = 1/2, \quad \psi_T = 1/2, \\ &\sigma_T = 1, \qquad \sigma_a = 1, \qquad q = 1. \end{aligned}$$

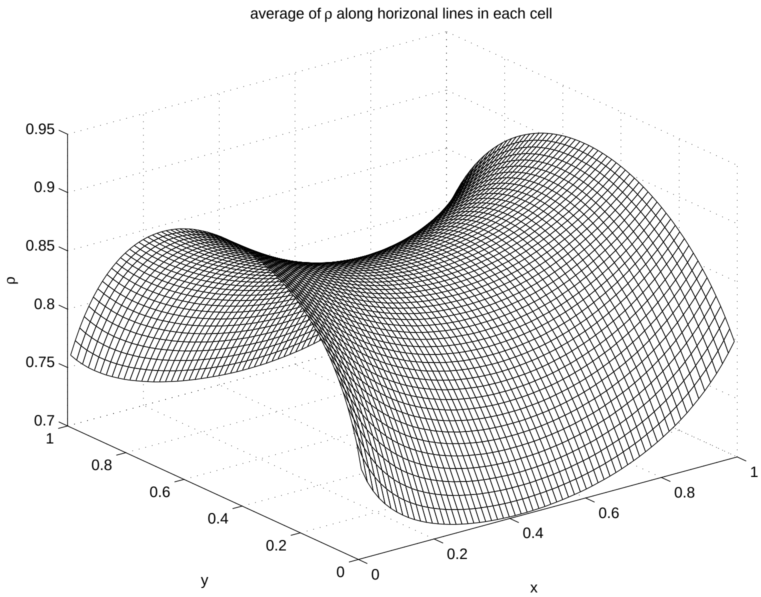

Fig. 5.1. Example 1. "Exact" solution which is calculated by our method with $\Delta x = \Delta y = 1/64$ for $\epsilon = 0.5$ are represented by the cell average along x-axis $\check{\rho}$.

ϵ	Δx	Δy	$\|\text{Error}\|_\infty$	ratio
0.5	2^{-2}	2^{-2}	$8.017 * 10^{-3}$	-
0.5	2^{-3}	2^{-3}	$4.035 * 10^{-3}$	2.0
0.5	2^{-4}	2^{-4}	$1.712 * 10^{-3}$	2.4
0.1	2^{-2}	2^{-2}	$1.966 * 10^{-2}$	-
0.1	2^{-3}	2^{-3}	$6.119 * 10^{-3}$	3.2
0.1	2^{-4}	2^{-4}	$3.456 * 10^{-3}$	1.8
0.02	2^{-2}	2^{-2}	$3.523 * 10^{-2}$	-
0.02	2^{-3}	2^{-3}	$8.739 * 10^{-3}$	4.0
0.02	2^{-4}	2^{-4}	$2.681 * 10^{-3}$	3.3

Table 5.1: Example 1. The error between the "exact" solution and the numerical solutions computed by the method with different Δx, Δy for different ϵ.

In this example the medium is homogeneous. The scattering coefficients and the source are constants in the computational domain and there are particles coming in from outside, which are isotropic in all directions. There is no boundary layer in this problem and the "exact" solution which is calculated by our method with $\Delta x = \Delta y = 1/64$ for $\epsilon = 0.5$ are represented in Fig 5.1. In Table 5.1, the relative error between the "exact" solution and the numerical solutions computed by the method with different Δx, Δy for different ϵ are shown and the uniform first order convergence can be seen easily.

Example 2

$$\begin{aligned}
&\psi_L = 0, \quad \psi_R = 0, \quad \psi_B = 0, \quad \psi_T = 0, \qquad (5.1)\\
&\sigma_T(x,y) = 1, \qquad \sigma_a(x,y) = 1, \qquad q(x,y) = 1, \qquad (x,y) \in [0,1/2] \times [0,1],\\
&\sigma_T(x,y) = 2, \qquad \sigma_a(x,y) = 2, \qquad q(x,y) = 0, \qquad (x,y) \in (1/2,1] \times [0,1].
\end{aligned}$$

The coefficients are piecewise constants and there is an interface along the vertical line $x = 1/2$.

