Asymptotic-Preserving Schemes for Multiscale Hyperbolic and Kinetic Equations

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Abstract

Hyperbolic and kinetic equations often possess small spatial and temporal scales that lead to various asymptotic limits. Numerical approximation of these equations is challenging due to presence of stiff source, collision, forcing terms, or when different scales co-exist. Asymptotic-preserving (AP) schemes are numerical methods that are efficient in these asymptotic regimes. They are designed to capture the asymptotic limit at the discrete level without resolving small scales. This paper aims to review the current status of AP schemes for a large class of hyperbolic and kinetic equations. We will first use simple models to illustrate the basic design principles, and then describe several generic AP strategies for handling general equations. Various aspects of the AP schemes for different asymptotic regimes, including some recent development, will be discussed as well.

Key words. Asymptotic-preserving, hyperbolic equations, kinetic equations, multi-scale, stiff relaxations.

AMS subject classifications. 35L02, 82B40, 65L04, 58J37.

1 Introduction

Hyperbolic and kinetic equations usually possess multiple or small spatial and temporal scales that pose tremendous numerical challenges. These scales arise very often from (possibly stiff) source terms, collision terms, strong forcing, etc., and a naive numerical discretization of these equations requires the mesh sizes and time steps smaller than the smallest scales of the problem which is often prohibitively expensive.

Typically, to efficiently compute problems with multiple scales one often couples a macroscopic model with a microscopic one through coupling conditions. One classical example is the coupling of the (microscopic) Boltzmann equation with the (macroscopic) fluid dynamic equations [5, 86, 30]. Such technique requires an interface or connection condition which transfers

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data between the macroscopic and the microscopic ones. This can be very difficult and is often ad-hoc. Another paradigm that has been very popular in the last two decades is the so-called *Asymptotic-Preserving (AP)* schemes [48] that bridge the two different scales in a seamless way: the transition between the two scales is realized *automatically*, in that a micro solver becomes a macro solver automatically if the numerical discretizations do not resolve the physically small scales. A requirement for AP schemes is to *preserve the asymptotic transition from the micro models to the macro ones at the discrete level*, which can be best illustrated in Figure 1. Such an AP property often leads to a uniform convergence in the scaling parameter [35, 49]. This article aims at introducing this framework and reviewing the current status of AP schemes for a large class of hyperbolic and kinetic equations, in various asymptotic regimes. Interested readers may also consult earlier reviews in this subject [49, 20, 21].

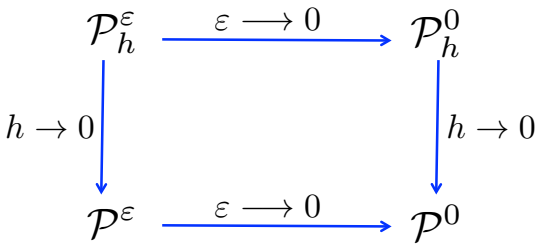

Figure 1: Illustration of AP schemes. $\mathcal{P}^\varepsilon$ is a microscopic equation that depends on the small scale ε (e.g., the Boltzmann equation with the small Knudsen number (ratio of the mean free path over the characteristic length)), and $\mathcal{P}^0$ is its macroscopic limit as $\varepsilon \to 0$ (e.g., the Euler equations). Denote the numerical approximation of $\mathcal{P}^\varepsilon$ by $\mathcal{P}_h^\varepsilon$, where h is the discretization parameter such as the time step or mesh size. The asymptotic limit of $\mathcal{P}_h^\varepsilon$ as $\varepsilon \to 0$ (with h fixed), if exists, is denoted by $\mathcal{P}_h^0$. If $\mathcal{P}_h^0$ is a good (consistent and stable) approximation of $\mathcal{P}^0$, then the scheme $\mathcal{P}_h^\varepsilon$ is called AP.

The rest of this paper is organized as follows. We will first use two simple models to illustrate the basic design principles of AP schemes in Section 2. Then in Section 3, using the nonlinear Boltzmann equation as an example, we discuss several generic AP strategies for handling general kinetic and hyperbolic equations. Section 4 summarizes some other asymptotic limits and the corresponding AP schemes. The paper is concluded in Section 5.

2 Basic design principles of AP schemes — two illustrative examples

While earlier attempts of AP schemes aim at stationary neutron transport in the diffusive regime [67, 66], the major challenges – and the most recent developments – in the design of AP schemes come from time discretizations [47, 8] or reformulation to a system that is insensitive to the specific spatial discretizations. As described in [48], a scheme is AP if

- it is a good discretization of the microscopic model; when the scaling parameter approaches zero, with numerical parameters fixed, it becomes a good macroscopic solver;

- an implicit discretization, which is necessary for uniform stability, can be implemented either explicitly or at least very efficiently (avoiding difficult nonlinear iterative algebraic solvers for instance).

We will first use two simple examples to illustrate the basic design principles of AP schemes. The first one is the Jin-Xin hyperbolic relaxation system proposed initially to solve the systems of conservation laws [59]. The second one is the Bhatnagar-Gross-Krook (BGK) model which is a kinetic equation introduced to simplify the complicated Boltzmann collision integral in rarefied gas dynamics [4].

2.1 The Jin-Xin relaxation model

The Jin-Xin semi-linear hyperbolic relaxation model in one spatial dimension reads as

$$\begin{cases} \partial_t u + \partial_x v = 0\,, \\ \partial_t v + a\partial_x u = \dfrac{1}{\varepsilon}(f(u) - v)\,, \end{cases} \tag{2.1}$$

where a is a constant, $f(u)$ is a nonlinear function of u, and ε is the relaxation time. As $\varepsilon \to 0$, the second equation above yields the local equilibrium $v = f(u)$, which, upon substitution to the first equation, gives the zero relaxation limit:

$$\partial_t u + \partial_x f(u) = 0\,. \tag{2.2}$$

When ε is small but not zero, numerically approximating the system (2.1) presents a challenge due to the stiff relaxation term. A naive explicit scheme would require the time step to resolve ε: $\Delta t = O(\varepsilon)$, which can be very time-consuming. A natural way is to treat this term implicitly which allows $\Delta t \gg \varepsilon$. The convection term, on the other hand, can be treated explicitly as it is not stiff. Leaving aside the spatial discretization, one can employ the following first-order time-splitting framework to solve (2.1):

$$\begin{cases} \dfrac{u^* - u^n}{\Delta t} = 0\,, \\ \dfrac{v^* - v^n}{\Delta t} = \dfrac{1}{\varepsilon}(f(u^*) - v^*)\,, \end{cases} \qquad \text{(relaxation step)} \tag{2.3}$$

$$\begin{cases} \dfrac{u^{n+1} - u^*}{\Delta t} + \partial_x v^* = 0\,, \\ \dfrac{v^{n+1} - v^*}{\Delta t} + a\partial_x u^* = 0\,. \end{cases} \qquad \text{(convection step)} \tag{2.4}$$

The second equation in (2.3) appears implicit at first sight, but note that from the first equation $u^* = u^n$, so the nonlinear term $f(u^*)$ is indeed explicit thus requires no Newton-type iterations. Therefore, although implicit, the whole scheme can be implemented explicitly — an important feature of AP schemes. For the spatial derivative in (2.4), one can apply the usual finite difference/volume schemes. For example, a first-order upwind scheme (applied to the Riemann invariants $u \pm \frac{1}{\sqrt{a}} v$) results in

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^*}{\Delta t} + \dfrac{v_{j+1}^* - v_{j-1}^*}{2\Delta x} = \dfrac{\sqrt{a}\Delta x}{2} \dfrac{u_{j+1}^* - 2u_j^* + u_{j-1}^*}{(\Delta x)^2}\,, \\ \dfrac{v_j^{n+1} - v_j^*}{\Delta t} + a\dfrac{u_{j+1}^* - u_{j-1}^*}{2\Delta x} = \dfrac{\sqrt{a}\Delta x}{2} \dfrac{v_{j+1}^* - 2v_j^* + v_{j-1}^*}{(\Delta x)^2}\,. \end{cases} \tag{2.5}$$

Now let us check the AP property of the above scheme. We will keep Δt, Δx fixed, and send $\varepsilon \to 0$. Then the relaxation step (2.3) (with the spatial index j added on) insures

$$u_j^* = u_j^n\,, \quad v_j^* = f(u_j^*) = f(u_j^n)\,,$$

which, when substituted to the first equation of (2.5), yields

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{f(u_{j+1}^n) - f(u_{j-1}^n)}{2\Delta x} = \frac{\sqrt{a}\Delta x}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\,. \tag{2.6}$$

The scheme (2.6) is nothing but the Lax-Friedrichs or Rusanov scheme applied to the limiting equation (2.2). Hence the scheme (2.3) (2.4) is AP, in both time and space.

