THE CONVERGENCE OF NUMERICAL TRANSFER SCHEMES IN DIFFUSIVE REGIMES I: DISCRETE-ORDINATE METHOD

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Abstract. In highly diffusive regimes, the transfer equation with anisotropic boundary conditions has an asymptotic behavior as the mean free path ϵ tends to zero that is governed by a diffusion equation and boundary conditions obtained through a matched asymptotic boundary layer analysis. A numerical scheme for solving this problem has a ϵ^{-1} contribution to the truncation error that generally gives rise to a nonuniform consistency with the transfer equation for small ϵ, thus degrading its performance in diffusive regimes. In this paper we show that whenever the discrete-ordinate method has the correct diffusion limit, both in the interior and at the boundaries, its solutions converge to the solution of the transport equation uniformly in ϵ. Our proof of the convergence is based on an asymptotic diffusion expansion and requires error estimates on a matched boundary layer approximation to the solution of the discrete-ordinate method.

Key words. transfer equation, diffusion approximation, boundary value problem, convergence, discrete-ordinate method, boundary layer, matched asymptotics.

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1. Introduction. We consider particles in a bounded domain that interact with a background medium through emission/absorption and scattering processes and whose phase space density is governed by a linear transport equation. A highly diffusive medium is characterized by a mean free path (the average distance a particle travels between interactions with the background medium) that is small compared to typical length scales of the domain. This small ratio is embodied by the introduction of a dimensionless parameter ϵ into the transport equation. By employing a singular asymptotic expansion in ϵ, it can be argued [13] that the leading behavior of the solution is governed by a diffusion equation in the interior of the spatial domain, i.e. away from boundaries and discontinuous material interfaces. Boundary conditions for this diffusion equation arise from an asymptotic boundary layer analysis [8,13,16]. The resulting equations are the basis for many numerical simulations of transport phenomena in purely diffusive regimes.

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In many applications the medium contains both diffusive and nondiffusive regions, requiring algorithms that handle these so-called transition regimes accurately. However, it is known from practical experience (for example, see [14]) that most numerical transport schemes do not converge uniformly as the mean free path becomes small and therefore suffer a corresponding degradation of both accuracy and efficiency in transition regimes. The criterion that determines whether a numerical transport scheme is well-behaved in diffusive regimes is that the scheme formally possesses a correct numerical diffusion approximation. This means that, for each fixed discretization, as ϵ tends to zero the leading asymptotic behavior of the numerical phase space density is governed by a discrete diffusion equation and boundary conditions that are a consistent and stable discretization of the diffusion equation and boundary conditions associated with the transport equation [9]. The correct boundary conditions for either the diffusion equation or its discrete analogue must be obtained through a matched asymptotic boundary layer analysis and not by simply insisting that the diffusion approximation be valid uniformly up to the boundary.

Although numerical experiments have shown the validity of this criterion [9], the question of whether the numerical solutions converge uniformly with respect to ϵ to the transport solution is still open, as is the uniform convergence rate of these numerical schemes. Of course, this question even makes perfect sense for the simpler stationary problem (the transfer equation) in one dimension. The analysis of this problem for transfer equation does not raise such technicalities as the merging of initial and boundary layers to match the interior diffusion approximation with the boundary and initial data, while the restriction to one spatial dimension greatly simplifies the attendant boundary layer analysis. However, certain time discretizations of the transport equation — for example, implicit ones — lead to transfer equations; it is therefore clear that much of what can be done on the transfer equation could be carried over to the transport equation.

We shall therefore consider the following one dimensional, one speed transfer equation in an isotropic medium for a particle density $\Psi^\epsilon = \Psi^\epsilon(x,\mu)$ over the single-particle phase space (x,μ), where $x_L \le x \le x_R$ is the particle position, and $-1 \le \mu \le 1$ the cosine of the angle between the direction of travel of the particle and the positive x-axis. In terms of the small parameter ϵ that is the ratio of the scale of particle mean free paths to that of typical gradient lengths of the problem, the scaled transfer equation is

$$\mu \partial_x \Psi^\epsilon + \frac{\sigma^T}{\epsilon}\Psi^\epsilon = \left[\frac{\sigma^T}{\epsilon} - \epsilon\,\sigma^A\right]\overline{\Psi^\epsilon} + \epsilon\, Q\,, \tag{1.1a}$$

over $(x_L, x_R) \times [-1,1]$, where the scalar density is defined by

$$\overline{\Psi^\epsilon}(x) = \frac{1}{2}\int_{-1}^{1} \Psi^\epsilon(x,\mu')\, d\mu'\,. \tag{1.1b}$$

Here $\sigma^T = \sigma^T(x) > 0$ is the transport coefficient, $\sigma^A = \sigma^A(x) > 0$ is the absorption coefficient, and $Q = Q(x) \ge 0$ is the isotropic source. The mean free path at x is given by $\epsilon/\sigma^T(x)$. We shall consider the boundary conditions

$$\Psi^\epsilon(\mu)\Big|_{x=x_L} = F_L(\mu)\,, \qquad \Psi^\epsilon(-\mu)\Big|_{x=x_R} = F_R(\mu)\,, \qquad \text{for } \mu > 0\,, \tag{1.1c}$$

where $F_L(\mu) \geq 0$ and $F_R(\mu) \geq 0$ specify the particles entering the domain at the left and right boundaries respectively whose direction of travel makes a cosine μ with the inward normal. Of course, our methods can be applied to a much wider class of boundary conditions.

The discrete-ordinate equation is an approximation of the transfer equation (1.1) in which only the μ variable is discretized. Specifically, the variable μ is discretized by a quadrature set consisting of $2M$ points $\mu_m \in (-1, 1)$ and $2M$ weights $\alpha_m > 0$ indexed by $m \in \mathcal{M} \equiv \{-M, \cdots, -1, 1, \cdots, M\}$ so as to respect the ordering

$$-1 < \mu_{-M} < \cdots < \mu_{-1} < 0 < \mu_1 < \cdots < \mu_M < 1\,, \tag{1.2a}$$

posses the symmetries

$$\mu_{-m} = -\mu_m\,, \qquad \text{and} \qquad \alpha_{-m} = \alpha_m\,, \qquad \text{for } m \in \mathcal{M}\,, \tag{1.2b}$$

and at least satisfy the quadrature conditions

$$\sum_{m=1}^{M} \mu_m^{2k}\,\alpha_m = \begin{cases} 1\,, & \text{for } k = 0\,, \\ \frac{1}{3}\,, & \text{for } k = 1\,. \end{cases} \tag{1.2c}$$

These conditions are met by many quadrature sets, for example, by Gauss quadrature over $[-1, 1]$ for $M \geq 1$, and by double Gauss quadrature over $[-1, 0] \cup [0, 1]$ for $M \geq 2$ [5].

The solution $\Psi^\epsilon = \Psi^\epsilon(x, \mu)$ of the transfer equation (1.1) evaluated at the quadrature points μ_m is then approximated formally by the discrete particle density $\psi^\epsilon = \psi^\epsilon_m(x)$ that satisfies the so-called discrete-ordinate equation given by

$$\mu_m \partial_x \psi^\epsilon_m + \frac{\sigma^T}{\epsilon}\psi^\epsilon_m = \left[\frac{\sigma^T}{\epsilon} - \epsilon\,\sigma^A\right]\overline{\psi^\epsilon} + \epsilon\, Q\,, \tag{1.3a}$$

over $(x_L, x_R) \times \mathcal{M}$, where the scalar density is defined by

$$\overline{\psi^\epsilon}(x) = \frac{1}{2}\sum_{n\in\mathcal{M}} \psi^\epsilon_n(x)\,\alpha_n\,, \tag{1.3b}$$

and the boundary conditions are

$$\psi^\epsilon_m\Big|_{x=x_L} = f_{L\,m}\,, \qquad \psi^\epsilon_{-m}\Big|_{x=x_R} = f_{R\,m}\,, \qquad \text{for } m > 0\,. \tag{1.3c}$$

In this paper, we always denote the angular averages with a bar as in either (1.1b) or (1.3b) depending on whether the angular variable is either continuous or discrete.

This paper examines the convergence of the discrete-ordinate method as the angular mesh is refined. Specifically, we consider quadrature sets that, in addition to (1.2), satisfy

$$\mu_m < \sum_{k=1}^{m} \alpha_k < \mu_{m+1}\,, \qquad \text{for } m = 1, \cdots, M-1. \tag{1.4}$$

This condition is satisfied by every commonly used quadrature set. We introduce points $\mu_{m+\frac{1}{2}}$ by $\mu_{\frac{1}{2}} \equiv 0$ and

$$\mu_{m+\frac{1}{2}} \equiv \mu_{m-\frac{1}{2}} + \alpha_m = \sum_{k=1}^{m} \alpha_k \,, \qquad \text{for } m = 1, \cdots, M. \tag{1.5}$$

In particular, $\mu_{M+\frac{1}{2}} = 1$ by (1.2c). By (1.4) the $\mu_{m+\frac{1}{2}}$ and μ_m interlace as

$$\mu_{m-\frac{1}{2}} < \mu_m < \mu_{m+\frac{1}{2}} \,, \qquad \text{for } m = 1, \cdots, M. \tag{1.6}$$

Now define δ, the resolution of the quadrature set, by

$$\delta \equiv \max_{1 \le m \le M} \left\{ \mu_{m+\frac{1}{2}} - \mu_m \,,\, \mu_m - \mu_{m-\frac{1}{2}} \right\} \le \max_{1 \le m \le M} \left\{ \alpha_m \right\}. \tag{1.7}$$

We shall consider a sequence of such quadrature sets $\{\mu_m^{(M)}, \alpha_m^{(M)}\}$ parameterized by M for which $\delta^{(M)} \to 0$ as $M \to \infty$, and for which there exists a constant $K < \infty$ such that

$$\frac{\mu_{m+\frac{1}{2}}^{(M)}}{\mu_m^{(M)}} \equiv \frac{1}{\mu_m^{(M)}} \sum_{k=1}^{m} \alpha_k^{(M)} \le K \qquad \text{uniformly over } M \text{ and } 1 \le m \le M. \tag{1.8}$$

Condition (1.8) is essential to the uniform arguments made later, but is satisfied by classical sequences of quadrature sets. We study the convergence of the corresponding solutions of (1.3) to solutions of (1.1) as $M \to \infty$. So as to keep the notation uncluttered, we will henceforth drop the superscript M in favor of refering to this limit as that of $\delta \to 0$.