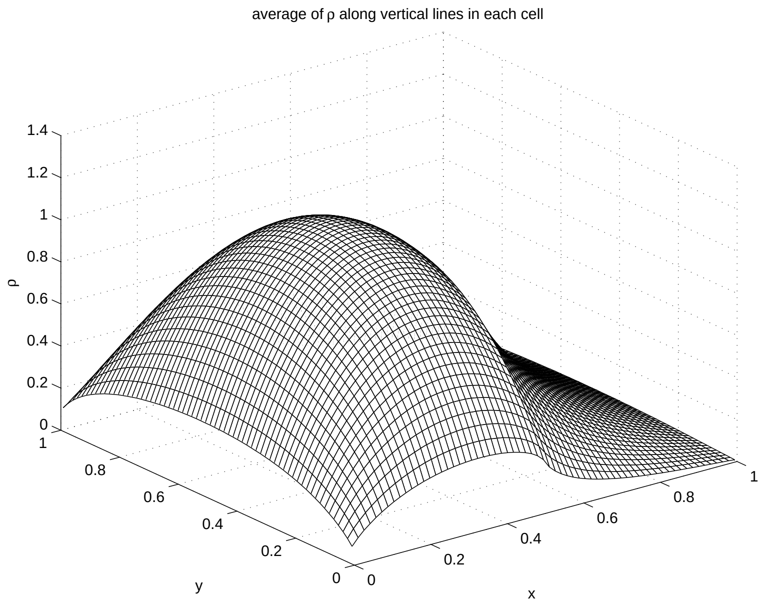

Fig. 5.2. Example 2. "Exact" solution which is calculated by our method with $\Delta x = \Delta y = 1/64$ for $\epsilon = 0.1$ are represented by the cell average along y-axis $\hat{\rho}$.

ϵ	Δx	Δy	$\|\text{Error}\|_\infty$	ratio
0.5	2^{-2}	2^{-2}	$1.755 * 10^{-2}$	-
0.5	2^{-3}	2^{-3}	$7.318 * 10^{-3}$	2.4
0.5	2^{-4}	2^{-4}	$3.689 * 10^{-3}$	2.0
0.1	2^{-2}	2^{-2}	$2.951 * 10^{-2}$	-
0.1	2^{-3}	2^{-3}	$8.359 * 10^{-3}$	3.5
0.1	2^{-4}	2^{-4}	$3.283 * 10^{-3}$	2.5
0.02	2^{-2}	2^{-2}	$4.069 * 10^{-2}$	-
0.02	2^{-3}	2^{-3}	$1.025 * 10^{-2}$	4.0
0.02	2^{-4}	2^{-4}	$2.497 * 10^{-3}$	4.1

Table 5.2: Example 2. The error between the "exact" solution and the numerical solutions computed by the method with different Δx, Δy for different ϵ.

The "exact" solution for $\epsilon = 0.1$ is presented in Fig 5.2 and the relative error between the "exact" solution and the numerical solutions computed by the method with different Δx, Δy for different ϵ are shown in Table 5.2. We can see the uniform first order convergence easily and when ϵ small the convergence order is even higher. This fact can also be seen in Table 5.

Example 3

$$
\begin{aligned}
&\psi_L = 0, \quad \psi_R = 0, \quad \psi_B = 0, \quad \psi_T = 0; \qquad (5.2)\\
&\sigma_T(x,y) = 1/2 + 2x + y^2, \quad \sigma_a(x,y) = 1, \quad q(x,y) = x,\\
&\qquad\qquad \text{for } (x,y) \in [0,1/4) \times [0,1] \cup [1/4,3/4) \cup [0,1/2),\\
&\sigma_T(x,y) = 2, \quad \sigma_a(x,y) = 2y, \quad q(x,y) = 0,\\
&\qquad\qquad \text{for } (x,y) \in [1/4,3/4) \times [1/2,1] \cup [3/4,1] \cup [0,1].
\end{aligned}
$$

The coefficients in this problem depend on the space variables and are discontinuous at the