2.2 The BGK model

The BGK model widely used in kinetic theory takes the following form:

$$\partial_t f + v\cdot\nabla_x f = \frac{1}{\varepsilon}(\mathcal{M} - f)\,, \tag{2.7}$$

where $f = f(t,x,v)$ is a probability distribution function of particles at time t, position $x \in \Omega \subset \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. ε is the Knudsen number defined as the ratio of the mean free path over the characteristic length. $\mathcal{M}$ is the local equilibrium or Maxwellian defined through the moments of f:

$$\mathcal{M}(v) = \frac{\rho}{(2\pi T)^{d/2}}\exp\left(-\frac{|v-u|^2}{2T}\right)\,, \tag{2.8}$$

where ρ, u, T are, respectively, the density, bulk velocity, and temperature:

$$\rho = \int_{\mathbb{R}^d} f\,\mathrm{d}v\,, \quad u = \frac{1}{\rho}\int_{\mathbb{R}^d} vf\,\mathrm{d}v\,, \quad T = \frac{1}{d\rho}\int_{\mathbb{R}^d}|v-u|^2 f\,\mathrm{d}v\,.$$

It is important to note that $\mathcal{M}$ so defined shares the same first $d+2$ moments with f:

$$U := (\rho, \rho u, E)^T = \int_{\mathbb{R}^d} f\phi(v)\,\mathrm{d}v = \int_{\mathbb{R}^d}\mathcal{M}\phi(v)\,\mathrm{d}v\,, \quad \phi(v) = (1, v, |v|^2/2)^T\,, \tag{2.9}$$

where $E = \frac{1}{2}\rho(|u|^2 + dT)$ is the total energy. Therefore, if one multiplies equation (2.7) by $\phi(v)$ and integrates over v, the right hand side will vanish and one obtains

$$\begin{cases} \partial_t\rho + \nabla_x\cdot(\rho u) = 0\,, \\ \partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u + \mathbb{P}) = 0\,, \\ \partial_t E + \nabla_x\cdot(Eu + \mathbb{P}u + q) = 0\,, \end{cases} \tag{2.10}$$

where $\mathbb{P}$ and q are the stress tensor and heat flux defined by

$$\mathbb{P} = \int_{\mathbb{R}^d}(v-u)\otimes(v-u)f\,\mathrm{d}v\,, \quad q = \frac{1}{2}\int_{\mathbb{R}^d}(v-u)|v-u|^2 f\,\mathrm{d}v\,.$$

The system (2.10) is the local conservation law which is not closed. However, when $\varepsilon \to 0$, (2.7) implies $f = \mathcal{M}$. This, substituted into (2.10), yields a closed system (the compressible Euler

equations):

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0\,, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + pI) = 0\,, \\ \partial_t E + \nabla_x \cdot ((E+p)u) = 0\,, \end{cases} \tag{2.11}$$

where $p = \rho T$ is the pressure and I is the identity matrix.

Our goal here is to design a numerical scheme for the BGK model (2.7) that is efficient when ε is small and can capture the asymptotic Euler limit (2.11) at the discrete level. Similarly as the Jin-Xin model, we adopt the time-splitting strategy and treat the stiff collision part implicitly and the non-stiff convection part explicitly ([16]):

$$\frac{f^* - f^n}{\Delta t} = \frac{1}{\varepsilon}(\mathcal{M}^* - f^*)\,, \quad \text{(collision step)} \tag{2.12}$$

$$\frac{f^{n+1} - f^*}{\Delta t} + v \cdot \nabla_x f^* = 0\,. \quad \text{(convection step)} \tag{2.13}$$

Although the collision step appears implicit ($\mathcal{M}^*$ is defined through f^* in a nonlinear way), it can be implemented explicitly. Indeed, taking the moments $\int_{\mathbb{R}^d} \cdot \phi(v)\,\mathrm{d}v$ on both sides of (2.12) gives $U^* = U^n$ due to property (2.9), that is, the macroscopic quantities ρ, u, and T are conserved in this step, so is the Maxwellian $\mathcal{M}^* = \mathcal{M}^n$. Therefore, one does not need any iterative solver for the collision step. For the spatial discretization in the convection step, one can still apply the upwind scheme (assume x and v are one-dimensional for simplicity):

$$\frac{f_j^{n+1} - f_j^*}{\Delta t} + \frac{\frac{v+|v|}{2}(f_j^* - f_{j-1}^*) + \frac{v-|v|}{2}(f_{j+1}^* - f_j^*)}{\Delta x} = 0\,. \tag{2.14}$$

Let us verify the AP property of the proposed scheme. Keeping Δt, Δx fixed, and sending $\varepsilon \to 0$ in (2.12) implies

$$f^* = \mathcal{M}^* = \mathcal{M}^n\,. \tag{2.15}$$

Now replacing f^* with $\mathcal{M}^n$ in (2.14), and taking the moments $\int_{\mathbb{R}} \cdot \phi(v)\,\mathrm{d}v$, one gets

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0\,, \tag{2.16}$$

where the flux $F_{j+\frac{1}{2}}$ is defined as

$$F_{j+\frac{1}{2}} = F_j^+ + F_{j+1}^- \quad \text{with} \quad F_j^\pm = \int_{\mathbb{R}^1} \frac{v \pm |v|}{2} \mathcal{M}_j \phi(v)\,\mathrm{d}v\,. \tag{2.17}$$

Thanks to the special form of the Maxwellian, $F_j^\pm$ can be represented in closed form in terms of the error function as

$$F_j^\pm = \begin{pmatrix} \rho_j u_j A_j^\pm \pm \rho_j B_j \\ \left(\rho_j T_j + \rho_j u_j^2\right) A_j^\pm \pm \rho_j u_j B_j \\ \left(\frac{3}{2}\rho_j T_j u_j + \frac{1}{2}\rho_j u_j^3\right) A_j^\pm \pm \left(\frac{1}{2}\rho_j u_j^2 + \rho_j T_j\right) B_j \end{pmatrix} \tag{2.18}$$

with

$$A_j^\pm = \frac{1}{2}\left(1 \pm \mathrm{erf}(s_j)\right)\,, \quad B_j = e^{-s_j^2}\sqrt{\frac{T_j}{2\pi}}\,, \quad s_j = \frac{u_j}{\sqrt{2T_j}}\,.$$

(2.16)–(2.18) is just the kinetic flux vector splitting (KFVS) scheme [83, 26] for the limiting compressible Euler equations (2.11). Hence the scheme (2.12) (2.13) is AP, in both time and space.

The above scheme is not necessarily the only AP scheme. Two variants are immediate.

1) Instead of applying the backward Euler scheme in (2.12), one can solve this step exactly

$$f^* = e^{-\frac{\Delta t}{\varepsilon}} f^n + \left(1 - e^{-\frac{\Delta t}{\varepsilon}}\right) \mathcal{M}^n \,,$$

where we again used the fact that $\mathcal{M}^n$ does not change over the collision step. This method is also AP since as $\varepsilon \to 0$, we still have (2.15).

2) Instead of time splitting, one can apply an implicit-explicit (IMEX) type scheme to the original equation (2.7):

$$\frac{f^{n+1} - f^n}{\Delta t} + v \cdot \nabla_x f^n = \frac{1}{\varepsilon}(\mathcal{M}^{n+1} - f^{n+1}) \,. \tag{2.19}$$

This scheme can still be solved in an explicit manner albeit $\mathcal{M}^{n+1}$ is implicit. Specifically, taking the moments $\int_{\mathbb{R}^d} \cdot \phi(v)\, \mathrm{d}v$ on both sides of (2.19) and using the property (2.9), one has

$$\frac{U^{n+1} - U^n}{\Delta t} + \nabla_x \cdot \int_{\mathbb{R}^d} v\phi(v)\, f^n \, \mathrm{d}v = 0 \,. \tag{2.20}$$

From (2.20), one can solve for U^{n+1}, which consequently defines $\mathcal{M}^{n+1}$. Thus f^{n+1} can be obtained from (2.19) explicitly. To see the AP property, as $\varepsilon \to 0$, (2.19) implies $f^n = \mathcal{M}^n$ for any n. Replacing f^n with $\mathcal{M}^n$, (2.20) is clearly a consistent discretization of the compressible Euler system (2.11).

Remark 2.1. *The numerical schemes presented in this section are all first-order accurate in both space and time. To achieve higher order in space, standard high-order spatial discretizations (e.g., weighted essentially non-oscillatory (WENO), discontinuous Galerkin (DG) schemes) can be used. The situation for time is, however, different. It is worthwhile to mention that standard high-order time-splitting methods such as the Strang splitting will suffer from order reduction when $\varepsilon \to 0$ [47]. Rather, one can use the Runge-Kutta splitting schemes [8] or the more general IMEX methods in the non-splitting framework [81, 84].*

In the above two examples, in spite of the nonlinear stiff terms which are treated implicitly, one can implement the schemes explicitly, thanks to the special structure of the relaxation terms. This will not be true for more general source or collision terms. In the next section we will see how these two special examples can be utilized to develop AP schemes for general hyperbolic and kinetic equations in which the nonlinear stiff terms can be implemented explicitly.

3 AP schemes for general hyperbolic and kinetic equations

In this section we discuss several generic AP strategies including the penalization, exponential reformulation, and micro-macro decomposition. Since the classical Boltzmann equation [10] constitutes the central model in kinetic theory, it will be used throughout to illustrate the ideas

of the three techniques. Application of each strategy to other equations will be mentioned at the end of each subsection.