The convergence of any numerical scheme is usually established by proving consistency and stability. In this spirit, the study of the convergence of the discrete-ordinate method was initiated by Keller [10] with the following basic stability estimate. The error $E^\epsilon = E_m^\epsilon(x)$ between the solution $\psi^\epsilon = \psi_m^\epsilon(x)$ of the discrete-ordinate equation (1.3) and the solution $\Psi^\epsilon = \Psi^\epsilon(x, \mu)$ of the transfer equation (1.1) restricted to the quadrature points μ_m is defined by

$$E_m^\epsilon(x) \equiv \psi_m^\epsilon(x) - \Psi^\epsilon(x, \mu_m) \,. \tag{1.9}$$

Introduce the so-called collocation operator $\mathcal{R}$ as the restriction of any function of μ to its evaluation at the quadrature points μ_m, so that, in particular, one has

$$(\mathcal{R}\Psi^\epsilon)_m(x) = \Psi^\epsilon(x, \mu_m) \,. \tag{1.10}$$

The discrete-ordinate error given by (1.9) can then be expressed compactly in terms of $\mathcal{R}$ as $E^\epsilon = \psi^\epsilon - \mathcal{R}\Psi^\epsilon$. By subtracting (1.1) from (1.3), the error E^ϵ is seen to satisfy the equation

$$\mu_m \partial_x E_m^\epsilon + \frac{\sigma^T}{\epsilon} E_m^\epsilon - \left[\frac{\sigma^T}{\epsilon} - \epsilon\, \sigma^A \right] \overline{E^\epsilon} = \left[\frac{\sigma^T}{\epsilon} - \epsilon\, \sigma^A \right] \left(\overline{\mathcal{R}\Psi^\epsilon} - \overline{\Psi^\epsilon} \right) , \tag{1.11a}$$

with the boundary conditions

$$
E^{\epsilon}_{m}\Big|_{x=x_L} = f_{L\,m} - F_L(\mu_m)\,, \qquad E^{\epsilon}_{-m}\Big|_{x=x_R} = f_{R\,m} - F_R(\mu_m)\,, \qquad \text{for } m > 0\,. \tag{1.11b}
$$

Upon applying the maximum principle to (1.11), one obtains the basic stability estimate

$$
\|E^{\epsilon}\|_{L^\infty([x_L,x_R]\times\mathcal{M})} \leq \max\left\{ \frac{1}{\epsilon^2}\left\|\frac{\sigma^S}{\sigma^A}\right\|_{L^\infty} \|\overline{\Psi^\epsilon} - \overline{\mathcal{R}\Psi^\epsilon}\|_{L^\infty}\,,\ \|f - \mathcal{R}F\|_{l^\infty} \right\}\,, \tag{1.12a}
$$

where $\sigma^S = \sigma^S(x) \equiv \sigma^T(x) - \epsilon^2\sigma^A(x) > 0$ is the scattering coefficient, and

$$
\|f - \mathcal{R}F\|_{l^\infty} \equiv \max\left\{ \|f_L - \mathcal{R}F_L\|_{l^\infty}\,,\ \|f_R - \mathcal{R}F_R\|_{l^\infty} \right\}\,. \tag{1.12b}
$$

Estimate (1.12a) shows that when one chooses $f_{L\,m} = F_L(\mu_m)$ and $f_{R\,m} = F_R(\mu_m)$, so that one has $\|f - \mathcal{R}F\|_{l^\infty} = 0$, the error of the discrete-ordinate method is bounded by a prefactor times the size of $\overline{\Psi^\epsilon} - \overline{\mathcal{R}\Psi^\epsilon}$, which is nothing but the truncation error of the quadrature scheme (1.2) applied to the transfer solution Ψ^ϵ. Keller [10,11,12] and Wendroff [20] used this estimate to show that the approximations converge uniformly on the quadrature points to the exact solution, provided that the truncation error for the exact solution converges to zero. These results were extended to general classes of scattering laws, boundary conditions, and quadrature rules by Anselone [1], Nestell [18], and Nelson [17].

Naively, one might expect from estimate (1.12) that the error of the discrete-ordinate method can be made spectrally accurate merely by choosing the quadrature rule to be so. However, this strategy will not succeed unless the solution of the transfer equation (1.1) is sufficiently regular so as to not degrade the spectral accuracy. For example, for functions $\Phi = \Phi(\mu)$ in $W^{1,1}([-1,1]) \equiv \{\Phi \in L^1([-1,1]) \,:\, \partial_\mu\Phi \in L^1([-1,1])\}$ the truncation error of the quadrature scheme can be written as

$$
\overline{\Phi} - \overline{\mathcal{R}\Phi} = \frac{1}{2}\int_{-1}^{1} R(\mu)\,\partial_\mu\Phi(\mu)\,d\mu\,, \tag{1.13a}
$$

where $R = R(\mu)$ is the odd, sawtoothed function of μ that for $\mu \in (0,1]$ is defined by

$$
\begin{aligned}
R(\mu) &\equiv \int_0^{\mu} \left(\sum_{m=1}^{M} \alpha_m \delta(\mu' - \mu_m) \right) - 1\, d\mu' \\
&= \begin{cases} \mu_{m-\frac{1}{2}} - \mu & \text{for } \mu_{m-\frac{1}{2}} < \mu \leq \mu_m, \\ \mu_{m+\frac{1}{2}} - \mu & \text{for } \mu_m < \mu \leq \mu_{m+\frac{1}{2}}. \end{cases}
\end{aligned} \tag{1.13b}
$$

By the Hölder inequality, one can therefore obtain the sharp error estimate

$$
|\overline{\Phi} - \overline{\mathcal{R}\Phi}| \leq \frac{\|R\|_{L^\infty}}{2}\int_{-1}^{1} |\partial_\mu\Phi(\mu)|\,d\mu = \frac{\delta}{2}\int_{-1}^{1} |\partial_\mu\Phi(\mu)|\,d\mu\,, \tag{1.14}
$$

where δ is the resolution of the quadrature set defined by (1.7). Estimate (1.14) suggests that, in order to obtain even a first order error estimate, one would like to control the L^1 norm of the first derivative of the integrand. When applied to solutions Ψ of the transfer equation, this means that one has to control the L^1 norm of the first derivative $\partial_\mu \Psi$. However, even this is too much to ask of solutions of the transfer equation (1.1) unless one is ready to make stringent assumptions on both the source term Q and the boundary data F_L and F_R. Indeed, Pitkäranta and Scott [19] showed that a general lack of regularity of solutions of the transfer equation would limit the convergence to first or second order, depending on the boundary conditions considered and the choice of quadrature sets.

Rather than try to consider every possible combination of regularity result and quadrature sets, we will work in the following abstract setting that is characteristic of a typical situation. We let $\mathcal{X}$ and $\mathcal{Y}$ be the domains of closed linear operators $D_{\mathcal{X}} : L^\infty([0,1]) \to L^1([0,1])$ and $D_{\mathcal{Y}} : L^\infty([-1,1]) \to L^1([-1,1])$ respectively equipped with the graph norms

$$\|G\|_{\mathcal{X}} \equiv \|G\|_{L^\infty} + \|D_{\mathcal{X}} G\|_{L^1} , \qquad \|\Phi\|_{\mathcal{Y}} \equiv \|\Phi\|_{L^\infty} + \|D_{\mathcal{Y}} \Phi\|_{L^1} , \tag{1.15}$$

for every $G \in \mathcal{X}$ and $\Phi \in \mathcal{Y}$. We assume that the solution $\Psi^\epsilon = \Psi^\epsilon(x,\mu)$ of (1.1) is in $L^\infty([x_L, x_R], \mathcal{Y})$ whenever $Q \in L^\infty([x_L, x_R])$ and F_L, $F_R \in \mathcal{X}$, satisfying a regularity estimate of the form

$$\sup_{x\in[x_L,x_R]} \Big\{ \|D_{\mathcal{Y}} \Psi^\epsilon(x,\cdot)\|_{L^1} \Big\} \le C_r , \tag{1.16}$$

where $C_r < \infty$ depends on the source Q and the boundary data F_L and F_R. We then assume that the quadrature sets, each of which satisfies (1.2) and (1.4), are such that every nonnegative $\Phi \in \mathcal{Y}$ satisfies

$$\left|\overline{\Phi} - \overline{\mathcal{R}\Phi}\right| \le \delta^q \, C_q \, \|D_{\mathcal{Y}} \Phi\|_{L^1} , \tag{1.17}$$

where $q > 0$ is the order of convergence and $C_q < \infty$. Finally, we assume that the discrete boundary data f_L and f_R, which are also parameterized by M, are chosen so that

$$\|f_L\|_{l^\infty} \le \|F_L\|_{L^\infty} , \qquad \|f_R\|_{l^\infty} \le \|F_R\|_{L^\infty} . \tag{1.18}$$

This condition is satisfied by all reasonable choices for the discrete boundary data.

As an example of one instance when estimates (1.16) and (1.17) are satisfied, in Appendix A we show that if $D_{\mathcal{X}} = \partial_\mu$, so that $\mathcal{X} = W^{1,1}([0,1]) \equiv \big\{ G \in L^1([0,1]) \,:\, \partial_\mu G \in L^1([0,1]) \big\}$ with

$$\|G\|_{\mathcal{X}} = \|G\|_{W^{1,1}} \equiv \|G\|_{L^\infty} + \|\partial_\mu G\|_{L^1} , \tag{1.19}$$

then for any $s \in (0,1)$ one may take $D_{\mathcal{Y}} = |\mu|^s \partial_\mu$, so that

$$\mathcal{Y} = \mathcal{Y}_s \equiv \left\{ \Phi \in L^\infty([-1,1]) \,:\, \int_{-1}^{1} |\mu|^s \big|\partial_\mu \Phi(\mu)\big| \, d\mu < \infty \right\} , \tag{1.20a}$$

$$\|\Phi\|_{\mathcal{Y}_s} \equiv \|\Phi\|_{L^\infty} + \frac{1}{2} \int_{-1}^{1} |\mu|^s \big|\partial_\mu \Phi(\mu)\big| \, d\mu . \tag{1.20b}$$

In that case (1.17) holds with $q = 1/(1+s)$ and $C_q = K^{s/(1+s)}$, while (1.16) holds with

$$C_r = \|F\|_{W^{1,1}} + \frac{2}{es}\max\left\{\|F\|_{L^\infty}, \left\|\frac{Q}{\sigma^A}\right\|_{L^\infty}\right\}, \tag{1.21}$$

where the norms of F denote the maximum of the corresponding norms of F_L and F_R:

$$\begin{aligned} \|F\|_{L^\infty} &\equiv \max\{\|F_L\|_{L^\infty}, \|F_R\|_{L^\infty}\}, \\ \|F\|_{W^{1,1}} &\equiv \max\{\|F_L\|_{W^{1,1}}, \|F_R\|_{W^{1,1}}\}. \end{aligned} \tag{1.22}$$

Regularity results in the spaces $\mathcal{Y}_s$ were first obtained by Pitkäranta and Scott [19], who also treated various other spaces. However, we can not directly appeal to their results because they did not treat the boundary conditions considered here and did not carry the dependence on ϵ in their estimates.

The assumed bounds (1.16) and (1.17) may be combined in (1.12) to obtain the convergence estimate

$$\left\|E^\epsilon\right\|_{L^\infty([x_L,x_R]\times\mathcal{M})} \le \max\left\{\frac{\delta^q}{\epsilon^2}C_c\,, \|f-\mathcal{R}F\|_{l^\infty}\right\}, \tag{1.23}$$

where

$$C_c \equiv \left\|\frac{\sigma^S}{\sigma^A}\right\|_{L^\infty} C_q\, C_r\,. \tag{1.24}$$

However, estimate (1.23) breaks down when $\epsilon \ll 1$ because of the factor ϵ^{-2}. Hence, this argument does not prove that the discrete-ordinate method converges uniformly as the mean free path becomes small — that is, as one approaches diffusive regimes. In this article we show that if the discrete-ordinate method has a correct diffusion approximation in the interior then its solution converges to the solution of the transfer equation *uniformly* in ϵ. Moreover, the estimates we find for the rate of this convergence are enhanced if the discrete-ordinate method has a correct diffusion approximation both in the interior and on the boundaries. Similar results for a fully discrete case will be presented in [7].

In order to prove that the convergence is uniform with respect to ϵ, we make explicit use of the diffusion approximation. Our strategy is best illustrated with the aid of Figure 1.1. Consider a family of solutions depending on two parameters, ϵ and the maximum quadrature increment δ, whose coordinate axes are depicted in Figure 1.1. The ϵ-axis ($\delta = 0$) corresponds to the continuum limit while the δ-axis ($\epsilon = 0$) corresponds to the diffusion approximation. The four nodes labeled Ψ^ϵ, Θ^ϵ, ψ^ϵ, and θ^ϵ represent respectively the solution of the continuum transfer equation (1.1), the solution of the diffusion equation associated with the continuum transfer equation, the solution of the discrete transfer equation (1.3), and the solution of the diffusion equation associated with the discrete transfer equation. The vertical line segments CDA and DDA represent the diffusion approximations for the continuum and discrete transfer equations. The horizontal line segments DNE and TNE represent the numerical errors of the discrete approximations to the continuum diffusion and transfer equations. In terms of Figure 1.1, our goal is to obtain an estimate for transfer numerical error (TNE) that is uniform in ϵ.