ϵ	Δx	Δy	$\|\text{Error}\|_\infty$	ratio
0.5	2^{-2}	2^{-2}	$4.604 * 10^{-2}$	-
0.5	2^{-3}	2^{-3}	$1.581 * 10^{-2}$	2.9
0.5	2^{-4}	2^{-4}	$5.107 * 10^{-3}$	3.1
0.1	2^{-2}	2^{-2}	$2.173 * 10^{-2}$	-
0.1	2^{-3}	2^{-3}	$8.269 * 10^{-3}$	2.6
0.1	2^{-4}	2^{-4}	$3.454 * 10^{-3}$	2.4
0.02	2^{-2}	2^{-2}	$3.019 * 10^{-2}$	-
0.02	2^{-3}	2^{-3}	$1.015 * 10^{-2}$	3.0
0.02	2^{-4}	2^{-4}	$2.587 * 10^{-3}$	3.9

Table 5.3: Example 3. The error between the "exact" solution and the numerical solutions computed by the method with different Δx, Δy for different ϵ.

Δx	Δy	$\|\text{Error}\|_\infty$
2^{-2}	2^{-2}	$1.111 * 10^{-2}$
2^{-3}	2^{-3}	$2.844 * 10^{-3}$
2^{-4}	2^{-4}	$4.873 * 10^{-3}$

Table 5.4: Example 4. The error between the "exact" solution and the numerical solutions computed by the method with different Δx, Δy for different ϵ. That the error for $\Delta x = \Delta y = 2^{-4}$ is bigger than $\Delta x = \Delta y = 2^{-3}$ is because $\psi(x, y)$ changes very fast at the boundary, we can no longer use the average along the grid lines to approximate $\psi(x, y)$ at the grid line as we do in (3.6). When we use coarser mesh, the average is less affected by the boundary layer.

interface. There is no incoming particles from outside, thus no boundary layer. The shape of the interface is displayed on the top of Fig. 5.3 while the bottom one shows the "exact" solution. The relative errors for different ϵ are displayed in Table 5.3, from which we can find that the convergence rate is between 1 and 2.

Example 4

$$\begin{aligned} &\psi_{Lm} = 5\mu_m, \quad \mu_m > 0, \quad \psi_{Rm} = -5\mu_m, \quad \mu_m < 0, \\ &\psi_{Bm} = 5\nu_m, \quad \nu_m > 0, \quad \psi_{Tm} = -5\nu_m, \quad \nu_m < 0. \\ &\sigma_T(x, y) = 1 + x + 2y, \quad \sigma_a(x, y) = 1/2 + x, \quad q(x, y) = 0 \quad \epsilon = 0.02. \end{aligned} \tag{5.3}$$

The incoming particle density are different in each direction so there are boundary layers in this problem. We can see the layers in Fig 5.4 and Table 5.4 gives the relative numerical errors calculated with different Δx and Δy. The error for $\Delta x = \Delta y = 2^{-4}$ is bigger than $\Delta x = \Delta y = 2^{-3}$ is because $\psi(x, y)$ changes very fast at the boundary, we can no longer use the average along the grid lines to approximate $\psi(x, y)$ at the grid line as we do in (3.6). In [15], the uniform second order convergence *up to the boundary* is proved for one dimension piecewise constant coefficient approximation even when boundary layers exist. This two dimensional method we present here does not have such good properties. Though we can no longer use coarse meshes to obtain good numerical approximations at the boundary layer, the error will decay exponentially away from the boundary layer.

Like the mesh displayed in Figure 5, we can resolve ϵ at the boundary and use coarse mesh inside. Because the relation (3.28) only depends on the cell itself, the coding is exactly the same. We can see in Figure 5 that good approximation is obtained. Similarly, for the transport diffusion coexisting case where interface layers exist, same strategy can be used.

Remark: Use the same idea in [15] as for the one dimensional method we can prove rigorously the uniform first order convergence with respect to ϵ. In fact, the error comes from approximating the coefficients by piecewise constants is uniformly second order and the cell edge average approximation has only uniform first order convergence.

Remark: We can see numerically in all these four examples when ϵ is small enough, the convergent order is almost two, which implies the limit scheme (4.7)(4.8) for the diffusion limit (2.10) is second order numerically. Using the idea in [9], we can also obtain the uniform first order convergence by its AP property.