We first briefly review the Boltzmann equation and its properties. The equation reads as

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \mathcal{Q}(f) \,, \tag{3.1}$$

where, compared with the BGK model (2.7), the only difference lies in the collision term $\mathcal{Q}(f)$ — a nonlinear integral operator modeling the binary interaction among particles:

$$\mathcal{Q}(f)(v) = \int_{\mathbb{R}^d} \int_{S^{d-1}} B(v - v_*, \sigma) \left[f(v') f(v'_*) - f(v) f(v_*) \right] \mathrm{d}\sigma \, \mathrm{d}v_* \,.$$

Here (v, v_*) and (v', v'_*) are the velocity pairs before and after a collision, during which the momentum and energy are conserved; hence (v', v'_*) can be represented in terms of (v, v_*) as

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2} \sigma \,, \\ v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2} \sigma \,, \end{cases}$$

with the parameter σ varying in the unit sphere S^{d-1}. The collision kernel $B(v - v_*, \sigma)$ is a non-negative function depending only on $|v - v_*|$ and cosine of the deviation angle $\frac{\sigma \cdot (v - v_*)}{|v - v_*|}$.

The collision operator $\mathcal{Q}(f)$ conserves mass, momentum, and energy:

$$\int_{\mathbb{R}^d} \mathcal{Q}(f) \phi(v) \, \mathrm{d}v = 0 \,, \quad \phi(v) = (1, v, |v|^2/2)^T \,. \tag{3.2}$$

It satisfies the celebrated Boltzmann's H-theorem:

$$- \int_{\mathbb{R}^d} \mathcal{Q}(f) \ln f \, \mathrm{d}v \geq 0 \,.$$

Moreover,

$$\int_{\mathbb{R}^d} \mathcal{Q}(f) \ln f \, \mathrm{d}v = 0 \Longleftrightarrow \mathcal{Q}(f) = 0 \Longleftrightarrow f = \mathcal{M} \,, \tag{3.3}$$

where $\mathcal{M}$ is given by (2.8). This means the entropy is maximized if and only if f reaches the local equilibrium.

As for the BGK model, if $\varepsilon \to 0$, the macroscopic limit of the Boltzmann equation is also the compressible Euler system (2.11) (which can be easily seen using (3.2) and (3.3)). Via the Chapman-Enskog expansion [13], one can derive from the Boltzmann equation the Navier-Stokes limit while retaining $O(\varepsilon)$ terms:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0 \,, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + p I) = \varepsilon \nabla_x \cdot (\mu \sigma(u)) \,, \\ \partial_t E + \nabla_x \cdot ((E + p) u) = \varepsilon \nabla_x \cdot (\mu \sigma(u) u + \kappa \nabla_x T) \,, \end{cases} \tag{3.4}$$

where $\sigma(u) = \nabla_x u + \nabla_x u^T - \frac{2}{d} \nabla_x \cdot u I$. μ and κ are the viscosity and heat conductivity, determined through the linearized Boltzmann collision operator [1].

The construction of AP schemes for the Boltzmann equation (3.1) is by no means trivial. Due to its complicated form, the implicit discretization of the collision operator will be difficult.

3.1 AP schemes based on penalization

The *penalization method*, introduced by Filbet and Jin [32], was the first AP scheme for the nonlinear Boltzmann equation. The idea is to penalize $\mathcal{Q}(f)$ by the BGK operator:

$$\partial_t f + v\cdot\nabla_x f = \underbrace{\frac{\mathcal{Q}(f)-\beta(\mathcal{M}-f)}{\varepsilon}}_{\text{less stiff, explicit}} + \underbrace{\frac{\beta(\mathcal{M}-f)}{\varepsilon}}_{\text{stiff, implicit}},$$

where β is some constant chosen properly to approximate the Frechet derivative of $\mathcal{Q}(f)$ around $\mathcal{M}$. After penalization, terms in the first brace become less stiff or non-stiff and can be treated explicitly. The other part has to be done implicitly but it is a BGK operator thus many techniques introduced in the previous section can be applied here.

A first-order IMEX discretization of (3.1) can thus be written as follows:

$$\frac{f^{n+1}-f^n}{\Delta t} + v\cdot\nabla_x f^n = \frac{\mathcal{Q}(f^n)-\beta(\mathcal{M}^n-f^n)}{\varepsilon} + \frac{\beta(\mathcal{M}^{n+1}-f^{n+1})}{\varepsilon}. \tag{3.5}$$

This is an implicit scheme but can be solved *explicitly* similarly as in (2.19). Indeed, taking the moments $\int_{\mathbb{R}^d}\cdot\,\phi(v)\,\mathrm{d}v$ on both sides of (3.5) and using the properties (2.9) and (3.2), we still have (2.20), from which one can solve for U^{n+1}, hence $\mathcal{M}^{n+1}$. Then f^{n+1} can be obtained from (3.5) explicitly. In practice, β can be roughly estimated as

$$\beta = \sup_v |\mathcal{Q}^-(f)|,$$

where $\mathcal{Q}^-$ is the loss part of the collision operator defined such that $\mathcal{Q}(f)=\mathcal{Q}^+(f)-f\mathcal{Q}^-(f)$. β can also be made time and spatially dependent for better numerical accuracy.

Concerning the AP property, the following results were established in [32].

Proposition 3.1. *Let f^n be the numerical solution given by the scheme (3.5).*

i) If $\varepsilon\to 0$ and $f^n=\mathcal{M}^n+O(\varepsilon)$, then $f^{n+1}=\mathcal{M}^{n+1}+O(\varepsilon)$.

ii) Assume $\varepsilon\ll 1$ and $f^n=\mathcal{M}^n+O(\varepsilon)$. If there exists a constant $C>0$ such that

$$\left\|\frac{f^{n+1}-f^n}{\Delta t}\right\| + \left\|\frac{U^{n+1}-U^n}{\Delta t}\right\| \le C,$$

then the scheme (3.5) asymptotically becomes a first order in time approximation of the compressible Navier-Stokes equations (3.4).

The property i) above is a weaker version of AP property since one requires the solution to be close to the equilibrium initially. Although hard to prove theoretically, extensive numerical results in [32] illustrate that the penalization scheme can achieve the following stronger AP property: regardless of the initial condition f^0, there exists an integer $N>0$ such that

$$f^n = \mathcal{M}^n + O(\varepsilon), \quad \text{for any } n\ge N. \tag{3.6}$$

Substituting (3.6) into (3.5) and taking the moments, one has

$$\frac{U^{n+1}-U^n}{\Delta t} + \nabla_x\cdot\int_{\mathbb{R}^d} v\phi(v)\,\mathcal{M}^n\,\mathrm{d}v + O(\varepsilon) = 0, \quad \text{for any } n\ge N,$$

which is a consistent discretization to the limiting Euler system (2.11). This means the scheme is AP after an initial transient time.

Remark 3.2. *A possible way to remove the initial layer hence achieve AP in one time step was suggested in [91], where the idea is to perform the penalization in two successive steps:*

$$\begin{cases} \dfrac{f^* - f^n}{\Delta t} + v \cdot \nabla_x f^n = \dfrac{\mathcal{Q}(f^n) - \beta(\mathcal{M}^n - f^n)}{\varepsilon} + \dfrac{\beta(\mathcal{M}^* - f^*)}{2\varepsilon}\,, \\ \dfrac{f^{n+1} - f^*}{\Delta t} = \dfrac{\beta(\mathcal{M}^{n+1} - f^{n+1})}{2\varepsilon}\,. \end{cases}$$

The penalization method was also applied to the nonlinear hyperbolic system with stiff relaxation in [32]:

$$\begin{cases} \partial_t u + \partial_x g(u,v) = 0\,, \\ \partial_t v + \partial_x h(u,v) = \dfrac{1}{\varepsilon} R(u,v)\,, \end{cases} \tag{3.7}$$

where the term R is dissipative: $\partial_v R \le 0$ and possesses a unique local equilibrium: $R(u,v) = 0$ implies $v = f(u)$. Then when $\varepsilon \to 0$, one has the macroscopic limit

$$\partial_t u + \partial_x g(u, f(u)) = 0\,. \tag{3.8}$$

Here inverting an implicit R is not as simple as that in the Jin-Xin model (2.1). Using the penalization, this term can be treated as follows:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} + \partial_x g(u^n, v^n) = 0\,, \\ \dfrac{v^{n+1} - v^n}{\Delta t} + \partial_x h(u^n, v^n) = \dfrac{1}{\varepsilon}[R(u^n, v^n) + \beta(v^n - f(u^n))] - \dfrac{\beta}{\varepsilon}[v^{n+1} - f(u^{n+1})]\,. \end{cases} \tag{3.9}$$

For this scheme, one can actually prove a similar AP property (3.6): for any initial condition v^0, u^0, as long as $\beta > \frac{1}{2}\sup|\partial_v R|$, there exists an integer $N > 0$ such that

$$v^n = f(u^n) + O(\varepsilon), \quad \text{for any } n \ge N. \tag{3.10}$$

Plugging (3.10) into the first equation of (3.9), we see that the scheme is AP beyond an initial layer.