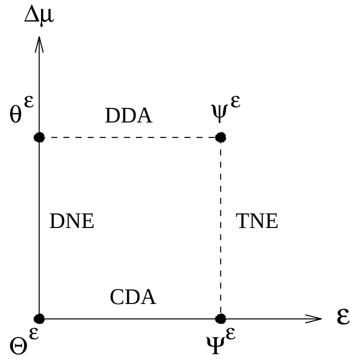

Figure 1.1

If the discrete-ordinate method has a correct diffusion approximation then solutions of the corresponding discrete diffusion equations will converge to the solution of the continuum diffusion equation associated with the transfer equation. In other words, the diffusion numerical error (DNE) vanishes as δ tends to zero. Thus, it suffices to show that as $\epsilon \to 0$ the discrete-ordinate solution is governed by its discrete diffusion approximation uniformly over refinements of the angular mesh (the upper horizontal line DDA). This we do following Bardos, Santos and Sentis [2], who have shown in the continuous case that the transfer solution is governed by its diffusion solution as $\epsilon \to 0$ (the bottom horizontal line CDA). A uniform estimate on the convergence will then be obtained by applying the triangle inequality along DDA, DNE and CDA, and comparing the result with the estimate of TNE given by (1.23).

This article is organized as follows. Section 2 states a result regarding how the transfer solution is governed by its diffusion solution as $\epsilon \to 0$ (the CDA). Section 3 states and proves the analogous result for the discrete-ordinate equation (the DDA). The key point here is that the estimates established there are uniform as the angular mesh is refined. With these results, the uniform convergence of the discrete-ordinate method is established in Section 4. The estimates for the order of convergence clearly show the role of the correct diffusion limit in proving the uniform convergence in diffusive regimes where the truncation error is not uniformly small in ϵ. Also, we point out that correct diffusion boundary conditions [9], increase the order of convergence. The detailed proofs of some technical statements concerning boundary layers are relegated to another appendix.

2. The Diffusion Approximation for the Transfer Equation. The diffusion approximation for problem (1.1) is valid in regimes where $\epsilon \ll 1$. A formal expansion in ϵ shows [8,13] that outside a boundary layer near $x = x_{_L}$ the solution of (1.1) has the approximate form

$$\Psi^\epsilon = \Theta^\epsilon - \epsilon \frac{\mu}{\sigma^T} \partial_x \Theta^\epsilon + O(\epsilon^2) \,, \tag{2.1}$$

where the function $\Theta^\epsilon = \Theta^\epsilon(x)$ satisfies the diffusion equation

$$-\partial_x \left(\frac{1}{3\sigma^T} \partial_x \Theta^\epsilon \right) + \sigma^A \Theta^\epsilon = Q \,, \tag{2.2a}$$

over $(x_{_L}, x_{_R})$ subject to the boundary conditions

$$\begin{aligned} \Theta^\epsilon - \epsilon \frac{\Lambda}{\sigma^T} \partial_x \Theta^\epsilon \bigg|_{x=x_L} &= \overline{F_{_L}} \equiv \int_0^1 F_{_L}(\mu) W(\mu)\, d\mu \,, \\ \Theta^\epsilon + \epsilon \frac{\Lambda}{\sigma^T} \partial_x \Theta^\epsilon \bigg|_{x=x_R} &= \overline{F_{_R}} \equiv \int_0^1 F_{_R}(\mu) W(\mu)\, d\mu \,. \end{aligned} \tag{2.2b}$$

Here Λ is the extrapolation length measured in mean free paths and $W = W(\mu)$ is the Case W-function [5], a positive density over $[0,1]$ that possesses the following properties:

$$\begin{aligned} &\text{a)} \quad \int_0^1 W(\mu)\, d\mu = 1 \,, \\ &\text{b)} \quad \int_0^1 \mu W(\mu)\, d\mu = \Lambda = 0.7104\cdots \,, \\ &\text{c)} \quad ⨍_0^1 \frac{W(\mu)}{1-\mu\xi}\, d\mu + 2\big(\xi - \tanh^{-1}(1/\xi)\big) W(1/\xi) = 0 \,, \quad \text{for every } \xi \in (1,\infty) \,, \end{aligned} \tag{2.3}$$

where $⨍$ denotes a principle-value integral.

Both the extrapolation length Λ and the Case W-function are components in the solution of half-space problems for a constant coefficient half-space transfer equation that arise in the formal boundary layer analysis for the diffusion limit of the transfer equation (1.1). Such an analysis is needed because the form of the diffusion approximation (2.1) is generally inconsistent with the transfer boundary conditions (1.1c). The analysis shows [8,13] that boundary layer correctors $\Gamma_{_L}^\epsilon$ and $\Gamma_{_R}^\epsilon$ may be constructed as functions of the stretched variables

$$z_{_L}^\epsilon \equiv \frac{1}{\epsilon} \int_{x_L}^x \sigma^T(s)\, ds \,, \qquad z_{_R}^\epsilon \equiv \frac{1}{\epsilon} \int_x^{x_R} \sigma^T(s)\, ds \,, \qquad \text{for } x \in [x_{_L}, x_{_R}] \,, \tag{2.4}$$

so that the solution Ψ^ϵ of the transfer problem (1.1) has the form

$$\Psi^\epsilon = \Theta^\epsilon - \epsilon \frac{\mu}{\sigma^T} \partial_x \Theta^\epsilon + \Gamma_{_L}^\epsilon + \Gamma_{_R}^\epsilon + O(\epsilon^2) \,, \tag{2.5}$$

where Θ^ϵ is the solution of the diffusion equation (2.3). The correctors have the form $\Gamma^\epsilon_L = \Gamma^\epsilon_L(z^\epsilon_L, \mu)$ and $\Gamma^\epsilon_R = \Gamma^\epsilon_R(z^\epsilon_R, -\mu)$, where $\Gamma^\epsilon_L(z,\mu)$ and $\Gamma^\epsilon_R(z,\mu)$ each satisfies a half-space problem of the form

$$\mu \partial_z \Gamma^\epsilon + \Gamma^\epsilon - \overline{\Gamma^\epsilon} = 0\,, \tag{2.6a}$$

over $(0,\infty)\times[-1,1]$ with boundary condition

$$\Gamma^\epsilon(0,\mu) = G^\epsilon(\mu) \qquad \text{for } \mu > 0\,, \tag{2.6b}$$

and decays exponentially to zero as $z \to \infty$. One obtains $\Gamma^\epsilon = \Gamma^\epsilon_L$ and $\Gamma^\epsilon = \Gamma^\epsilon_R$ respectively by setting $G^\epsilon = G^\epsilon_L$ and $G^\epsilon = G^\epsilon_R$ where

$$G^\epsilon_L(\mu) \equiv F_L(\mu) - \Theta^\epsilon(x_L) + \epsilon\,\frac{\mu}{\sigma^T(x_L)}\partial_x\Theta^\epsilon(x_L)\,, \qquad \text{for } \mu > 0\,, \tag{2.7a}$$

$$G^\epsilon_R(\mu) \equiv F_R(\mu) - \Theta^\epsilon(x_R) - \epsilon\,\frac{\mu}{\sigma^T(x_R)}\partial_x\Theta^\epsilon(x_R)\,, \qquad \text{for } \mu > 0\,. \tag{2.7b}$$

The key point of the boundary layer analysis is that the solutions of (2.6) associated with these boundary data will decay to zero if and only if Θ^ϵ satisfies the boundary conditions (2.2b).

Before proceeding to the justification of the formal asymptotic result stated above, we will give some estimates on Θ^ϵ, the solution of the diffusion equation (2.2), that play a central role in what follows. The ϵ dependence of Θ^ϵ arises solely due to the explicit appearance of ϵ in the boundary condition (2.2b). This simple dependence allows Θ^ϵ or any of its derivatives to be bounded uniformly over x in $[x_L, x_R]$ and ϵ. In particular, recalling that we have assumed $\sigma^T > 0$ over $[x_L, x_R]$ we have the following lemma.

Lemma 2.1: The maximum principle applied to (2.2) yields

$$0 \le \Theta^\epsilon \le C^{(0)}_\Theta \equiv \max\left\{ \|F\|_{L^\infty}\,,\ \left\|\frac{Q}{\sigma^A}\right\|_{L^\infty} \right\}. \tag{2.8a}$$

Furthermore, if σ^T, σ^A and Q are sufficiently smooth functions of x then the derivatives of Θ^ϵ satisfy uniform (over ϵ) bounds of the form

$$\left\| \left(\frac{1}{\sigma^T}\partial_x\right)^k \Theta^\epsilon \right\|_{L^\infty} \le C^{(k)}_\Theta\,, \qquad \text{for } k = 1, 2, \cdots\,, \tag{2.8b}$$

where the constants $C^{(k)}_\Theta < \infty$ depend on $\|F\|_{L^\infty}$ and the maximum norms of a finite number (depending on k) of derivatives of σ^T, σ^A, and Q. In particular, one can take

$$
\begin{aligned}
&C_\Theta^{(1)} \equiv \left(\frac{1}{\tau} + \frac{3\tau}{2} \left\| \frac{\sigma^A}{\sigma^T} \right\|_{L^\infty} \right) C_\Theta^{(0)} \,, \\
&C_\Theta^{(2)} \equiv 3 \left\| \frac{\sigma^A}{\sigma^T} \right\|_{L^\infty} C_\Theta^{(0)} \,, \\
&C_\Theta^{(3)} \equiv 3 \left\| \frac{1}{\sigma^T} \partial_x \frac{\sigma^A}{\sigma^T} \right\|_{L^\infty} C_\Theta^{(0)} + 3 \left\| \frac{\sigma^A}{\sigma^T} \right\|_{L^\infty} \left(C_\Theta^{(1)} + \left\| \frac{1}{\sigma^T} \partial_x \frac{Q}{\sigma^A} \right\|_{L^\infty} \right) , \\
&C_\Theta^{(4)} \equiv 3 \left\| \left(\frac{1}{\sigma^T} \partial_x \right)^2 \frac{\sigma^A}{\sigma^T} \right\|_{L^\infty} C_\Theta^{(0)} + 6 \left\| \frac{1}{\sigma^T} \partial_x \frac{\sigma^A}{\sigma^T} \right\|_{L^\infty} \left(C_\Theta^{(1)} + \left\| \frac{1}{\sigma^T} \partial_x \frac{Q}{\sigma^A} \right\|_{L^\infty} \right) \\
&\qquad + 3 \left\| \frac{\sigma^A}{\sigma^T} \right\|_{L^\infty} \left(C_\Theta^{(2)} + \left\| \left(\frac{1}{\sigma^T} \partial_x \right)^2 \frac{Q}{\sigma^A} \right\|_{L^\infty} \right) ,
\end{aligned}
\tag{2.8c}
$$

where τ is ϵ times the problem thickness in mean free paths:

$$
\tau \equiv \int_{x_L}^{x_R} \sigma^T(x)\, dx \,. \tag{2.9}
$$

Proof: The bound (2.8a) follows from the maximum principle. The value for $C_\Theta^{(2)}$ then follows directly from the diffusion equation (2.2a). The value for $C_\Theta^{(1)}$ is then obtained from those for $C_\Theta^{(0)}$ and $C_\Theta^{(2)}$ by a standard interpolation argument. The values for $C_\Theta^{(3)}$, $C_\Theta^{(4)}$ and higher $C_\Theta^{(k)}$ can then be obtained by taking the appropriate derivatives of (2.2a) and using the previous estimates. //

Equipped with the above estimates, one can establish the following theorem regarding how the transfer solution is governed by the uniform diffusion approximation (2.5) as $\epsilon \to 0$.