6. Conclusion

A uniformly first order convergent numerical method for the discete-ordinate transport equation in the rectangle geometry is presented in this paper. This method is an extension of the method that is proposed in [15] for one dimensional discrete ordinate transport equation. Firstly we approximate the scattering coefficients and source term by piecewise constants determined by their cell averages, then the solution at the cell edges are approximated by their average along the cell edges. Solve analytically the system of equations for the cell edge averages in each cell and piece together the numerical solution with the neighboring cells using the interface conditions. When there is no interface or boundary layer, we prove that this method is asymptotic-preserving and its first order accuracy with respect to the mean free path is shown numerically.

The convergence rate can be improved by interpolating the cell edge averages between neighboring cells. Moreover we can extend this method to other kinds of meshes like parallelogram. For problems with interface or boundary layers, the method we discuss here can not use coarse mesh to get good approximation, unlike its one dimensional case. We'll try to find a way to solve this problem in the future and extend this method to some other related transport equations.

References

[1] M. P. D. Abreu, H. A. Filho, F. C. D. Silva and R. C. Barros, On the spectral nodal methods applied to discrete ordinates eigenvalue problems in cartestian geometry, *Proceedings of the* 2^{nd} *international conference on computational heat and mass transfer*, Federal University of Rio de Janeiro, Brazil, October 22-26, 2001.

[2] G. Bal, L. Ryzhik, Diffusion approximation of radiative transfer problems with interfaces, *SIAM J. App. Math.*, 60 (2000), 1887–1912

[3] R. Caflisch, S. Jin and G. Russo, Uniformly accurate schemes for hyperbolic systems with relaxations, *SIAM J. Num. Anal.*, 34 (1997), 246-281

[4] S. Chandrasekhar, *Radiative Transfer*,Dover, New York, (1960)

[5] P. Crispel, P. Degond and M.-H. Vignal, An asymptotic preserving scheme for the two-fluid Euler-Poisson model in the quasineutral limit, *J. Comp. Phys.*, 223 (2007), 208-234,

[6] P. Degond, S. Jin and J. G. Liu, Mach-number uniform asymptotic-preserving gauge schemes for compressible flows, *Bulletin of the Institute of Mathematics, Academia Sinica, New Series*, **2**:4 (2007), 851-892,

[7] P. Degond, S. Jin and M. Tang, On the time-splitting spectral method for the complex Ginzburg-Landau equation in the large time and space scale limit. *SIAM J. Sci. Comp.*, 30 (2008), 2466-2487.

[8] P. Edstrom, A fast and stable solution method for the radiative transfer problem, *SIAM Rev.*, **47**:3 (2005), 447–468

[9] F. Golse, S. Jin and C. D. Levermore, The convergence of numerical transfer schemes in diffusive regimes I: discrete-ordinate method, *SIAM J. Numer. Anal.*, **36**:5 (1999), 1333–1369

[10] S. Jin, Efficient Asymptotic-Preserving (AP) Schemes for Some Multiscale Kinetic Equations, *SIAM J. Sci. Comp.*, 21 (1999), 441–454

[11] S. Jin, Runge-Kutta methods for hyperbolic conservation laws with stiff relaxation terms, *J. Comp. Phys.*, 122 (1995), 51–67.

[12] S. Jin and C. D. Levermore, Numerical schemes for hyperbolic conservation laws with stiff relaxation terms, *J. Comp. Phys.*, 126 (1996), 449–467.

[13] S. Jin, C. D. Levermore, The discrete-ordinate method in diffusive regimes, *Transport theory and statistical physics*, **20**:5 (1991), 413-439

[14] S. Jin, X. Liao and X. Yang, Computation of interface reflection and regular or diffuse transmission of the planar symmetric radiative transfer equation with isotropic scattering and its diffusion limit, *SIAM J. Sci. Comp.*, 30 (2008), 1992-2017.

[15] S. Jin, M. Tang, H. Han, A uniformly second order numerical method for the one-dimensional discrete-ordinate transport equation and its diffusion limit with interface, *Networks and Heterogeneous Media* 4 (2009), 35-65.

[16] S. Jin, X. Yang and G.W. Yuan, A domain decomposition method for a two-scale transport equation with energy flux conserved at the interface, *Kinetic and Related Models*, 1 (2008), 65-84.