Remark 3.3. *Numerical methods for nonlinear hyperbolic systems with stiff relaxation/source terms were among the earliest AP schemes for time-dependent problems. The limit (3.8) is an analogy of the Euler limit of the Boltzmann equation. If one uses the Chapman-Enskog expansion to $O(\varepsilon)$ term, then (3.7) can be approximated by a parabolic equation, an analogy of the Navier-Stokes limit to the Boltzmann equation. Numerical study of system of the type (3.7) begun in the works [47, 50], where the AP principle was applied to design numerical schemes to handle the stiff relaxation term. While an IMEX type Runge-Kutta method was used in [47] to capture the solution of (3.8), to capture the diffusion limit is much more difficult, since the numerical viscosity of $O(\Delta x)$ in shock capturing methods easily dominates the physical viscosity term of $O(\varepsilon)$, unless special flux is used that builds in the limit (see [50] and its further study and extension to unstructured meshes in [7, 3]). For AP schemes for gas dynamics with external force and frictions, see [12]. In principle one cannot expect to take $\Delta x, \Delta t \gg O(\varepsilon)$ and still capture the solution of the diffusion limit (likewise, the compressible Navier-stokes solution in a Boltzmann solver), unless special numerical viscosity can be chosen so it does not pollute the*

physical viscosity. For smooth solutions this should not be a problem. However, for shocks and boundary layers, which have thickness depending on ε, one cannot obtain reliable solution to the diffusion or compressible Navier-Stokes equations with mesh size much larger than ε.

The idea of using a linear/simpler operator to penalize the nonlinear/complicated operator turns out to be a generic approach. For specific problems, one needs to seek appropriate penalization operator that serves the purpose, and this usually relies on the knowledge of the original operator. For example, consider the nonlinear Fokker-Planck-Landau equation whose collision operator is given by

$$\mathcal{Q}(f)(v) = \nabla_v \cdot \int_{\mathbb{R}^d} A(v - v_*) \left[f(v_*) \nabla_v f(v) - f(v) \nabla_{v_*} f(v_*) \right] \, \mathrm{d}v_* \,,$$

where A is a semi-positive definite matrix. This equation is relevant in the study of Coulomb interaction [87]. The diffusive nature of the collision operator introduces more stiffness. An explicit scheme would require $\Delta t = O(\varepsilon(\Delta v)^2)$ which is more restrictive than the Boltzmann case. It was shown in [61] that the BGK operator is no longer suitable and the following Fokker-Planck operator was proposed as a penalization:

$$\mathcal{P}_{\mathrm{FP}} = \nabla_v \cdot \left(\mathcal{M} \nabla_v \left(\frac{f}{\mathcal{M}} \right) \right) .$$

Similar approaches, with variant penalties, have been proposed for the quantum Boltzmann equation [31], the quantum Fokker-Planck-Landau equation [42], the multi-species Boltzmann equation [51], and the two-scale collisions for semiconductor equations [44, 41].

Another AP scheme, developed later in [72], relies on the integral representation of the BGK equation. The final form of the scheme also ends up with a linear combination – with slightly different coefficients – of the Boltzmann collision operator and the BGK operator. The coefficients are discontinuous so one turns off the Boltzmann operator and the solver becomes a pure BGK solver when the mean free path is small.

3.2 AP schemes based on exponential reformulation

Another class of AP schemes is the *exponential method*. Unlike the penalization idea, this method does not directly look for numerical schemes for the original equation, but instead reformulates the equation into an exponential form, with the equilibrium function embedded in, before applying the standard explicit Runge-Kutta method. Because the new formulation has the Maxwellian function embedded, it is easier to capture the asymptotic limit: in fact a large class of standard numerical methods achieves AP properties automatically if applied to the new equation. Such flexibility allows one to seek for more properties that are hard to obtain with the original form, including the high order of accuracy in both time and space, the strong-AP property, positivity and many others.

Early studies on the homogeneous Boltzmann equation trace back to the Wild sum approach [33]. It was further elaborated in the IMEX Runge-Kutta framework in [29] for homogeneous case and extended to treat the nonhomogenous case in [69].

Consider the space homogeneous Boltzmann equation:

$$\partial_t f = \frac{1}{\varepsilon} \mathcal{Q}(f) \,. \tag{3.11}$$

Dimarco and Pareschi in [29] introduced the following reformulation:

$$\partial_t \left[(f-\mathcal{M}) e^{\beta t/\varepsilon} \right] = \partial_t f e^{\beta t/\varepsilon} + \frac{\beta(f-\mathcal{M})}{\varepsilon} e^{\beta t/\varepsilon} = \frac{\mathcal{Q} - \beta(\mathcal{M}-f)}{\varepsilon} e^{\beta t/\varepsilon} . \tag{3.12}$$

Here β is an auxiliary parameter and as in the penalization method, $\beta(\mathcal{M}-f)$ is used to approximate the Frechet derivative of $\mathcal{Q}$. Numerically β could be any $O(1)$ number, but one chooses the smallest value that preserves the positivity of f.

Equation (3.12) is fully equivalent to the original problem (3.11), but since the scheme is essentially applied to update the difference between the distribution function and the Maxwellian, it removes the stiffness and forces the convergence of f to $\mathcal{M}$ numerically, thus easily guarantees the AP property. It can be shown that all *explicit* Runge-Kutta methods, when applied, not only achieve the high order convergence, but also obtains AP property automatically.

For the nonhomogenous case, the equilibrium is convecting, making it difficult to extend the scheme. To overcome that, in [69], Li and Pareschi explored the possibility of using an evolving Maxwellian function *within* each time step. The Boltzmann equation is reformulated as:

$$\partial_t \left[(f-\mathcal{M}) e^{\beta t/\varepsilon} \right] = \left(\frac{\mathcal{P} - \beta \mathcal{M}}{\varepsilon} - v \cdot \nabla_x f - \partial_t \mathcal{M} \right) e^{\beta t/\varepsilon} ,$$

where $\mathcal{P} = \mathcal{Q} + \beta f$. Meanwhile the moment equations are obtained by taking the moments of the original Boltzmann equation (3.1):

$$\partial_t U + \int \phi(v)\, v \cdot \nabla_x f \, \mathrm{d}v = 0 . \tag{3.13}$$

To compute $\partial_t \mathcal{M}$, note that

$$\partial_t \mathcal{M} = \partial_\rho \mathcal{M} \partial_t \rho + \nabla_u \mathcal{M} \cdot \partial_t u + \partial_T \mathcal{M} \partial_t T ,$$

where $\partial_\rho \mathcal{M}$, $\nabla_u \mathcal{M}$ and $\partial_T \mathcal{M}$ can all be expressed explicitly, and the time derivatives of the other three macroscopic quantities ρ, u, T can be obtained from (3.13).

Using the Runge-Kutta method on this formulation, one obtains the following scheme:

- Step i

$$\begin{cases} \left(f^{(i)} - \mathcal{M}^{(i)} \right) e^{c_i \lambda} & = (f^n - \mathcal{M}^n) + \sum_{j=1}^{i-1} a_{ij} \frac{h}{\varepsilon} \left[\mathcal{P}^{(j)} - \beta \mathcal{M}^{(j)} - \varepsilon v \cdot \nabla_x f^{(j)} - \varepsilon \partial_t \mathcal{M}^{(j)} \right] e^{c_j \lambda} \\ \int \phi f^{(i)} \mathrm{d}v & = \int \phi f^n \mathrm{d}v + \sum_{j=1}^{i-1} a_{ij} \left(-h \int \phi v \cdot \nabla_x f^{(j)} \mathrm{d}v \right) \end{cases} ;$$

- Final step

$$\begin{cases} \left(f^{n+1} - \mathcal{M}^{n+1} \right) e^{\lambda} & = (f^n - \mathcal{M}^n) + \sum_{i=1}^{\nu} b_i \frac{h}{\varepsilon} \left[\mathcal{P}^{(i)} - \beta \mathcal{M}^{(i)} - \varepsilon v \cdot \nabla_x f^{(i)} - \varepsilon \partial_t \mathcal{M}^{(i)} \right] e^{c_i \lambda} \\ \int \phi f^{n+1} \mathrm{d}v & = \int \phi f^n \mathrm{d}v + \sum_{i=1}^{\nu} b_i \left(-h \int \phi v \cdot \nabla_x f^{(i)} \mathrm{d}v \right) \end{cases} ,$$

in which we denote $\lambda = \beta \Delta t / \varepsilon$ and use the coefficients from the following Butcher tableaux:

$$\begin{array}{c|cccc} c_1 & & & & \\ c_2 & a_{21} & & & \\ \vdots & \ddots & \ddots & & \\ c_\nu & a_{\nu,1} & a_{\nu,2} & \cdots & \\ \hline & b_1 & b_2 & \cdots & \cdots \end{array}$$

This method preserves positivity, high order accuracy, and the strong AP properties can also be proved (i.e., f is driven to $\mathcal{M}$ in one time step). It has been successfully extended to treat the quantum Boltzmann equation [43], the multi-species Boltzmann equation [71] and the Fokker-Planck-Landau equation [27].