Theorem 2.2: Suppose $\Psi^\epsilon(x,\mu)$ is the solution of the transfer equation (1.1), and $\Theta^\epsilon(x)$ is the solution of the corresponding diffusion equation (2.2). Then within the space $L^\infty([0,\infty)\times[-1,1])$ there exists unique solutions $\Gamma^\epsilon = \Gamma^\epsilon_L(z,\mu)$ and $\Gamma^\epsilon = \Gamma^\epsilon_R(z,\mu)$ of the half-space problem (2.6) for the boundary data G^ϵ_L and G^ϵ_R given by (2.7) such that for some $C_b < \infty$ one has the bound

$$
\left\| \Psi^\epsilon - \Theta^\epsilon + \epsilon \frac{\mu}{\sigma^T} \partial_x \Theta^\epsilon - \Gamma^\epsilon_L - \Gamma^\epsilon_R \right\|_{L^\infty([x_L, x_R]\times[-1,1])} \le \epsilon^2 C_b \,, \tag{2.10}
$$

where $\Gamma^\epsilon_L = \Gamma^\epsilon_L(z^\epsilon_L, \mu)$ and $\Gamma^\epsilon_R = \Gamma^\epsilon_R(z^\epsilon_R, -\mu)$ with z^ϵ_L and z^ϵ_R defined in (2.4). Moreover, Γ^ϵ_L and Γ^ϵ_R each satisfy the bounds

$$
|\overline{\Gamma^\epsilon}(z)| \le 2A \exp\left(-\tfrac{1}{2} z \right) , \qquad \text{over } [0,\infty) \,, \tag{2.11a}
$$

$$
|\Gamma^\epsilon(z,\mu)| \le \frac{4A}{2-\mu} \exp\left(-\tfrac{1}{2} z \right) , \qquad \text{over } [0,\infty)\times[-1,1] \,, \tag{2.11b}
$$

where A is given by

$$A = C_\Theta^{(0)} + \epsilon\, C_\Theta^{(1)} . \tag{2.12}$$

This result is similar to that first proved by Blankenship and Papanicolaou [4] in a stochastic setting, and to that proved by Bardos, Santos, and Sentis [2] in a deterministic setting. The main result of [2] is included as (2.14) in the following corollary, which is a direct consequence of Theorem 2.2.

Corollary 2.3: Suppose $\Psi^\epsilon(x,\mu)$ is the solution of the transfer equation (1.1) and $\Theta^\epsilon(x)$ is the solution of the corresponding diffusion equation (2.2). Then the difference between Ψ^ϵ and its diffusion approximation (2.1) is second order in ϵ as $\epsilon \to 0$ in the L^∞ norm away from the boundaries, i.e., for every $[x_{IL}, x_{IR}] \subset (x_L, x_R)$ one has the estimate

$$\left\| \Psi^\epsilon - \Theta^\epsilon + \epsilon \frac{\mu}{\sigma^T} \partial_x \Theta^\epsilon \right\|_{L^\infty([x_{IL}, x_{IR}]\times[-1,1])} \leq \epsilon^2\, C_I , \tag{2.13}$$

where $C_I < \infty$ is a constant depending on $[x_{IL}, x_{IR}]$. Moreover, the difference between Ψ^ϵ and Θ^ϵ converges to zero in the L^p norm for every $p \in [1,\infty)$ as

$$\left\| \Psi^\epsilon - \Theta^\epsilon \right\|_{L^p([x_L, x_R]\times[-1,1])} \leq \epsilon^{\frac{1}{p}}\, C_p , \tag{2.14}$$

where $C_p < \infty$ is a positive constant depending on p.

Neither Theorem 2.2 nor Corollary 2.3 will be proved directly here, a proof of assertion (2.14) being found in [2]. Rather, both results will follow from the arguments used in the next section to establish the analogous results for the discrete-ordinate equation. Indeed, the constants C_b, C_I and C_p appearing above are bounded by the values of the corresponding constants in that section.

3. The Diffusion Approximation for the Discrete-Ordinate Equation. The diffusion equation corresponding to the discrete-ordinate equation (1.3) is (see [9,15])

$$-\partial_x\left(\frac{1}{3\sigma^T}\partial_x\theta^\epsilon\right)+\sigma^A\theta^\epsilon=Q\,, \tag{3.1a}$$

over (x_L, x_R) with the boundary conditions

$$\begin{aligned} \theta^\epsilon-\epsilon\frac{\lambda}{\sigma^T}\partial_x\theta^\epsilon\Big|_{x=x_L}&=\overline{f_L}\equiv\sum_{m=1}^{M}f_{L\,m}w_m\,,\\ \theta^\epsilon+\epsilon\frac{\lambda}{\sigma^T}\partial_x\theta^\epsilon\Big|_{x=x_R}&=\overline{f_R}\equiv\sum_{m=1}^{M}f_{R\,m}w_m\,. \end{aligned} \tag{3.1b}$$

Here the extrapolation length λ of the discrete-ordinate method measured in mean free paths is given by

$$\lambda=\sum_{m=1}^{M}\mu_m-\sum_{n=1}^{M-1}\frac{1}{\xi_{n+\frac12}}\,, \tag{3.2a}$$

while the discrete W-function w_m of the discrete-ordinate method is given by

$$w_m=\prod_{n=1}^{M-1}\left(\mu_m-\frac{1}{\xi_{n+\frac12}}\right)\prod_{\substack{k=1\\k\neq m}}^{M}\left(\frac{1}{\mu_m-\mu_k}\right)>0\,, \tag{3.2b}$$

where for each $n=1,\cdots,M-1$ we determine $\xi=\xi_{n+\frac12}$ as the unique (positive, simple) root of

$$1=\frac12\sum_{m\in\mathcal{M}}\frac{\alpha_m}{1-\mu_m\xi} \tag{3.3}$$

that lies in the open interval $(1/\mu_{n+1}, 1/\mu_n)$. These expressions for λ and the w_m are derived in [9] through a discrete boundary layer analysis.

Just as the classical W-function plays an important role in the boundary layer analysis for the diffusion limit of the transfer equation, the discrete W-function determined by (3.2b) plays a similar role for the discrete-ordinate equation. The following properties of w_m parallel the properties of the classical W-function given in (2.3):

$$\begin{aligned} &\text{a)}\quad \sum_{m=1}^{M}w_m=1\,,\\ &\text{b)}\quad \sum_{m=1}^{M}\mu_m w_m=\lambda=\sum_{m=1}^{M}\mu_m-\sum_{n=1}^{M-1}\frac{1}{\xi_{n+\frac12}}\,,\\ &\text{c)}\quad \sum_{m=1}^{M}\frac{w_m}{1-\mu_m\xi_{n+\frac12}}=0\,,\qquad \text{for } n=1,\cdots,M-1\,. \end{aligned} \tag{3.4}$$

These identities will prove useful in the subsequent analysis. We refer the reader to [9] for their proof.

A formal boundary layer analysis shows [9] that boundary layer correctors γ_L^ϵ and γ_R^ϵ may be constructed as a function of the stretched variables z_L^ϵ and z_R^ϵ defined in (2.4) so that the solution ψ^ϵ of the discrete-ordinate problem (1.3) has the form

$$\psi^\epsilon = \theta^\epsilon - \epsilon \frac{\mu_m}{\sigma^T} \partial_x \theta^\epsilon + \gamma_L^\epsilon + \gamma_R^\epsilon + O(\epsilon^2) \,, \tag{3.5}$$

where θ^ϵ is the solution of the diffusion equation (3.1). The correctors have the form $\gamma_L^\epsilon = \gamma_{L\,m}^\epsilon(z_L^\epsilon)$ and $\gamma_R^\epsilon = \gamma_{R\,-m}^\epsilon(z_R^\epsilon)$, where $\gamma_{L\,m}^\epsilon(z)$ and $\gamma_{R\,m}^\epsilon(z)$ each satisfies a half-space problem of the form

$$\mu_m \partial_z \gamma_m^\epsilon + \gamma_m^\epsilon - \overline{\gamma^\epsilon} = 0 \,, \tag{3.6a}$$

over $(0, \infty) \times \mathcal{M}$ with boundary conditions

$$\gamma_m^\epsilon(0) = g_m^\epsilon \,, \qquad \text{for } m > 0 \,, \tag{3.6b}$$

and decays exponentially to zero as $z \to \infty$. One obtains $\gamma^\epsilon = \gamma_L^\epsilon$ and $\gamma^\epsilon = \gamma_R^\epsilon$ respectively by setting $g^\epsilon = g_L^\epsilon$ and $g^\epsilon = g_R^\epsilon$ where

$$g_{L\,m}^\epsilon \equiv f_{L\,m} - \theta^\epsilon(x_L) + \epsilon \frac{\mu_m}{\sigma^T(x_L)} \partial_x \theta^\epsilon(x_L) \,, \qquad \text{for } m > 0 \,, \tag{3.7a}$$

$$g_{R\,m}^\epsilon \equiv f_{R\,m} - \theta^\epsilon(x_R) - \epsilon \frac{\mu_m}{\sigma^T(x_R)} \partial_x \theta^\epsilon(x_R) \,, \qquad \text{for } m > 0 \,. \tag{3.7b}$$

As before, the key point of the boundary layer analysis is that the solution of (3.6) associated with these boundary data will decay to zero if and only if θ^ϵ satisfies the boundary conditions (3.1b).

Before proceeding to the discrete-ordinate analog of Theorem 2.2, we observe that θ^ϵ and its derivatives satisfy the same estimates as those given by Lemma 2.1 for Θ^ϵ and its derivatives.

Lemma 3.1: The maximum principle applied to (3.1) yields

$$0 \le \theta^\epsilon \le C_\Theta^{(0)} \equiv \max\left\{ \|F\|_{L^\infty} \,, \left\| \frac{Q}{\sigma^A} \right\|_{L^\infty} \right\} \,, \tag{3.8a}$$

Furthermore, if σ^T, σ^A and Q are sufficiently smooth functions of x then the derivatives of θ^ϵ satisfy uniform (over the quadrature set and ϵ small) bounds of the form

$$\left\| \left(\frac{1}{\sigma^T} \partial_x \right)^k \theta^\epsilon \right\|_{L^\infty} \le C_\Theta^{(k)} \,, \qquad \text{for } k = 1, 2, \cdots \,, \tag{3.8b}$$

where the constants $C_\Theta^{(k)} < \infty$ are the same ones that appear in (2.8) of Lemma 2.1.

Proof: The bound (3.8a) follows from the maximum principle and the fact the discrete boundary data is assumed to satisfy (1.18). The remaining estimates then are derived exactly as in Lemma 2.1, using only the diffusion equation (3.1a). //

Our main result in this section regards how the discrete-ordinate solution is governed by the corresponding diffusion approximation (3.5) as $\epsilon \to 0$ uniformly in the angular grid; it clearly parallels Theorem 2.2.

Theorem 3.2: Suppose $\psi_m^\epsilon(x)$ is the solution of the discrete-ordinate equation (1.3) and $\theta^\epsilon(x)$ is the solution of the corresponding diffusion equation (3.1). Then within the space $L^\infty([0,\infty) \times \mathcal{M})$ there exists a unique solutions $\gamma^\epsilon = \gamma_{L\,m}^\epsilon(z)$ and $\gamma^\epsilon = \gamma_{R\,m}^\epsilon(z)$ of the half-space problem (3.6) for the boundary data g_L^ϵ and g_R^ϵ given by (3.7) such that for some $C_b < \infty$ one has the bound

$$(3.9) \qquad \left\| \psi^\epsilon - \theta^\epsilon + \epsilon \frac{\mu_m}{\sigma^T} \partial_x \theta^\epsilon - \gamma_L^\epsilon - \gamma_R^\epsilon \right\|_{L^\infty([x_L, x_R] \times \mathcal{M})} \leq \epsilon^2 C_b \,,$$

where $\gamma_L^\epsilon = \gamma_{L\,m}^\epsilon(z_L^\epsilon)$ and $\gamma_R^\epsilon = \gamma_{R\,-m}^\epsilon(z_R^\epsilon)$ with z_L^ϵ and z_R^ϵ defined in (2.4). Moreover, γ_L^ϵ and γ_R^ϵ each satisfy the bounds

$$(3.10a) \qquad |\overline{\gamma^\epsilon}(z)| \leq 2A\sqrt{K} \exp\left(-\tfrac{1}{2}z\right), \qquad \text{over } [0,\infty)\,,$$

$$(3.10b) \qquad |\gamma_m^\epsilon(z)| \leq \frac{4A\sqrt{K}}{2-\mu_m} \exp\left(-\tfrac{1}{2}z\right), \qquad \text{over } [0,\infty) \times \mathcal{M}\,,$$

where $K > 1$ was introduced in (1.8) and A is given by (2.12).