[17] E. B. Hauser, M. T.M.B. Vilhena and R. C. Barros, The LTS_N exponential nodal method for one-speed X,Y,Z geometry discrete ordinates problems, 2007 international nuclear atlantic conference, Santos, SP, September 30 to October 5, 2007.

[18] E. W. Larsen, J. E. Morel and W. F. Miller Jr., Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes, *J. Comput. Phys.*, 69 (1987), 283-324

[19] J. A. M. de Mello and R. C. Barros, An exponential spectral nodal method for one-speed X,Y-geometry deep penetration discrete ordinate problems, *Annals of Nuclear Energy*, 29 (2002), 1855-1869

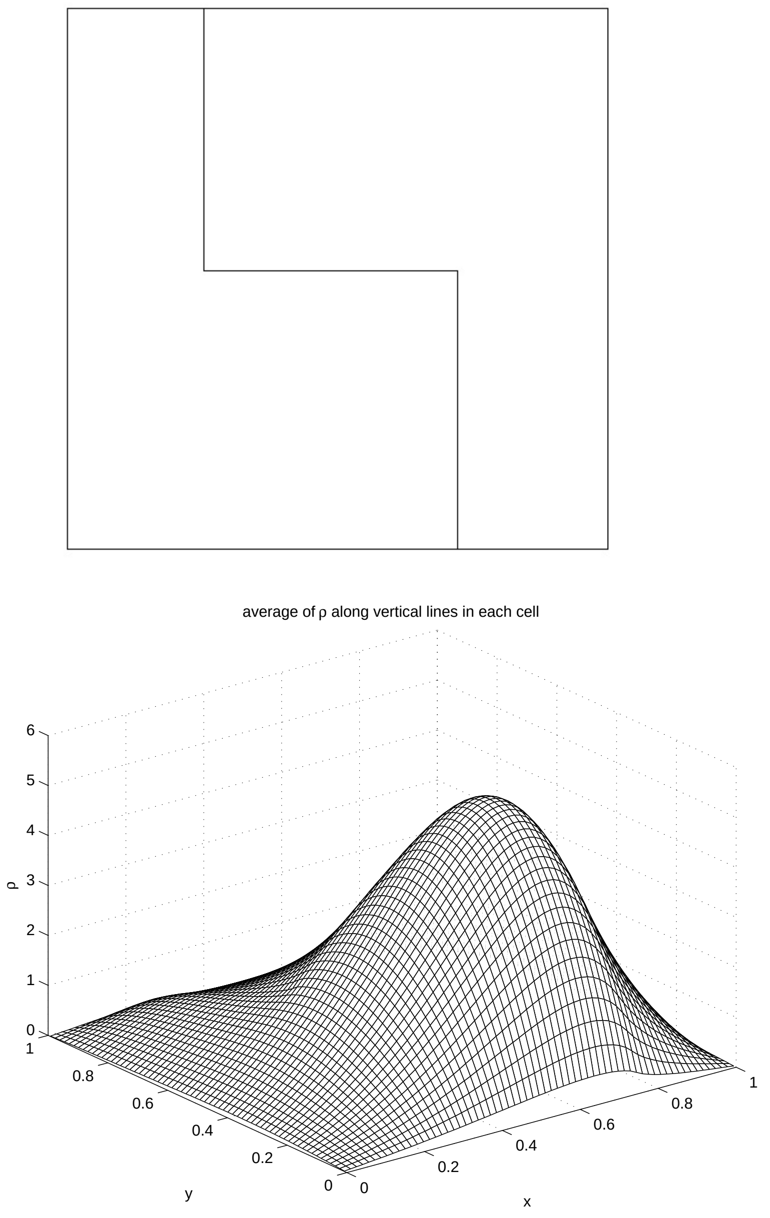

Fig. 5.3. Example 3. "Exact" solution which is calculated by our method with $\Delta x = \Delta y = 1/64$ for $\epsilon = 0.02$ are represented by the cell average along y-axis $\hat{\rho}$. Top: the shape of the interface; Bottom: the exact solution