3.3 AP schemes based on micro-macro decomposition

Another framework of AP schemes for the Boltzmann-like equations is termed the "micro-macro" decomposition, in which one decomposes the distribution function into the local Maxwellian, plus the deviation. It was used by Liu and Yu in [74] for analyzing the shock propagation of the Euler equations in passing the fluid limit of the Boltzmann equation. One early attempt of using such a decomposition to design an AP scheme was considered by Klar and Schmeiser for the radiative heat transfer equations [64]. Its application to the nonlinear Boltzmann equation started with the work of Bennoune-Lemou-Mieussens in [2].

The main idea begins with decomposing f into the Maxwellian and the remainder:

$$f = \mathcal{M} + \varepsilon g\,, \quad \text{with} \quad \int \phi(v) g \,\mathrm{d}v = 0\,.$$

Since the collision operator $\mathcal{Q}(f) := \mathcal{Q}[f,f]$ is bilinear, the linearized collision operator, which depends on $\mathcal{M}$, reads as:

$$\mathcal{L}_{\mathcal{M}} g = \mathcal{Q}[\mathcal{M}, g] + \mathcal{Q}[g, \mathcal{M}]\,.$$

With several lines of calculation, one gets

$$\begin{cases} \partial_t g + (\mathrm{I} - \Pi_{\mathcal{M}})(v\cdot\nabla_x g) - \mathcal{Q}[g,g] = \dfrac{1}{\varepsilon}\left[\mathcal{L}_{\mathcal{M}} g - (\mathrm{I}-\Pi_{\mathcal{M}})(v\cdot\nabla_x \mathcal{M})\right]\,, \\ \partial_t \displaystyle\int \phi \mathcal{M}\,\mathrm{d}v + \int \phi\, v\cdot\nabla_x \mathcal{M}\,\mathrm{d}v + \varepsilon \nabla_x \cdot \langle v\phi g\rangle = 0\,. \end{cases} \tag{3.14}$$

Here $\Pi_{\mathcal{M}}$ is a projection operator that maps arbitrary $\mathcal{M}$-weighted L_2 function into the null space of $\mathcal{L}_{\mathcal{M}}$, namely, for any $\psi \in L_2(\mathcal{M}^{-1}\mathrm{d}v)$:

$$\langle \Pi_{\mathcal{M}}(\psi)\,, \phi\rangle = \langle \psi\,, \phi\mathcal{M}\rangle\,, \quad \text{with} \quad \Pi_{\mathcal{M}}(\psi) \in \mathrm{Null}(\mathcal{L}_{\mathcal{M}}) = \mathrm{Span}\{\mathcal{M}, v\mathcal{M}, |v|^2\mathcal{M}\}\,.$$

For the Boltzmann equation one can write down the projection operator explicitly:

$$\Pi_{\mathcal{M}}(\psi) = \frac{1}{\rho}\left[\langle\psi\rangle + \frac{(v-u)\cdot\langle(v-u)\psi\rangle}{T} + \left(\frac{|v-u|^2}{2T} - \frac{d}{2}\right)\frac{2}{d}\langle\left(\frac{|v-u|^2}{2T} - \frac{d}{2}\right)\psi\rangle\right]\mathcal{M}\,.$$

Unlike the original Boltzmann equation with stiff term $\mathcal{Q}[f,f]$, the two stiff terms here are both linear thus their implicit discretization can be inverted easily.

In [2], the following discretization is taken:

$$\begin{cases} \dfrac{g^{n+1}-g^n}{\Delta t} + (\mathrm{I}-\Pi_{\mathcal{M}^n})(v\cdot\nabla_x g^n) - \mathcal{Q}[g^n, g^n] = \dfrac{1}{\varepsilon}\left[\mathcal{L}_{\mathcal{M}^n} g^{n+1} - (\mathrm{I}-\Pi_{\mathcal{M}^n})(v\cdot\nabla_x \mathcal{M}^n)\right]\,, \\ \displaystyle\int \phi\,\mathcal{M}^{n+1}\,\mathrm{d}v + \Delta t\varepsilon \int \phi\, v\cdot\nabla_x g^{n+1}\,\mathrm{d}v = \int \phi\,\mathcal{M}^n\,\mathrm{d}v - \Delta t \int \phi\, v\cdot\nabla_x \mathcal{M}^n\,\mathrm{d}v\,. \end{cases} \tag{3.15}$$

Obviously the only term that needs to be inverted is $\mathrm{I} - \frac{\Delta t}{\varepsilon}\mathcal{L}_{\mathcal{M}}$ in the first equation. It is a linear operator, and the negative spectrum guarantees the invertibility. The quadratic operator $\mathcal{Q}[f,f]$ is no longer stiff and is treated explicitly.

We list the AP property proved in [2]:

Proposition 3.4. *The scheme is AP, more specifically:*

i) The time discretization (3.15) *of the system* (3.14) *gives in the limit* $\varepsilon \to 0$ *a scheme consistent to the compressible Euler equations (2.11).*

ii) For small ε*, the scheme* (3.15) *is asymptotically equivalent, up to* $\mathcal{O}(\varepsilon^2)$*, to an explicit time discretization of the Navier-Stokes equations (3.4).*

Although $\mathcal{L}_{\mathcal{M}}$ is a linear operator, the computation, however, is far from being satisfactory. In fact many techniques, including the Carleman representation, used to speed up the computation of the Boltzmann collision operator [78], cannot be applied for the linear operator. It is perhaps for this reason, so far this method has only been applied to the BGK model, as was done in [2] (see an extension to the multi-species BGK model in [56]). Higher order schemes can be achieved by combining IMEX scheme in time and DG discretization in space [88].

4 Other asymptotic limits and AP schemes

In this section, we briefly describe several other asymptotic limits and the construction of corresponding AP schemes.

4.1 Diffusion limit of linear transport equation

In many applications, such as neutron transport and radiative transfer, the collision operator is linear. The interesting scaling is the diffusive scaling. A typical such equation has the form of

$$\varepsilon\, \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \sigma(v, w)[M(v)f(w) - M(w)f(v)]\, \mathrm{d}w\,, \tag{4.1}$$

where M is the normalized Maxwellian

$$M(v) = \frac{1}{\pi^{d/2}} \exp(-|v|^2)\,.$$

The anisotropic scattering kernel σ is rotationally invariant and satisfies

$$\sigma(v, w) = \sigma(w, v) > 0\,.$$

Define the collision frequency λ as

$$\lambda(v) = \int_{\mathbb{R}^d} \sigma(v, w) M(w)\, \mathrm{d}w \le \mu\,.$$

One can show that as $\varepsilon \to 0$, $f \to \rho(t, x)M(v)$ in (4.1), where $\rho(t, x) = \int f(v)\, \mathrm{d}v$ satisfies the diffusion equation [82, 76]:

$$\partial_t \rho = \nabla_x \cdot (D \nabla_x \rho)\,,$$

with the diffusion coefficient matrix

$$D = \int \frac{M(v)}{\lambda(v)} v \otimes v\, \mathrm{d}v\,.$$

Developing numerical schemes for such equations that are efficient in diffusive regimes constitutes the earliest works in AP schemes. It was carried out first in [67, 66] for stationary

transport equation in which some spatial discretizations were studied. As mentioned earlier, the most challenging issue for AP schemes is the time discretization, or rather, a reformulation of the equation so that it becomes suitable for most spatial or velocity discretizations. It begins independently with the work of Jin-Pareschi-Toscani [54, 55] and Klar [63]. The idea in [55] was to use an even and odd-decomposition to write f as a linear combination of its even and odd parts (in velocity) satisfying a hyperbolic system with stiff relaxation which can be solved using the ideas developed earlier [47, 59, 8]. In [68], the micro-macro decomposition approach was used (see also related works [64, 9]). A uniform in ε stability for this method was proved in [73]. A high-order DG-IMEX scheme based on micro-macro decomposition was proposed in [46], and the uniform stability was also established [45]. Most of the schemes designed, though successfully relaxed the ε dependence, still suffer from the parabolic scaling $\Delta t = O(\Delta x^2)$, and it was solved in [70], where fully implicit scheme is used. Another interesting approach using well-balanced scheme based on non-conservative product was given by Gosse and Toscani [36], although the idea has been developed only for one space dimension.

4.2 High-field limit

In kinetic equations, often there is a strong external field, such as the electric or magnetic field, that balances the collision term, leading to the so-called high field limit [11]. Consider for example the interaction between the electrons and a surrounding bath through Coulomb force in electrostatic plasma, where the electron distribution $f(t, x, v)$ is governed by the Vlasov-Poisson-Fokker-Planck system:

$$\partial_t f + v \cdot \nabla_x f - \frac{1}{\varepsilon} \nabla_x \phi \cdot \nabla_v f = \frac{1}{\varepsilon} \nabla_v \cdot (v f + \nabla_v f) \,, \tag{4.2}$$

$$-\Delta_x \phi = \rho - h \,, \tag{4.3}$$

where $\varepsilon = \left(\frac{l_e}{\Lambda}\right)^2$ is the ratio between the mean free path and the Debye length. Let $\varepsilon \to 0$ in (4.2), one obtains the high-field limit [79, 38]:

$$\partial_t \rho - \nabla_x \cdot (\rho \nabla_x \phi) = 0 \,,$$
$$-\Delta_x \phi = \rho - h \,.$$

One can combine the force term with the Fokker-Planck term in (4.2) as

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \nabla_v \cdot \left[M \nabla_v (M f) \right] \,,$$

where $M := e^{-|v + \nabla_x \phi|^2/2}$. This is the starting point of two exisiting AP schemes [57, 19], based on which other well-developed AP frameworks can be used. For more general collision operator, for example, the semiconductor Boltzmann collision operator, this trick does not apply and one needs other ideas [58]. So far there have not been many AP schemes for this limit and there remain many interesting open questions [37].