Remark: The bounds (3.10) are uniform in the angular grid — reducing to the bounds for the continuous case (2.11) upon setting $K = 1$.

Proof: Let $\mathcal{T}_\epsilon$ denote the discrete-ordinate transfer operator that acts on any ψ as

$$(3.11) \qquad \mathcal{T}_\epsilon \psi = \mu_m \partial_x \psi + \frac{\sigma^T}{\epsilon} \psi - \left[\frac{\sigma^T}{\epsilon} - \epsilon\, \sigma^A \right] \overline{\psi}\,.$$

We employ the classical Bensoussan-Lions-Papanicolaou technique [3,4]. Namely, we construct an approximate solution $\widetilde{\psi}^\epsilon$ of the discrete-ordinate equation (1.3) such that

$$(3.12) \qquad \left\| \widetilde{\psi}^\epsilon - \theta^\epsilon + \epsilon \frac{\mu_m}{\sigma^T} \partial_x \theta^\epsilon - \gamma_L^\epsilon - \gamma_R^\epsilon \right\|_{L^\infty([x_L, x_R] \times \mathcal{M})} = O(\epsilon^2)\,,$$

whose transfer error T^ϵ satisfies

$$(3.13a) \qquad T^\epsilon \equiv \mathcal{T}_\epsilon\left(\psi^\epsilon - \widetilde{\psi}^\epsilon\right) = O(\epsilon^3)\,,$$

and boundary errors b^ϵ_{L} and b^ϵ_{R} satisfy

$$\text{(3.13b)}\qquad \begin{aligned} b^\epsilon_{L\,m} &\equiv \psi^\epsilon_m - \widetilde{\psi}^\epsilon_m\Big|_{x=x_L} = O(\epsilon^2)\,, &&\text{for } m>0\,,\\ b^\epsilon_{R\,m} &\equiv \psi^\epsilon_{-m} - \widetilde{\psi}^\epsilon_{-m}\Big|_{x=x_R} = O(\epsilon^2)\,, &&\text{for } m>0\,. \end{aligned}$$

Since by assumption $\sigma^A > 0$, the inverse of $\mathcal{T}_\epsilon$ is of order ϵ^{-1} in any L^p space. In particular, when the maximum principle is applied to (3.13), one obtains the estimate

$$\text{(3.14a)}\qquad \left\|\psi^\epsilon - \widetilde{\psi}^\epsilon\right\|_{L^\infty} \le \max\left\{ \frac{1}{\epsilon}\left\|\frac{T^\epsilon}{\sigma^A}\right\|_{L^\infty},\ \left\|b^\epsilon\right\|_{l^\infty}\right\} = O(\epsilon^2)\,.$$

where $L^\infty = L^\infty([x_L, x_R]\times\mathcal{M})$ and

$$\text{(3.14b)}\qquad \left\|b^\epsilon\right\|_{l^\infty} \equiv \max\left\{\left\|b^\epsilon_L\right\|_{l^\infty},\ \left\|b^\epsilon_R\right\|_{l^\infty}\right\}\,.$$

The result will then follow from the triangle inequality applied to (3.12) and (3.14a).

The construction of $\widetilde{\psi}^\epsilon$ is motivated by the formal solution of the discrete-ordinate equation (1.3a) that is obtained by inserting the expansion

$$\text{(3.15)}\qquad \psi^\epsilon \sim \psi^{(0)} + \epsilon\,\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \cdots\,,$$

into the equation and balancing terms order by order in ϵ. This can be done either in the style of Hilbert or that of Chapman-Enskog. We adopt the later approach, in which each $\psi^{(k)}$ is expressed formally in terms of $\overline{\psi^\epsilon}$ and its derivatives subject to the constraint

$$\text{(3.16)}\qquad \partial_x\big(\,\overline{\mu\psi^\epsilon}\,\big) + \epsilon\,\sigma^A\,\overline{\psi^\epsilon} = \epsilon\, Q\,.$$

When this constraint is incorporated into it, the discrete-ordinate equation (1.3a) becomes

$$\text{(3.17)}\qquad \psi^\epsilon_m - \overline{\psi^\epsilon} = -\epsilon\,\frac{1}{\sigma^T}\partial_x\big(\mu_m\psi^\epsilon_m - \overline{\mu\psi^\epsilon}\,\big)\,.$$

Upon inserting (3.15) into (3.17) and balancing order by order in ϵ, one easily obtains

$$\text{(3.18)}\qquad \begin{aligned} \psi^\epsilon_m = \overline{\psi^\epsilon} &- \epsilon\,\frac{\mu_m}{\sigma^T}\partial_x\overline{\psi^\epsilon} + \epsilon^2\frac{\mu_m^2-\frac{1}{3}}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\overline{\psi^\epsilon}\right)\\ &- \epsilon^3\frac{\mu_m\big(\mu_m^2-\frac{1}{3}\big)}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\overline{\psi^\epsilon}\right)\right) + O(\epsilon^4)\,, \end{aligned}$$

whereby the constraint (3.16) formally becomes

$$\text{(3.19)}\qquad -\partial_x\left(\frac{1}{3\sigma^T}\partial_x\overline{\psi^\epsilon}\right) + \sigma^A\,\overline{\psi^\epsilon} = Q + O(\epsilon^2)\,.$$

Up to this point the discrete-ordinate boundary conditions (1.3b), which determine the solution ψ^ϵ, have played no role.

Motivated by (3.18) and (3.19), we construct an approximate interior solution $\widetilde{\psi}_I^\epsilon$ of the discrete-ordinate equation from the solution θ^ϵ of the diffusion approximation (3.1) by

$$
\begin{aligned}
\widetilde{\psi}_{I\,m}^\epsilon \equiv \theta^\epsilon - \epsilon\,\frac{\mu_m}{\sigma^T}\partial_x\theta^\epsilon + \epsilon^2\,\frac{\mu_m^2-\frac{1}{3}}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\theta^\epsilon\right)\\
- \epsilon^3\,\frac{\mu_m\left(\mu_m^2-\frac{1}{3}\right)}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\theta^\epsilon\right)\right).
\end{aligned}
\tag{3.20}
$$

By (3.9b) of Lemma 3.1 it is easily checked that

$$
\mathcal{T}_\epsilon\big(\psi^\epsilon-\widetilde{\psi}_I^\epsilon\big) = \epsilon^3\mu_m^2\left(\mu_m^2-\tfrac{1}{3}\right)\partial_x\left(\frac{1}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\theta^\epsilon\right)\right)\right) = O(\epsilon^3)\,,
\tag{3.21}
$$

whereby (3.13a) is satisfied by $\widetilde{\psi}^\epsilon = \widetilde{\psi}_I^\epsilon$. When the boundary conditions (1.3c) are applied however, the difference $\psi^\epsilon - \widetilde{\psi}_I^\epsilon$ is seen to satisfy

$$
\begin{aligned}
\psi_m^\epsilon - \widetilde{\psi}_{I\,m}^\epsilon\Big|_{x=x_L} &= f_{L\,m} - \theta^\epsilon + \epsilon\,\frac{\mu_m}{\sigma^T}\partial_x\theta^\epsilon - \epsilon^2\,\frac{\mu_m^2-\frac{1}{3}}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\theta^\epsilon\right)\\
&\quad + \epsilon^3\,\frac{\mu_m\left(\mu_m^2-\frac{1}{3}\right)}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\theta^\epsilon\right)\right)\Bigg|_{x=x_L}, \qquad \text{for } m>0\,,
\end{aligned}
\tag{3.22a}
$$

$$
\begin{aligned}
\psi_{-m}^\epsilon - \widetilde{\psi}_{I\,-m}^\epsilon\Big|_{x=x_R} &= f_{R\,m} - \theta^\epsilon - \epsilon\,\frac{\mu_m}{\sigma^T}\partial_x\theta^\epsilon - \epsilon^2\,\frac{\mu_m^2-\frac{1}{3}}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\theta^\epsilon\right)\\
&\quad - \epsilon^3\,\frac{\mu_m\left(\mu_m^2-\frac{1}{3}\right)}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\theta^\epsilon\right)\right)\Bigg|_{x=x_R}, \qquad \text{for } m>0\,,
\end{aligned}
\tag{3.22b}
$$

whereby (3.13b) is not satisfied by $\widetilde{\psi}^\epsilon = \widetilde{\psi}_I^\epsilon$. Indeed, if an incoming flux, $f_{L\,m}$ or $f_{R\,m}$, is anisotropic then the corresponding right side of (3.22) is at best $O(1)$. Even if an incoming flux is isotropic then the corresponding right side of (3.22) is at best $O(\epsilon)$. Boundary layer correctors are therefore needed to reduce these contributions to $O(\epsilon^2)$.

Boundary layer correctors χ_L^ϵ and χ_R^ϵ are constructed as functions of the stretched variables z_L^ϵ and z_R^ϵ defined by (2.4) in such a way that (3.13) is satisfied by

$$
\widetilde{\psi}_m^\epsilon(x) \equiv \widetilde{\psi}_{I\,m}^\epsilon(x) + \chi_{L\,m}^\epsilon\big(z_L^\epsilon(x)\big) + \chi_{R\,-m}^\epsilon\big(z_R^\epsilon(x)\big)\,.
\tag{3.23}
$$

Moreover, χ_L^ϵ and χ_R^ϵ decay exponentially away from their respective boundaries. In order to eliminate both the $O(1)$ and $O(\epsilon)$ terms in (3.22), we require that $\chi_{L\,m}^\epsilon(z)$ and $\chi_{R\,m}^\epsilon(z)$ satisfy

$$
\chi_{L\,m}^\epsilon(0) = f_{L\,m} - \theta^\epsilon(x_L) + \epsilon\,\frac{\mu_m}{\sigma^T(x_L)}\partial_x\theta^\epsilon(x_L)\,, \qquad \text{for } m>0\,,
\tag{3.24a}
$$

$$
\chi_{R\,m}^\epsilon(0) = f_{R\,m} - \theta^\epsilon(x_R) - \epsilon\,\frac{\mu_m}{\sigma^T(x_R)}\partial_x\theta^\epsilon(x_R)\,, \qquad \text{for } m>0\,.
\tag{3.24b}
$$

By doing so, (3.22) implies that the boundary errors, b^ϵ_{L} and b^ϵ_{R} of (3.13b), become

$$
\begin{aligned}
b^\epsilon_{L\,m} = & - \epsilon^2 \frac{\mu_m^2 - \frac{1}{3}}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right) \\
& + \epsilon^3 \frac{\mu_m \left(\mu_m^2 - \frac{1}{3}\right)}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right) \right) \\
& - \chi^\epsilon_{R\,-m}(z^\epsilon_{R}) \bigg|_{x=x_L} , \qquad \text{for } m > 0 ,
\end{aligned}
\tag{3.25a}
$$

$$
\begin{aligned}
b^\epsilon_{R\,m} = & - \epsilon^2 \frac{\mu_m^2 - \frac{1}{3}}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right) \\
& - \epsilon^3 \frac{\mu_m \left(\mu_m^2 - \frac{1}{3}\right)}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right) \right) \\
& - \chi^\epsilon_{L\,-m}(z^\epsilon_{L}) \bigg|_{x=x_R} , \qquad \text{for } m > 0 ,
\end{aligned}
\tag{3.25b}
$$