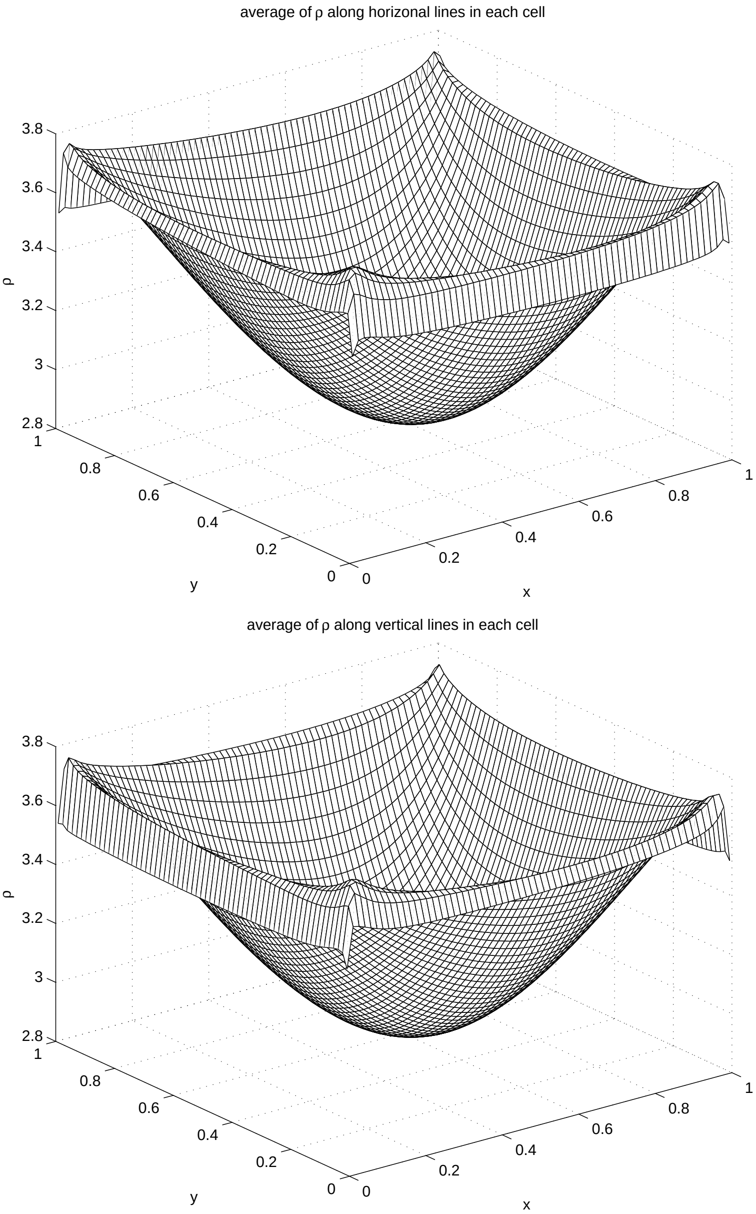

Fig. 5.4. Example 4. "Exact" solution which is calculated by our method with $\Delta x = \Delta y = 1/64$ for $\epsilon = 0.02$ are represented, Top: the cell average along x-axis $\breve{\rho}$; Bottom: the cell average along y-axis $\hat{\rho}$.