4.3 Quasi-neutral limit in plasmas

Consider the one-species recaled Euler-Poisson equations for charged particles:

$$\partial_t n + \nabla_x \cdot q = 0\,, \tag{4.4}$$

$$\partial_t q + \nabla_x \cdot \left(\frac{q \otimes q}{n}\right) + \nabla_x p = n \nabla_x \phi\,, \tag{4.5}$$

$$\varepsilon^2 \Delta_x \phi = n - 1\,, \tag{4.6}$$

where n is the particle number density, q is the momentum, $p = n^\gamma$ is the pressure law with $\gamma \geq 1$, and ϕ is the electric potential. Here the negatively charged electrons with scaled charge equal to -1 is considered, with a uniform ion background density set to 1. The dimensionless parameter $\varepsilon = \lambda_D / L$ is the scaled Debye length, i.e., the ratio of the actual Debye length λ_D to the macroscopic length scale L. In many applications, $\varepsilon \ll 1$. This is the so-called *quasi-neutral* regime. When $\varepsilon \to 0$ in (4.6), one has $n = 1$, and the following quasi-neutral limiting equations arise [6]:

$$\nabla_x \cdot q = 0\,, \tag{4.7}$$

$$\partial_t q + \nabla_x \cdot (q \otimes q) = \nabla_x \phi\,, \tag{4.8}$$

which is the incompressible Euler equations.

Since the Poisson equation (4.6) becomes degenerate when $\varepsilon \to 0$, a direct discretization of the Euler-Poisson system performs poorly in the quasi-neutral regime. A key idea introduced by Degond et al. in [17] is to reformulate the system to a new one that remains *uniformly elliptic*. Notice that when taking the limit $\varepsilon \to 0$, the electric potential ϕ undergoes drastic changes from the Poisson equation (4.6) into

$$\Delta_x \phi = \nabla_x^2 : (q \otimes q)\,, \tag{4.9}$$

where ∇_x^2 denotes the Hessian and ":" the contracted product of rank two tensors. (4.9) is obtained by taking the divergence of (4.8). A uniformly stable scheme should have a formulation that is consistent to (4.9) when $\varepsilon \to 0$. To achieve this, one can take ∂_t on (4.4), $\nabla_x \cdot$ on (4.5) and ∂_{tt} on (4.6) to get

$$\partial_{tt} n + \nabla_x \cdot \partial_t q = 0\,, \tag{4.10}$$

$$\nabla_x \cdot \partial_t q + \nabla_x^2 : \left(\frac{q \otimes q}{n} + p\mathrm{I}\right) = \nabla_x \cdot (n \nabla_x \phi)\,, \tag{4.11}$$

$$\varepsilon^2 \Delta_x \partial_{tt} \phi = \partial_{tt} n\,. \tag{4.12}$$

Eliminating $\nabla_x \cdot \partial_t q$ by combining (4.10) and (4.11) and using (4.12), one obtains

$$-\nabla_x \cdot [(n + \varepsilon^2 \partial_{tt}) \nabla_x \phi] + \nabla_x^2 : \left[\frac{q \otimes q}{n} + p\mathrm{I}\right] = 0\,. \tag{4.13}$$

Note now the equation (4.13) is uniformly elliptic and does not degenerate, and in fact approaches to (4.9) as $\varepsilon \to 0$. Thus, one can expect the asymptotic stability with respect to ε if a suitable time discretization is used.

This approach has also been generalized to two-fluid model [18], Particle-in-Cell method for Vlasov-Poisson system [22], Euler-Maxwell system [23], among other plasma models. For more details, see recent comprehensive reviews [20, 21].

4.4 Low Mach number limit of compressible flows

Recently there has been increasing research activities in developing Mach number uniform fluid solvers. Consider the case of isentropic Navier-Stokes equations:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0\,, \tag{4.14}$$

$$\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \frac{1}{\varepsilon^2}\nabla_x p = \frac{1}{\mathrm{Re}}\Delta_x u\,, \tag{4.15}$$

where ρ is the density, u is the velocity, $p = \rho^\gamma$ is the pressure, ε is the Mach number and Re the Reynolds number. When $\varepsilon \ll 1$, one seeks for asymptotic expansions: $\rho = \rho^{(0)} + \varepsilon^2\rho^{(2)} + \cdots$, $u = u^{(0)} + \varepsilon^2 u^{(2)} + \cdots$ and $p = p^{(0)} + \varepsilon^2 p^{(2)} + \cdots$ which then yield [62]:

$$\nabla_x \cdot u^{(0)} = 0\,,$$

$$\partial_t u^{(0)} + \left(u^{(0)} \cdot \nabla_x\right) u^{(0)} + \frac{1}{\rho^{(0)}}\nabla_x p^{(2)} = \frac{1}{\rho^{(0)}\mathrm{Re}}\Delta_x u^{(0)}\,.$$

The standard explicit numerical method, when applied to the compressible equations (4.14) (4.15), requires $\Delta t = O(\varepsilon \Delta x)$ for stability and $\Delta x = o(\varepsilon)$ to reduce numerical dissipation [39, 25]. This imposes tremendous computational cost in the low Mach number or incompressible regime.

In developing a numerical scheme that is efficient for all Mach numbers, one needs to efficiently handle the fast moving acoustic waves that travel with speed $O(1/\varepsilon)$ such that one can use mesh size and time step independent of ε. This is usually achieved by splitting the flux into fast moving (corresponding to the acoustic waves) and a slowly moving one. One such approach was introduced in [40] (see a related method in [24] and its extension to the full Euler and Navier-Stokes systems [15]), which takes the following splitting:

$$\begin{cases} \partial_t \rho + \alpha \nabla_x \cdot (\rho u) + (1-\alpha)\nabla_x \cdot (\rho u) = 0\,, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \left(\dfrac{p(\rho) - a(t)\rho}{\varepsilon^2}\right) + \dfrac{a(t)}{\varepsilon^2}\nabla_x \rho = \dfrac{1}{\mathrm{Re}}\Delta_x u\,, \end{cases}$$

where α and $a(t)$ are artificial parameters. By choosing $a(t)$ well approximates $p'(\rho)$, the third term in the second equation is non-stiff, and will be treated explicitly. The term $\nabla_x \rho$ calls for implicit treatment, but it could be done easily due to the linearity, so only Poisson solvers are needed like in a projection method for incompressible Navier-Stokes equations [14, 85]. The scheme is shock-capturing in the high Mach number regime, and reduces to a projection method when $\varepsilon \to 0$.

This is a direction being rapidly developed. One can find other techniques such as splitting by Klein [65, 80], Lagrange-projection scheme [92], a modification of the Roe solver [77] with applications to astrophysics problems, careful choice of numerical viscosity [28], etc.

4.5 Stochastic AP schemes

Our discussion so far has been retained exclusively to deterministic equations. In practical applications, kinetic and hyperbolic problems almost always contain parameters that are uncertain, due to modeling and/or measurement errors. Initiated by the work [60], there has been increasing interest recently in the development of AP schemes for quantifying uncertainties in

kinetic equations. We take the following Goldstein-Taylor model as an example:

$$\begin{cases} \partial_t u + \partial_x v = 0\,, \\ \partial_t v + \dfrac{1}{\varepsilon} a(x,z)\partial_x u = -\dfrac{1}{\varepsilon} v\,, \end{cases} \tag{4.16}$$

where $a(x,z)$ is a random wave speed with $z \in I_z \subset \mathbb{R}^d$, a set of random variables equipped with probability density function $\pi(z)$. These random variables characterize the random inputs in the system. In the diffusion limit $\varepsilon \to 0$, one has $v = -a(x,z)\partial_x u$ from the second equation above, which, upon substitution to the first equation, gives a heat equation with random diffusion coefficient:

$$\partial_t u = \partial_x (a(x,z)\partial_x u)\,. \tag{4.17}$$

To deal with the random parameters, a popular approach is the stochastic Galerkin method based on generalized polynomial chaos expansion (gPC-sG) [90], which has been successfully applied to many physical and engineering problems [34, 89, 75]. The gPC-sG is essentially a spectral method in the random domain with the expansion basis chosen as orthogonal polynomials with weight being the probability density $\pi(z)$.

Now to solve the system (4.16) under both uncertainty and diffusive scaling, a natural way would be to combine the gPC-sG with the deterministic AP scheme properly. According to the definition in [60], a scheme is *stochastic AP* if a gPC-sG method for the equation (4.16) becomes a gPC-sG approximation for the limiting equation (4.17) as $\varepsilon \to 0$, with the gPC order (the highest degree of polynomials used to discretize z), Δt and Δx fixed. Under the gPC-sG approximation, the discrete system is a deterministic set of equations, thus often allows straightforward extension of the well-developed deterministic AP schemes. For recent development of uncertainty quantification for kinetic equations, see [60, 52, 53].