However, the introduction of χ^ϵ_{L} and χ^ϵ_{R} adds new terms to the transfer error (3.13a). Specifically, because by (2.4) one has $\partial_x z^\epsilon_{L} = \sigma^T/\epsilon$ and $\partial_x z^\epsilon_{R} = -\sigma^T/\epsilon$, one obtains

$$
\begin{aligned}
T^\epsilon &= \mathcal{T}_\epsilon \big(\psi^\epsilon - \widetilde{\psi}^\epsilon_{I}\big) - \mathcal{T}_\epsilon \chi^\epsilon_{L} - \mathcal{T}_\epsilon \chi^\epsilon_{R} \\
&= \epsilon^3 \mu_m^2 \big(\mu_m^2 - \tfrac{1}{3}\big) \partial_x \left(\frac{1}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right) \right) \right) \\
&\quad - \frac{\sigma^T}{\epsilon} \big(\mu_m \partial_z \chi^\epsilon_{L\,m} + \chi^\epsilon_{L\,m} - \overline{\chi^\epsilon_{L}} \big) - \epsilon\, \sigma^A \overline{\chi^\epsilon_{L}} \\
&\quad - \frac{\sigma^T}{\epsilon} \big(- \mu_m \partial_z \chi^\epsilon_{R\,-m} + \chi^\epsilon_{R\,-m} - \overline{\chi^\epsilon_{R}} \big) - \epsilon\, \sigma^A \overline{\chi^\epsilon_{R}} ,
\end{aligned}
\tag{3.26}
$$

In order to be formally consistent with (3.13a), one must impose

$$
- \frac{\sigma^T}{\epsilon} \big(\mu_m \partial_z \chi^\epsilon_{L\,m} + \chi^\epsilon_{L\,m} - \overline{\chi^\epsilon_{L}} \big) - \epsilon\, \sigma^A \overline{\chi^\epsilon_{L}} = O(\epsilon^3) ,
\tag{3.27a}
$$

$$
- \frac{\sigma^T}{\epsilon} \big(- \mu_m \partial_z \chi^\epsilon_{R\,-m} + \chi^\epsilon_{R\,-m} - \overline{\chi^\epsilon_{R}} \big) - \epsilon\, \sigma^A \overline{\chi^\epsilon_{R}} = O(\epsilon^3) ,
\tag{3.27b}
$$

or equivalently, that $\chi^\epsilon_{L\,m}(z)$ and $\chi^\epsilon_{R\,m}(z)$ satisfy

$$
\mu_m \partial_z \chi^\epsilon_{L\,m} + \chi^\epsilon_{L\,m} - \overline{\chi^\epsilon_{L}} = -\epsilon^2 \frac{\sigma^A}{\sigma^T} \overline{\chi^\epsilon_{L}} + O(\epsilon^4) ,
\tag{3.28a}
$$

$$
\mu_m \partial_z \chi^\epsilon_{R\,m} + \chi^\epsilon_{R\,m} - \overline{\chi^\epsilon_{R}} = -\epsilon^2 \frac{\sigma^A}{\sigma^T} \overline{\chi^\epsilon_{R}} + O(\epsilon^4) ,
\tag{3.28b}
$$

for z in the range determined by (2.4) and all $m \in \mathcal{M}$.

Condition (3.28) is realized by decomposing χ^ϵ_{L} and χ^ϵ_{R} as

$$
\chi^\epsilon_{L} = \gamma^\epsilon_{L} + \epsilon^2 \beta^\epsilon_{L} , \qquad \chi^\epsilon_{R} = \gamma^\epsilon_{R} + \epsilon^2 \beta^\epsilon_{R} ,
\tag{3.29}
$$

where $\gamma^\epsilon_{L\,m}(z)$ and $\gamma^\epsilon_{R\,m}(z)$ each satisfies the half-space problem (3.6) with boundary data (3.8) and decays exponentially as $z \to \infty$, while $\beta^\epsilon_{L\,m}(z)$ and $\beta^\epsilon_{R\,m}(z)$ each satisfies a half-space problem of the form

$$\mu_m \partial_z \beta^\epsilon_m + \beta^\epsilon_m - \overline{\beta^\epsilon} = S^\epsilon(z)\,, \tag{3.30a}$$

over $(0,\infty)\times\mathcal{M}$ with the homogeneous boundary condition

$$\beta^\epsilon_m(0) = 0\,, \qquad \text{for } m > 0\,, \tag{3.30b}$$

and remains bounded as $z \to \infty$. One obtains $\beta^\epsilon = \beta^\epsilon_{L}$ and $\beta^\epsilon = \beta^\epsilon_{R}$ respectively by setting $S^\epsilon = S^\epsilon_{L}$ an $S^\epsilon = S^\epsilon_{R}$ where

$$S^\epsilon_{L} = -\frac{\sigma^A}{\sigma^T}\overline{\gamma^\epsilon_{L}}\,, \qquad S^\epsilon_{R} = -\frac{\sigma^A}{\sigma^T}\overline{\gamma^\epsilon_{R}}\,. \tag{3.31}$$

Here we continuously extend the function σ^A/σ^T to those values of z that lie outside the domain given by (2.4) by assigning it to take the value at either x_{R} or x_{L} as appropriate.

The existence and uniqueness of γ^ϵ_{L}, γ^ϵ_{R}, β^ϵ_{L}, and β^ϵ_{R} are corollaries of more general theorems on half-space problems stated and proved in Appendix B. Moreover, there it will be shown that γ^ϵ_{L} and γ^ϵ_{R} satisfy the bounds (3.10), and that β^ϵ_{L} and β^ϵ_{R} consequently satisfy the bounds

$$\left\|\beta^\epsilon\right\|_{L^\infty([x_{L},x_{R}]\times\mathcal{M})} \le C_\beta \equiv 44A\sqrt{K}\left\|\frac{\sigma^A}{\sigma^T}\right\|_{L^\infty([x_{L},x_{R}])}\,, \tag{3.32}$$

where K was introduced in (1.8) and A was defined in (2.12). Supposing that such γ^ϵ_{L}, γ^ϵ_{R}, β^ϵ_{L}, and β^ϵ_{R} exist, we shall complete the proof of Theorem 3.2.

With the boundary layer correctors constructed in (3.29), the approximate solution (3.23) satisfies (3.12) because (3.8) of Lemma 3.1 and (3.32) show that

$$\frac{1}{\epsilon^2}\left\|\widetilde{\psi}^\epsilon - \theta^\epsilon + \epsilon\frac{\mu_m}{\sigma^T}\partial_x\theta^\epsilon - \gamma^\epsilon_{L} - \gamma^\epsilon_{R}\right\|_{L^\infty([x_{L},x_{R}]\times\mathcal{M})} \le \frac{2}{3}C^{(2)}_\Theta + \epsilon\,\frac{2}{3}C^{(3)}_\Theta + 2C_\beta\,. \tag{3.33}$$

The transfer error (3.26) becomes

$$T^\epsilon = \epsilon^3\mu_m^2\left(\mu_m^2 - \tfrac{1}{3}\right)\partial_x\left(\frac{1}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\left(\frac{1}{\sigma^T}\partial_x\theta^\epsilon\right)\right)\right) + \epsilon^3\sigma^A\overline{\beta^\epsilon_{L}} + \epsilon^3\sigma^A\overline{\beta^\epsilon_{R}}\,, \tag{3.34}$$

whereby (3.13a) is satisfied because (3.8) and (3.32) show that

$$\frac{1}{\epsilon^3}\left\|\frac{T^\epsilon}{\sigma^A}\right\|_{L^\infty} \le \frac{2}{3}\left\|\frac{\sigma^T}{\sigma^A}\right\|_{L^\infty} C^{(4)}_\Theta + 2C_\beta\,. \tag{3.35}$$

The boundary errors (3.25) become

$$
\begin{aligned}
b^\epsilon_{L\,m} &= -\epsilon^2 \frac{\mu_m^2 - \frac{1}{3}}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right) \\
&\quad + \epsilon^3 \frac{\mu_m \left(\mu_m^2 - \frac{1}{3}\right)}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right)\right) \\
&\quad - \gamma^\epsilon_{R\,-m}(z^\epsilon_R) - \epsilon^2 \beta^\epsilon_{R\,-m}(z^\epsilon_R) \Big|_{x=x_L}, \qquad \text{for } m > 0\,, \\
b^\epsilon_{R\,m} &= -\epsilon^2 \frac{\mu_m^2 - \frac{1}{3}}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right) \\
&\quad - \epsilon^3 \frac{\mu_m \left(\mu_m^2 - \frac{1}{3}\right)}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \left(\frac{1}{\sigma^T} \partial_x \theta^\epsilon \right)\right) \\
&\quad - \gamma^\epsilon_{L\,-m}(z^\epsilon_L) - \epsilon^2 \beta^\epsilon_{L\,-m}(z^\epsilon_L) \Big|_{x=x_R}, \qquad \text{for } m > 0\,,
\end{aligned}
\tag{3.36}
$$

whereby (3.13b) is satisfied because (3.8), (3.10) and (3.32) show that

$$
\frac{1}{\epsilon^2} \left\| b^\epsilon \right\|_{L^\infty} \le \frac{2}{3} C^{(2)}_\Theta + \epsilon \, \frac{2}{3} C^{(3)}_\Theta + \frac{1}{\epsilon^2} 4A\sqrt{K} \exp\left(-\frac{\tau}{2\epsilon} \right) + C_\beta\,,
\tag{3.37}
$$

which is bounded for small ϵ. This completes the proof of Theorem 3.2. //

The following corollary is an easy consequence of Theorem 3.2. It is the analog for the discrete-ordinate equation of Corollary 2.3.

Corollary 3.3: Suppose $\psi^\epsilon_m(x)$ is the solution of the discrete-ordinate equation (1.3) and $\theta^\epsilon(x)$ is the solution of the corresponding diffusion equation (3.1). Then the difference between ψ^ϵ and its diffusion approximation (3.5) is second order in ϵ as $\epsilon \to 0$ in the L^∞ norm away from the boundaries, i.e., for every $[x_{I\,L}, x_{I\,R}] \subset (x_L, x_R)$ one has the estimate

$$
\left\| \psi^\epsilon - \theta^\epsilon + \epsilon \frac{\mu_m}{\sigma^T} \partial_x \theta^\epsilon \right\|_{L^\infty([x_{I\,L}, x_{I\,R}] \times \mathcal{M})} \le \epsilon^2 \, C_I\,,
\tag{3.38}
$$

where $C_I < \infty$ is a constant depending on $[x_{I\,L}, x_{I\,R}]$. Moreover, the difference between ψ^ϵ and θ^ϵ converges to zero in the L^p norm for every $p \in [1, \infty)$ as

$$
\left\| \psi^\epsilon - \theta^\epsilon \right\|_{L^p([x_L, x_R] \times \mathcal{M})} \le \epsilon^{\frac{1}{p}} \, C_p\,.
\tag{3.39}
$$

where $C_p < \infty$ is a positive constant depending on p.