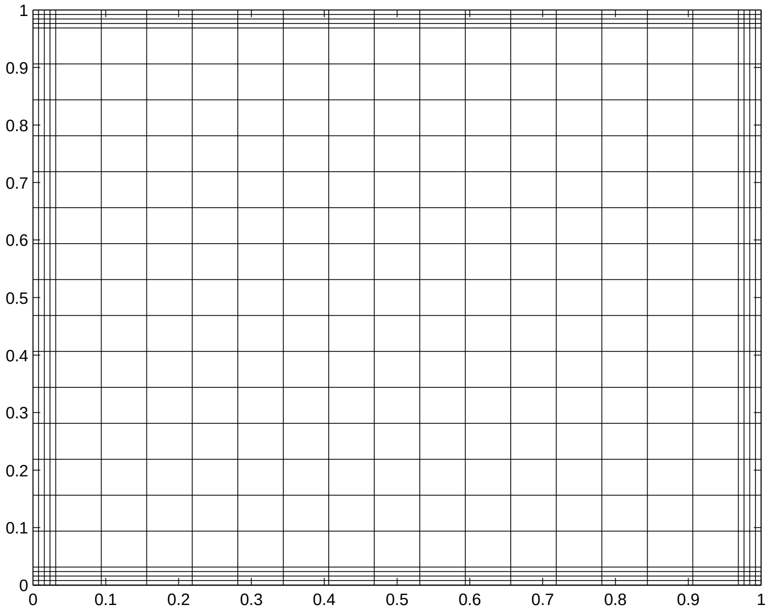

Fig. 5.5. Example 4. Mesh resolving the boundary layers, for $x, y \in [0, 1/16] \cup [15/16, 1]$, $\Delta x, \Delta y = 2^{-7}$ and for $x, y \in (1/16, 15/16)$, $\Delta x, \Delta y = 2^{-4}$.

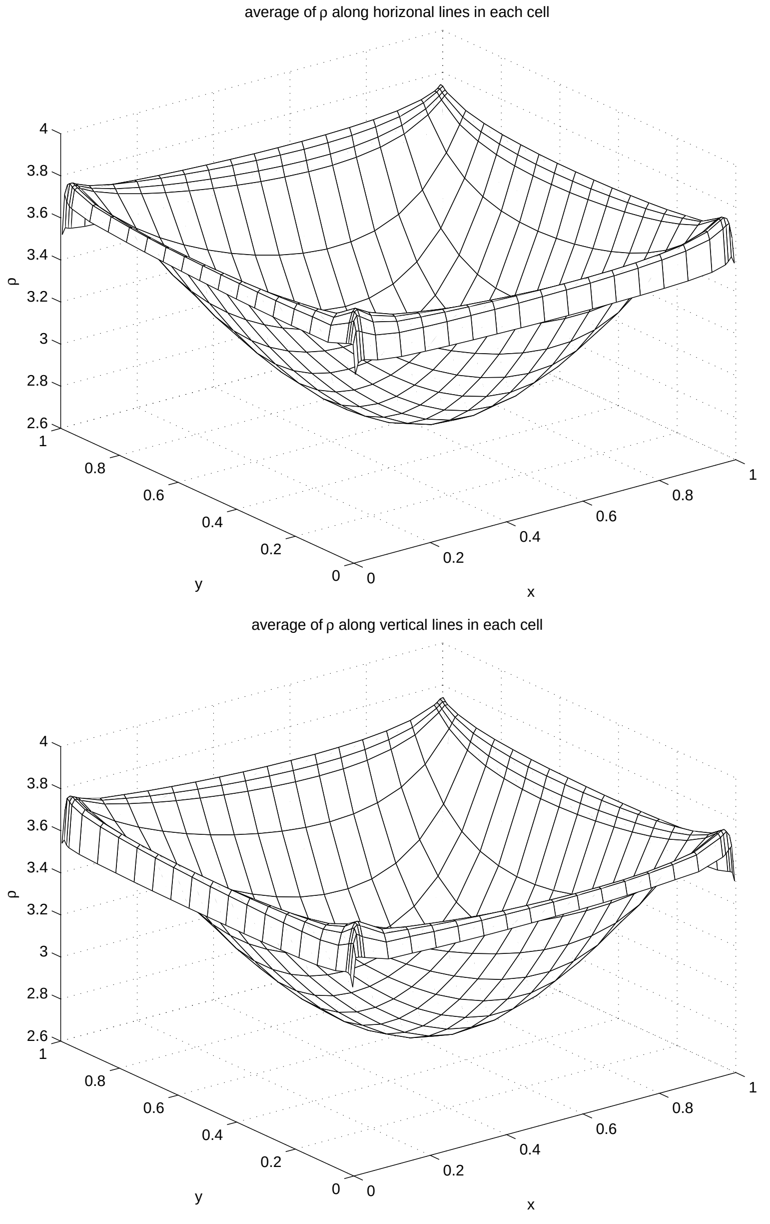

Fig. 5.6. Example 4. Numerical solution that is calculated with the mesh displayed in Figure 5 for $\epsilon = 0.02$, Top: the cell average along x-axis $\check{\rho}$; Bottom: the cell average along y-axis $\hat{\rho}$.