5 Conclusion

In this paper, we have reviewed the basic design principles and several generic strategies of the construction of AP schemes for multi-scale hyperbolic and kinetic equations. To handle multiple temporal or spatial scales, unlike a typical multi-scale and multi-physical approach that requires the coupling of microscopic and macroscopic solvers, the AP schemes solve exclusively the microscopic equations. They allow the discretization parameters free of the small scale constraints, while capture the coarse scale structure when the small physical scale parameter approaches zero. This is usually achieved by some implicit treatment or reformulation of the original equations, guided by the underlying asymptotic limit. Although the classical Boltzmann equation was mainly used to illustrate the ideas, the techniques presented can be applied to a large class of kinetic and hyperbolic equations. Other asymptotic limits and AP schemes were discussed as well.

References

[1] C. Bardos, F. Golse, and D. Levermore. Fluid dynamic limits of kinetic equations. I. Formal derivations. *J. Stat. Phys.*, 63:323–344, 1991.

[2] M. Bennoune, M. Lemou, and L. Mieussens. Uniformly stable numerical schemes for the Boltzmann equation preserving the compressible Navier-Stokes asymptotics. *J. Comput. Phys.*, 227:3781–3803, 2008.

[3] C. Berthon, G. Moebs, C. Sarazin-Desbois, and R. Turpault. An asymptotic-preserving scheme for systems of conservation laws with source terms on 2D unstructured meshes. *Commun. Appl. Math. Comput. Sci.*, 11(1):55–77, 2016.

[4] P. L. Bhatnagar, E. P. Gross, and M. Krook. A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems. *Phys. Rev.*, 94:511–525, 1954.

[5] J.-F. Bourgat, P. Le Tallec, B. Perthame, and Y. Qiu. Coupling Boltzmann and Euler equations without overlapping. In *Domain decomposition methods in science and engineering (Como, 1992)*, volume 157 of *Contemp. Math.*, pages 377–398. Amer. Math. Soc., Providence, RI, 1994.

[6] Y. Brenier. Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 25(3-4):737–754, 2000.

[7] C. Buet, B. Després, and E. Franck. Design of asymptotic preserving finite volume schemes for the hyperbolic heat equation on unstructured meshes. *Numer. Math.*, 122(2):227–278, 2012.

[8] R. E. Caflisch, S. Jin, and G. Russo. Uniformly accurate schemes for hyperbolic systems with relaxation. *SIAM J. Numer. Anal.*, 34:246–281, 1997.

[9] J. A. Carrillo, T. Goudon, P. Lafitte, and F. Vecil. Numerical schemes of diffusion asymptotics and moment closures for kinetic equations. *J. Sci. Comput.*, 36(1):113–149, 2008.

[10] C. Cercignani. *The Boltzmann Equation and Its Applications*. Springer-Verlag, New York, 1988.

[11] C. Cercignani, I. M. Gamba, and C. D. Levermore. High field approximations to a Boltzmann-Poisson system and boundary conditions in a semiconductor. *Appl. Math. Lett.*, 10(4):111–117, 1997.

[12] C. Chalons, M. Girardin, and S. Kokh. Large time step and asymptotic preserving numerical schemes for the gas dynamics equations with source terms. *SIAM J. Sci. Comput.*, 35(6):A2874–A2902, 2013.

[13] S. Chapman and T. G. Cowling. *The Mathematical Theory of Non-Uniform Gases*. Cambridge University Press, Cambridge, third edition, 1991.

[14] A. J. Chorin. Numerical solution of the Navier-Stokes equations. *Math. Comp.*, 22:745–762, 1968.

[15] F. Cordier, P. Degond, and A. Kumbaro. An asymptotic-preserving all-speed scheme for the Euler and Navier-Stokes equations. *J. Comput. Phys.*, 231(17):5685–5704, 2012.

[16] F. Coron and B. Perthame. Numerical passage from kinetic to fluid equations. *SIAM J. Numer. Anal.*, 28:26–42, 1991.

[17] P. Crispel, P. Degond, and M.-H. Vignal. An asymptotically stable discretization for the Euler-Poisson system in the quasi-neutral limit. *C. R. Math. Acad. Sci. Paris*, 341(5):323–328, 2005.

[18] P. Crispel, P. Degond, and M.-H. Vignal. An asymptotic preserving scheme for the two-fluid Euler-Poisson model in the quasineutral limit. *J. Comput. Phys.*, 223(1):208–234, 2007.

[19] N. Crouseilles and M. Lemou. An asymptotic preserving scheme based on a micro-macro decomposition for collisional Vlasov equations: diffusion and high-field scaling limits. *Kinet. Relat. Models*, 4(2):441–477, 2011.

[20] P. Degond. Asymptotic-preserving schemes for fluid models of plasmas. In *Numerical models for fusion*, volume 39/40 of *Panor. Synthèses*, pages 1–90. Soc. Math. France, Paris, 2013.

[21] P. Degond and F. Deluzet. Asymptotic-preserving methods and multiscale models for plasma physics. *preprint*, 2016.

[22] P. Degond, F. Deluzet, L. Navoret, A.-B. Sun, and M.-H. Vignal. Asymptotic-preserving particle-in-cell method for the Vlasov-Poisson system near quasineutrality. *J. Comput. Phys.*, 229(16):5630–5652, 2010.

[23] P. Degond, F. Deluzet, and D. Savelief. Numerical approximation of the Euler-Maxwell model in the quasineutral limit. *J. Comput. Phys.*, 231(4):1917–1946, 2012.

[24] P. Degond and M. Tang. All speed scheme for the low Mach number limit of the isentropic Euler equations. *Commun. Comput. Phys.*, 10(1):1–31, 2011.

[25] S. Dellacherie. Analysis of Godunov type schemes applied to the compressible Euler system at low Mach number. *J. Comput. Phys.*, 229(4):978–1016, 2010.

[26] S. M. Deshpande. Kinetic theory based new upwind methods for inviscid compressible flows. *AIAA Paper 86-0275*, 1986.

[27] G. Dimarco, Q. Li, L. Pareschi, and B. Yan. Numerical methods for plasma physics in collisional regimes. *J. Plasma Phys.*, 81(1):1–31, 2015.

[28] G. Dimarco, R. Loubere, and M.-H. Vignal. Study of a new asymptotic-preserving scheme for the Euler system in the low Mach number limit. *preprint*, 2016.

[29] G. Dimarco and L. Pareschi. Exponential Runge-Kutta methods for stiff kinetic equations. *SIAM J. Numer. Anal.*, 49(1):2057–2077, 2011.

[30] W. E and B. Engquist. The heterogeneous multiscale methods. *Commun. Math. Sci.*, 1(1):87–132, 2003.

[31] F. Filbet, J. Hu, and S. Jin. A numerical scheme for the quantum Boltzmann equation with stiff collision terms. *ESAIM Math. Model. Numer. Anal.*, 46:443–463, 2012.

[32] F. Filbet and S. Jin. A class of asymptotic-preserving schemes for kinetic equations and related problems with stiff sources. *J. Comput. Phys.*, 229:7625–7648, 2010.

[33] E. Gabetta, L. Pareschi, and G. Toscani. Relaxation schemes for nonlinear kinetic equations. *SIAM J. Numer. Anal.*, 34(6):2168–2194, 1997.

[34] R. G. Ghanem and P. D. Spanos. *Stochastic Finite Elements: A Spectral Approach*. Springer-Verlag, New York, 1991.

[35] F. Golse, S. Jin, and C. D. Levermore. The convergence of numerical transfer schemes in diffusive regimes. I. Discrete-ordinate method. *SIAM J. Numer. Anal.*, 36(5):1333–1369, 1999.

[36] L. Gosse and G. Toscani. Asymptotic-preserving & well-balanced schemes for radiative transfer and the Rosseland approximation. *Numer. Math.*, 98(2):223–250, 2004.

[37] L. Gosse and N. Vauchelet. Numerical high-field limits in two-stream kinetic models and 1D aggregation equations. *SIAM J. Sci. Comput.*, 38(1):A412–A434, 2016.

[38] T. Goudon, J. Nieto, F. Poupaud, and J. Soler. Multidimensional high-field limit of the electrostatic Vlasov-Poisson-Fokker-Planck system. *J. Differential Equations*, 213(2):418–442, 2005.

[39] H. Guillard and C. Viozat. On the behaviour of upwind schemes in the low Mach number limit. *Comput. & Fluids*, 28(1):63–86, 1999.

[40] J. Haack, S. Jin, and J.-G. Liu. An all-speed asymptotic-preserving method for the isentropic Euler and Navier-Stokes equations. *Commn. Comput. Phys.*, 12:955–980, 2012.

[41] J. Hu, S. Jin, and L. Wang. An asymptotic-preserving scheme for the semiconductor Boltzmann equation with two-scale collisions: a splitting approach. *Kinet. Relat. Models*, 8:707–723, 2015.