Proof: The definitions of $z_{_L}^\epsilon$ and $z_{_R}^\epsilon$ given by (2.4) and the bounds on $\gamma^\epsilon_{_L\,m}(z)$ and $\gamma^\epsilon_{_R\,m}(z)$ that follow from (3.10b) yield the exponential decay estimates

$$
\begin{aligned}
\Big|\gamma^\epsilon_{_L\,m}\big(z_{_L}^\epsilon(x)\big)\Big| &\le 4A\sqrt{K}\exp\bigg(-\frac{\sigma(x-x_{_L})}{2\epsilon}\bigg)\,,\\
\Big|\gamma^\epsilon_{_L\,m}\big(z_{_R}^\epsilon(x)\big)\Big| &\le 4A\sqrt{K}\exp\bigg(-\frac{\sigma(x_{_R}-x)}{2\epsilon}\bigg)\,,
\end{aligned}
\tag{3.40}
$$

where $\sigma \equiv \inf\big\{\sigma^{T}(x) \,:\, x\in[x_{_L},x_{_R}]\big\} > 0$. Here K was introduced in (1.8) and A was defined in (2.12). These bounds immediately gives the L^∞ estimates

$$
\begin{aligned}
\big\|\gamma^\epsilon_{_L}\big\|_{L^\infty([x_{I\,L},x_{I\,R}]\times\mathcal{M})} &\le 4A\sqrt{K}\exp\bigg(-\frac{\sigma(x_{I\,L}-x_{_L})}{2\epsilon}\bigg)\,,\\
\big\|\gamma^\epsilon_{_R}\big\|_{L^\infty([x_{I\,L},x_{I\,R}]\times\mathcal{M})} &\le 4A\sqrt{K}\exp\bigg(-\frac{\sigma(x_{_R}-x_{I\,R})}{2\epsilon}\bigg)\,,
\end{aligned}
\tag{3.41}
$$

and the L^p estimates

$$
\begin{aligned}
\big\|\gamma^\epsilon_{_L}\big\|_{L^p([x_{_L},x_{_R}]\times\mathcal{M})} &= \bigg(\int_{x_L}^{x_R}\sum_m\Big|\gamma^\epsilon_{_L\,m}\big(z_{_L}^\epsilon(x)\big)\Big|^p\alpha_m\,dx\bigg)^{\frac1p}\\
&\le 4A\sqrt{K}\left(\int_{x_L}^{x_R}2\exp\bigg(-\frac{p\sigma(x-x_{_L})}{2\epsilon}\bigg)\,dx\right)^{\frac1p} \le 4A\sqrt{K}\left(\frac{4\epsilon}{p\sigma}\right)^{\frac1p}\,,\\
\big\|\gamma^\epsilon_{_R}\big\|_{L^p([x_{_L},x_{_R}]\times\mathcal{M})} &\le 4A\sqrt{K}\left(\int_{x_L}^{x_R}2\exp\bigg(-\frac{p\sigma(x_{_R}-x)}{2\epsilon}\bigg)\,dx\right)^{\frac1p} \le 4A\sqrt{K}\left(\frac{4\epsilon}{p\sigma}\right)^{\frac1p}\,.
\end{aligned}
\tag{3.42}
$$

The L^∞ bound (3.38) then follows from (3.41) and (3.10), while the L^p bound (3.39) follows from (3.42), (3.10) and (3.8). //

4. Uniform Convergence of the Discrete-Ordinate Method. We are now ready for the endgame of the strategy outlined in the introduction. Recall that in Figure 1.1 the TNE line represents the error of solutions to the numerical transfer equation (1.3), one estimate of which was given in the introduction by (1.23). We will now apply the triangle inequality on the deviations represented by the DDA, DNE and CDA lines to obtain a second estimate on the error represented by TNE. We will then combine this new estimate with the previous one (1.23) to obtain a uniform estimate on the error of solutions to the numerical transfer equation.

The CDA and DDA lines in Figure 1.1 represent the diffusion approximations for the continuous equation and the discrete-ordinate method. An estimate of the first of these was stated as (2.13) of Corollary 2.3, while an estimate of the second was proved as (3.33) of Corollary 3.3. The bounds so obtained were

$$
\begin{aligned}
\left\| \Psi^\epsilon - \Theta^\epsilon + \epsilon \frac{\mu}{\sigma^T} \partial_x \Theta^\epsilon \right\|_{L^\infty([x_{IL}, x_{IR}] \times [-1,1])} &\le \epsilon^2 C_I \,, \\
\left\| \psi^\epsilon - \theta^\epsilon + \epsilon \frac{\mu_m}{\sigma^T} \partial_x \theta^\epsilon \right\|_{L^\infty([x_{IL}, x_{IR}] \times \mathcal{M})} &\le \epsilon^2 C_I \,.
\end{aligned}
\tag{4.1}
$$

Here C_I is independent of ϵ and the quadrature set, and is the same for both bounds.

The DNE line in Figure 1.1 represents the difference between the interior diffusion approximation associated with the solution θ^ϵ of the discrete diffusion equation (3.1) and that associated with the solution Θ^ϵ of continuous diffusion equation (2.2). This difference can be expressed in terms of $\mathcal{E}^\epsilon \equiv \theta^\epsilon - \Theta^\epsilon$, and is estimated by our first result of this section.

Lemma 4.1: The difference between the discrete and continuous interior diffusion approximations satisfies the estimate

$$
\left\| \mathcal{E}^\epsilon - \epsilon \frac{\mu_m}{\sigma^T} \partial_x \mathcal{E}^\epsilon \right\|_{L^\infty(\Omega_I)} \le C_D(\epsilon, \delta) \,,
\tag{4.2a}
$$

where

$$
C_D(\epsilon, \delta) \equiv \left[1 + \epsilon \left(\frac{1}{\tau} + \frac{3\tau}{2} \left\| \frac{\sigma^A}{\sigma^T} \right\|_{L^\infty} \right) \right] \left(\left| \overline{f} - \overline{F} \right|_\infty + \epsilon \, C_\Theta^{(1)} \left| \lambda - \Lambda \right| \right) ,
\tag{4.2b}
$$

with $C_\Theta^{(1)}$ given by (2.8c), τ given by (2.9), and

$$
\left| \overline{f} - \overline{F} \right|_\infty \equiv \max \left\{ \left| \overline{f_L} - \overline{F_L} \right| , \left| \overline{f_R} - \overline{F_R} \right| \right\} .
\tag{4.2c}
$$

Proof: Upon taking the difference of the discrete diffusion equation (3.1) and the continuous diffusion equation (2.2), the diffusion error, $\mathcal{E}^\epsilon \equiv \theta^\epsilon - \Theta^\epsilon$, is found to satisfy the diffusion equation

$$
-\partial_x \left(\frac{1}{3\sigma^T} \partial_x \mathcal{E}^\epsilon \right) + \sigma^A \mathcal{E}^\epsilon = 0 \,,
\tag{4.3a}
$$

with the boundary conditions

$$
\begin{aligned}
\mathcal{E}^\epsilon - \epsilon\,\frac{\lambda}{\sigma^T}\partial_x\mathcal{E}^\epsilon\Big|_{x=x_L} &= \overline{f_L} - \overline{F_L} + \epsilon\,\frac{\lambda-\Lambda}{\sigma^T}\partial_x\Theta^\epsilon\Big|_{x=x_L}, \\
\mathcal{E}^\epsilon + \epsilon\,\frac{\lambda}{\sigma^T}\partial_x\mathcal{E}^\epsilon\Big|_{x=x_R} &= \overline{f_R} - \overline{F_R} - \epsilon\,\frac{\lambda-\Lambda}{\sigma^T}\partial_x\Theta^\epsilon\Big|_{x=x_R}.
\end{aligned}
\tag{4.3b}
$$

The maximum principle applied to (4.4) yields

$$
\|\mathcal{E}^\epsilon\|_{L^\infty} \le \left|\overline{f} - \overline{F}\right|_\infty + \epsilon\, C_\Theta^{(1)}\,|\lambda - \Lambda|\,. \tag{4.4}
$$

It then follows from the diffusion equation (4.3a) that

$$
\left\|\left(\frac{1}{\sigma^T}\partial_x\right)^2\mathcal{E}^\epsilon\right\|_{L^\infty} \le 3\left\|\frac{\sigma^A}{\sigma^T}\right\|_{L^\infty}\|\mathcal{E}^\epsilon\|_{L^\infty}\,. \tag{4.5}
$$

By applying a standard interpolation argument to (4.4) and (4.5), one obtains the estimate

$$
\left\|\frac{1}{\sigma^T}\partial_x\mathcal{E}^\epsilon\right\|_{L^\infty} \le \left(\frac{1}{\tau} + \frac{3\tau}{2}\left\|\frac{\sigma^A}{\sigma^T}\right\|_{L^\infty}\right)\|\mathcal{E}^\epsilon\|_{L^\infty}\,. \tag{4.6}
$$

Estimate (4.2) now follows from (4.4) and (4.6) by the triangle inequality. //

We now estimate the error of the discrete-ordinate method over $\Omega_I = [x_{IL}, x_{IR}] \times \mathcal{M}$ for any $[x_{IL}, x_{IR}] \subset (x_L, x_R)$ by applying the triangle inequality along the DDA, DNE and CDA lines in Figure 1.1, bounds for which are given by (4.1) and (4.2). We obtain

$$
\begin{aligned}
\|E^\epsilon\|_{L^\infty(\Omega_I)} &\le \left\|\psi^\epsilon - \theta^\epsilon + \epsilon\,\frac{\mu_m}{\sigma^T}\partial_x\theta^\epsilon\right\|_{L^\infty(\Omega_I)} + \left\|\mathcal{E}^\epsilon - \epsilon\,\frac{\mu_m}{\sigma^T}\partial_x\mathcal{E}^\epsilon\right\|_{L^\infty(\Omega_I)} \\
&\quad + \left\|\mathcal{R}\Psi^\epsilon - \Theta^\epsilon + \epsilon\,\frac{\mu_m}{\sigma^T}\partial_x\Theta^\epsilon\right\|_{L^\infty(\Omega_I)} \\
&\le \epsilon^2 2C_I + C_D(\epsilon,\delta)\,.
\end{aligned}
\tag{4.7}
$$

It is clear from (4.2b) that this bound on E^ϵ will be dominated by the $C_D(\epsilon,\delta)$ term as ϵ tends to zero. So the rate at which the discrete-ordinate method will converge in diffusive regimes will be determined by the rate at which $C_D(\epsilon,\delta)$ vanishes as δ goes to zero (uniformily over small ϵ).

Let us suppose for now that for some $\epsilon_o > 0$ we have a uniform estimate

$$
C_D(\epsilon,\delta) \le \delta^d C_u \qquad \text{over } 0 \le \epsilon \le \epsilon_o\,, \tag{4.8}
$$

where $d > 0$ is the order of convergence and $C_u < \infty$. Upon comparing (4.7) and (1.23), we then obtain the error estimate

$$
\begin{aligned}
\|E^\epsilon\|_{L^\infty(\Omega_I)} &\le \min\left\{\epsilon^2 2C_I + \delta^d C_u\,,\ \max\left\{\frac{\delta^q}{\epsilon^2}C_c\,,\ \|f - \mathcal{R}F\|_{l^\infty}\right\}\right\} \\
&\le \max\left\{\|f - \mathcal{R}F\|_{l^\infty}\,,\ \min\left\{\epsilon^2 2C_I + \delta^d C_u\,,\ \frac{\delta^q}{\epsilon^2}C_c\right\}\right\}.
\end{aligned}
\tag{4.9}
$$

The last minimum can be computed by determining the δ dependent ϵ at which the two terms are equal and substituting the result into either term. One finds that

$$
\begin{aligned}
\min\left\{\epsilon^2 2C_{_I} + \delta^d C_u\,,\ \frac{\delta^q}{\epsilon^2}C_c\right\} &= \frac{1}{2}\left[\left(\delta^{2d}C_u^2 + \delta^q 8C_{_I}C_c\right)^{\frac{1}{2}} + \delta^d C_u\right] \\
&\le \delta^d C_u + \delta^{q/2}\sqrt{2C_{_I}C_c}\,.
\end{aligned}
\tag{4.10}
$$

Hence, establishing the uniform convergence of the discrete-ordinate method comes down to obtaining a uniform estimate like (4.8) and a convergence estimate for $\|f - \mathcal{R}F\|_{l^\infty}$.