[42] J. Hu, S. Jin, and B. Yan. A numerical scheme for the quantum Fokker-Planck-Landau equation efficient in the fluid regime. *Commun. Comput. Phys.*, 12:1541–1561, 2012.

[43] J. Hu, Q. Li, and L. Pareschi. Asymptotic-preserving exponential methods for the quantum Boltzmann equation with high-order accuracy. *J. Sci. Comput.*, 62:555–574, 2015.

[44] J. Hu and L. Wang. An asymptotic-preserving scheme for the semiconductor Boltzmann equation toward the energy-transport limit. *J. Comput. Phys.*, 281:806–824, 2015.

[45] J. Jang, F. Li, J.-M. Qiu, and T. Xiong. Analysis of asymptotic preserving DG-IMEX schemes for linear kinetic transport equations in a diffusive scaling. *SIAM J. Numer. Anal.*, 52:2048–2072, 2014.

[46] J. Jang, F. Li, J.-M. Qiu, and T. Xiong. High order asymptotic preserving DG-IMEX schemes for discrete-velocity kinetic equations in a diffusive scaling. *J. Comput. Phys.*, 281:199–224, 2015.

[47] S. Jin. Runge-Kutta methods for hyperbolic conservation laws with stiff relaxation terms. *J. Comput. Phys.*, 122:51–67, 1995.

[48] S. Jin. Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations. *SIAM J. Sci. Comput.*, 21:441–454, 1999.

[49] S. Jin. Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review. *Riv. Mat. Univ. Parma*, 3:177–216, 2012.

[50] S. Jin and C. D. Levermore. Numerical schemes for hyperbolic conservation laws with stiff relaxation terms. *J. Comput. Phys.*, 126(2):449–467, 1996.

[51] S. Jin and Q. Li. A BGK-penalization-based asymptotic-preserving scheme for the multispecies Boltzmann equation. *Numer. Methods Partial Differential Equations*, 29:1056–1080, 2013.

[52] S. Jin and L. Liu. An asymptotic-preserving stochastic Galerkin method for the semiconductor Boltzmann equation with random inputs and diffusive scalings. *preprint*, 2016.

[53] S. Jin and H. Lu. An asymptotic-preserving stochastic Galerkin method for radiative heat transfer equations with random inputs and diffusive scalings. *preprint*, 2016.

[54] S. Jin, L. Pareschi, and G. Toscani. Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations. *SIAM J. Numer. Anal.*, 35(6):2405–2439 (electronic), 1998.

[55] S. Jin, L. Pareschi, and G. Toscani. Uniformly accurate diffusive relaxation schemes for multiscale transport equations. *SIAM J. Numer. Anal.*, 38(3):913–936 (electronic), 2000.

[56] S. Jin and Y. Shi. A micro-macro decomposition-based asymptotic-preserving scheme for the multispecies Boltzmann equation. *SIAM J. Sci. Comput.*, 31(6):4580–4606, 2009/10.

[57] S. Jin and L. Wang. An asymptotic preserving scheme for the Vlasov-Poisson-Fokker-Planck system in the high field regime. *Acta Math. Sci. Ser. B Engl. Ed.*, 31(6, [November 2010 on cover]):2219–2232, 2011.

[58] S. Jin and L. Wang. Asymptotic-preserving numerical schemes for the semiconductor Boltzmann equation efficient in the high field regime. *SIAM J. Sci. Comput.*, 35:B799–B819, 2013.

[59] S. Jin and Z. P. Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Commun. Pure Appl. Math.*, 48:235–276, 1995.

[60] S. Jin, D. Xiu, and X. Zhu. Asymptotic-preserving methods for hyperbolic and transport equations with random inputs and diffusive scalings. *J. Comput. Phys.*, 289:35–52, 2015.

[61] S. Jin and B. Yan. A class of asymptotic-preserving schemes for the Fokker-Planck-Landau equation. *J. Comput. Phys.*, 230:6420–6437, 2011.

[62] S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.*, 34(4):481–524, 1981.

[63] A. Klar. An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit. *SIAM J. Numer. Anal.*, 35(3):1073–1094 (electronic), 1998.

[64] A. Klar and C. Schmeiser. Numerical passage from radiative transfer to nonlinear diffusion models. *Math. Models Meth. Appl. Sci.*, 11(05):749–767, 2001.

[65] R. Klein. Semi-implicit extension of a Godunov-type scheme based on low Mach number asymptotics. I. One-dimensional flow. *J. Comput. Phys.*, 121(2):213–237, 1995.

[66] E. W. Larsen and J. E. Morel. Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes. II. *J. Comput. Phys.*, 83(1):212–236, 1989.

[67] E. W. Larsen, J. E. Morel, and W. F. Miller Jr. Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes. *J. Comput. Phys.*, 69(2):283–324, 1987.

[68] M. Lemou and L. Mieussens. A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit. *SIAM J. Sci. Comput.*, 31:334–368, 2008.

[69] Q. Li and L. Pareschi. Exponential Runge-Kutta for the inhomogeneous Boltzmann equations with high order of accuracy. *J. Comput. Phys.*, 259:402–420, 2014.

[70] Q. Li and L. Wang. Implicit asymptotic preserving method for linear transport equations. *preprint*, 2016.

[71] Q. Li and X. Yang. Exponential Runge-Kutta methods for the multi-species Boltzmann equation. *Commun. Comput. Phys.*, 15:996–1011, 2014.

[72] C. Liu, K. Xu, Q. Sun, and Q. Cai. A unified gas-kinetic scheme for continuum and rarefied flow IV: Full Boltzmann and model equations. *J. Comput. Phys.*, 314:305–340, 2016.

[73] J.-G. Liu and L. Mieussens. Analysis of an asymptotic preserving scheme for linear kinetic equations in the diffusion limit. *SIAM J. Numer. Anal.*, 48(4):1474–1491, 2010.

[74] T.-P. Liu and S.-H. Yu. Boltzmann equation: micro-macro decompositions and positivity of shock profiles. *Commun. Math. Phys.*, 246:133–179, 2004.

[75] O. P. Le Maitre and O. M. Knio. *Spectral Methods for Uncertainty Quantification: With Applications to Computational Fluid Dynamics*. Springer, 2010.

[76] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser. *Semiconductor equations*. Springer-Verlag, Vienna, 1990.

[77] F. Miczek, F. G. Ropke, and P. V. Edelmann. New numerical solver for flows at various Mach numbers. *Astronomy and Astrophysics*, 578:A50, 2015.

[78] C. Mouhot and L. Pareschi. Fast algorithms for computing the Boltzmann collision operator. *Math. Comp.*, 75:1833–1852, 2006.

[79] J. Nieto, F. Poupaud, and J. Soler. High-field limit for the Vlasov-Poisson-Fokker-Planck system. *Arch. Ration. Mech. Anal.*, 158(1):29–59, 2001.

[80] S. Noelle, G. Bispen, K. R. Arun, M. Lukacova-Medvidova, and C.-D. Munz. A weakly asymptotic preserving low Mach number scheme for the Euler equations of gas dynamics. *SIAM J. Sci. Comput.*, 36(6):B989–B1024, 2014.

[81] L. Pareschi and G. Russo. Implicit-explicit Runge-Kutta schemes and applications to hyperbolic systems with relaxation. *J. Sci. Comput.*, 25:129–155, 2005.

[82] F. Poupaud. Diffusion approximation of the linear semiconductor Boltzmann equation: analysis of boundary layers. *Asymptotic Anal.*, 4(4):293–317, 1991.

[83] D. I. Pullin. Direct simulation methods for compressible inviscid ideal-gas flow. *J. Comput. Phys.*, 34:231–244, 1980.

[84] G. Puppo and S. Pieraccini. Implicit-explicit schemes for BGK kinetic equations. *J. Sci. Comput.*, 32:1–28, 2007.

[85] R. Témam. Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires. II. *Arch. Rational Mech. Anal.*, 33:377–385, 1969.

[86] S. Tiwari and A. Klar. An adaptive domain decomposition procedure for Boltzmann and Euler equations. *J. Comput. Appl. Math.*, 90(2):223–237, 1998.

[87] C. Villani. A review of mathematical topics in collisional kinetic theory. In S. Friedlander and D. Serre, editors, *Handbook of Mathematical Fluid Mechanics*, volume I, pages 71–305. North-Holland, 2002.

[88] T. Xiong, J. Jang, F. Li, and J-M. Qiu. High order asymptotic preserving nodal discontinuous Galerkin IMEX schemes for the BGK equation. *J. Comput. Phys.*, 284:70–94, 2015.

[89] D. Xiu. *Numerical Methods for Stochastic Computations*. Princeton University Press, New Jersey, 2010.

[90] D. Xiu and G. E. Karniadakis. The Wiener-Askey polynomial chaos for stochastic differential equations. *SIAM J. Sci. Comput.*, 24:619–644, 2002.

[91] B. Yan and S. Jin. A successive penalty-based asymptotic-preserving scheme for kinetic equations. *SIAM J. Sci. Comput.*, 35:A150–A172, 2013.

[92] H. Zakerzadeh. On the Mach-uniformity of the Lagrange-projection scheme. *Math. Model Num. Anal.*, 2016.