We will examine the consequences of choosing the discrete boundary data values by collocation, namely, by setting $f_{_L} = \mathcal{R}F_{_L}$ and $f_{_R} = \mathcal{R}F_{_R}$. This is the discrete boundary data that is most commonly used. Because $\|f - \mathcal{R}F\|_{l^\infty}$ is then identically zero, we only have to find the uniform bound (4.8). Furthermore, that problem reduces to questions about how the quadrature sets $\{\mu_m, w_m\}_{m=1}^M$ approximate integrals over $[0,1]$ weighted by $W(\mu)$. More specifically, for various choices of $G(\mu)$ we wish to estimate

$$
\overline{\mathcal{R}G} - \overline{G} = \sum_{m=1}^{M} G(\mu_m)w_m - \int_0^1 G(\mu)W(\mu)\,d\mu\,.
\tag{4.11}
$$

For example, because $\lambda - \Lambda = \overline{\mathcal{R}\mu} - \overline{\mu}$, we see that one will have $\lambda = \Lambda$ if and only if the quadrature set integrates linear functions exactly. One will then have $\left|\overline{\mathcal{R}F} - \overline{F}\right|_\infty = 0$ whenever $F_{_L}$ and $F_{_R}$ are linear in μ, in which case (4.8) holds with $C_u = 0$. As was pointed out in [9] however, it is not the case that $\lambda = \Lambda$ when $\{\mu_m, w_m\}$ is derived from classical quadrature sets. While new quadrature sets were found in [9] that satisfy $\lambda = \Lambda$ as well as (1.2) and (1.4), this was not done for arbitrary M. For the purposes of a convergence study it is therefore not now meaningful to assume that $\lambda = \Lambda$ while $\delta \to 0$.

Let us suppose that the quadrature sets $\{\mu_m, w_m\}$ are such that every nonnegative $G \in \mathcal{X}$ satisfies

$$
\left|\overline{\mathcal{R}G} - \overline{G}\right| \le \delta^d\, C_d\, \|D_{\mathcal{X}}G\|_{L^1}\,,
\tag{4.12}
$$

where $d > 0$ is the order of convergence and $C_d < \infty$. In light of the previous paragraph, the best we can expect is $d \le 1$ when $\{\mu_m, w_m\}$ is derived from classical quadrature sets. From (4.12) it immediately follows that

$$
\begin{aligned}
\left|\overline{f} - \overline{F}\right|_\infty &= \left|\overline{\mathcal{R}F} - \overline{F}\right|_\infty \le \delta^d\, C_d\, \|D_{\mathcal{X}}F\|_{L^1}\,, \\
|\lambda - \Lambda| &= \left|\overline{\mathcal{R}\mu} - \overline{\mu}\right| \quad\ \le \delta^d\, C_d\, \|D_{\mathcal{X}}\mu\|_{L^1}\,,
\end{aligned}
\tag{4.13}
$$

whereby the uniform estimate (4.8) is satified with

$$
C_u = \left[1 + \epsilon_o\left(\frac{1}{\tau} + \frac{3\tau}{2}\left\|\frac{\sigma^A}{\sigma^T}\right\|_{L^\infty}\right)\right]\left(\|D_{\mathcal{X}}F\|_{L^1} + \epsilon_o\, C_\Theta^{(1)}\, \|D_{\mathcal{X}}\mu\|_{L^1}\right) C_d\,.
\tag{4.14}
$$

This completes the proof of the main theorem of this paper.

Theorem 4.2: Consider a sequence of such quadrature sets $\{\mu_m^{(M)}, \alpha_m^{(M)}\}$ parameterized by M for which $\delta^{(M)} \to 0$ as $M \to \infty$, and for which conditions (1.2), (1.4) and (1.8) hold.

Suppose furthermore that spaces $\mathcal{X}$ and $\mathcal{Y}$ exist that satisfy (1.16), (1.17) and (4.12).

For every $Q \in L^\infty([x_L, x_R])$ and F_L, $F_R \in \mathcal{X}$ let $\Psi^\epsilon(x,\mu)$ be the solution of the transfer equation (1.1).

For each member of the family $\{\mu_m^{(M)}, \alpha_m^{(M)}\}$ let $\psi_m^\epsilon(x)$ be the solution of the discrete-ordinate equation (1.3) with boundary data given by $f_L = \mathcal{R}F_L$ and $f_R = \mathcal{R}F_R$.

Then $\psi_m^\epsilon(x)$ converges to $\Psi^\epsilon(x,\mu_m)$ in Ω_I uniformly in ϵ as $\delta \to 0$. The rate of uniform convergence given by (4.10).

Remark: We have shown the existence of families of quadrature sets that satisfy all the above hypotheses save (4.12). If we again choose $D_{\mathcal{X}} = \partial_\mu$, so that $\mathcal{X} = W^{1,1}([0,1])$, then by (4.11) we see that any $G = G(\mu)$ in $\mathcal{X}$ satisfies

$$\overline{\mathcal{R}G} - \overline{G} = \int_0^1 U(\mu)\partial_\mu G(\mu)\, d\mu\,, \tag{4.15a}$$

where $U(\mu)$ is defined by

$$U(\mu) \equiv \int_0^\mu \left(W(\mu') - \sum_{m=1}^{M} w_m \delta(\mu' - \mu_m) \right) d\mu'\,. \tag{4.15b}$$

By the Hölder inequality, one can then obtain the estimate

$$\left|\overline{\mathcal{R}G} - \overline{G}\right| \le \|U\|_{L^\infty} \|\partial_\mu G\|_{L^1}\,, \tag{4.16}$$

whereby one can establish (4.12) for any family of quadrature sets for which one can find $d > 0$ and $C_d < \infty$ such that

$$\|U\|_{L^\infty} \le \delta^d\, C_d\,. \tag{4.17}$$

As $q = 1/(1+s) < 1$ for $\mathcal{Y} = \mathcal{Y}_s$ given by (1.20), by (4.10) we give nothing away if (4.17) can be established for $d = 1/2$.

5. Conclusions. Although the truncation error of the discrete-ordinate method is not uniformly small as the mean free path tends to zero. here we have shown that it converges uniformly to the solution of the transfer equation in diffusive regimes (the limit of vanishing mean free path). This result shows that thick meshes — that is, meshes having sizes of the order of many mean free paths — can be used to discretize diffusive media. The possibility of using the same numerical scheme to simulate both diffusive and nondiffusive media is important for applications where it can help in reducing the task of simulating nonhomogeneous media with components having sharply contrasting cross-sections.

The uniform convergence rests on the fact that the discrete-ordinate method has the correct diffusion limit. The correct diffusion limit requires both the correct interior diffusion limit *and* the correct diffusion boundary conditions. In Section 4 we have shown that the correct diffusion boundary conditions improve the accuracy of the scheme. In particular, this shows that, as indicated asymptotically and numerically in [9], the effect of the extrapolation length condition, as well as the discrete boundary data, should not be neglected.

Estimate (4.10) suggests that the best situation is when $q = 2d$. This situation would essentially be realized by the following. If we could find quadrature sets $\{\mu_m, \alpha_m\}_{m\in\mathcal{M}}$ such that for some integer K with $2 \le K \le M$ we have

$$
\begin{aligned}
&\sum_{m=1}^{M} \mu_m^{2k}\alpha_m = \frac{1}{2k+1}\,, &&\text{for } k = 0,\ldots,K-1\,,\\
&\sum_{m=1}^{M} \mu_m^{k} w_m = \int_0^1 \mu^k W(\mu)\,d\mu\,, &&\text{for } k = 1,\ldots,K\,,
\end{aligned}
\tag{5.1}
$$

then for $F \in W^{2K,1}$ one should be able to establish (1.17) for any $q < 2K$ and (4.12) for $d = K$. This would mean the uniform estimate (4.10) would be $O(\delta^r)$ for any $r < K$. The best one can hope for is that one can take $K = M$. In that case (1) will become $2M$ equations that will completely determine the $2M$ basic unknowns, namely $\{\mu_m, \alpha_m\}_{m=1}^{M}$. Already for $K = M = 2$ this gives a set that is a bit different than the set B_4 found in [9]. The question of the existence of a quadrature set satisfying (5.1) for general $K = M$ is open. Finally, it should be remarked that because $W(\mu)$ is well approximated by polynomials of low degree, the double Gaussian quadrature set comes fairly close to satisfying (5.1) with $K = M$. This observation may account in part for its reported success (cf. references given in [5]).

Appendix A: Regularity and Quadrature Estimates. In this appendix we extend the convergence study of Pitkäranta and Scott [19] to include boundary terms and to reflect the proper ϵ scaling. Our proof is self-contained and differs from that in [19]. We will first estmiate the regularity of Ψ in μ and then obtain the corresponding error estimate for the quadrature rule. It will be clear that the reason why this estimate is weaker than the one suggested by (1.14) is the singularity at $\mu = 0$ in the transfer equation.

The basic regularity result for $\Psi(x, \mu)$ that we will use is the following.

Theorem A.1: Let $s \in (0, 1)$. For any F_L, $F_R \in W^{1,1}([0, 1])$, and $Q \in L^\infty([x_L, x_R])$, the corresponding solution Ψ^ϵ of (1.1) satisfies

$$\text{(A.1)} \qquad \frac{1}{2}\int_{-1}^{1} |\mu|^s |\partial_\mu \Psi^\epsilon(x, \mu)|\, d\mu \leq \|F\|_{W^{1,1}} + \frac{2}{es} \max\left\{ \|F\|_{L^\infty}, \left\| \frac{Q}{\sigma^A} \right\|_{L^\infty} \right\},$$

where the norms of F indicate the maximum of the corresponding norms of F_L and F_R, as defined in (1.22). In particular, one has an estimate of the form (1.16) for $\mathcal{Y} = \mathcal{Y}_s$ as defined in (1.20) and with C_r given by (1.21).

Proof: We first introduce

$$\text{(A.2)} \qquad S^\epsilon \equiv \left[\frac{\sigma^T}{\epsilon} - \epsilon\, \sigma^A \right] \overline{\Psi^\epsilon} + \epsilon\, Q\,,$$

whereby (1.1) becomes

$$\text{(A.3)} \qquad \mu \partial_z \Psi^\epsilon + \Psi^\epsilon = \frac{\epsilon}{\sigma^T} S^\epsilon\,.$$

The integral formulation of (A.3) can be expressed as

$$\text{(A.4)} \qquad \begin{aligned} \Psi(x, \mu) &= \exp\left(- \frac{z_L^\epsilon(x)}{\mu} \right) F_L(\mu) + \frac{1}{\mu} \int_{x_L}^{x} \exp\left(- \frac{z_L^\epsilon(x) - z_L^\epsilon(x_1)}{\mu} \right) S^\epsilon(x_1)\, dx_1\,, \\ \Psi(x, -\mu) &= \exp\left(- \frac{z_R^\epsilon(x)}{\mu} \right) F_R(\mu) + \frac{1}{\mu} \int_{x}^{x_R} \exp\left(- \frac{z_R^\epsilon(x) - z_R^\epsilon(x_1)}{\mu} \right) S^\epsilon(x_1)\, dx_1\,, \end{aligned}$$

for each $\mu \in (0, 1]$, where z_L^ϵ and z_R^ϵ are the stretched variables defined in (2.4).

We split the integral on the left side of (A.1) as $\int_{-1}^{0} + \int_0^1$ and give the estimate for $\int_0^1$, the one for $\int_{-1}^0$ being completely similar. It follows from the first equation in (A.4) that

$$\begin{aligned} \partial_\mu \Psi^\epsilon(x, \mu) = \frac{z_L^\epsilon(x)}{\mu^2} \exp\left(- \frac{z_L^\epsilon(x)}{\mu} \right) F_L(\mu) + \exp\left(- \frac{z_L^\epsilon(x)}{\mu} \right) \partial_\mu F_L(\mu) \\ + \int_{x_L}^{x} \left(\frac{z_L^\epsilon(x) - z_L^\epsilon(x_1)}{\mu^3} - \frac{1}{\mu^2} \right) \exp\left(- \frac{z_L^\epsilon(x) - z_L^\epsilon(x_1)}{\mu} \right) S(x_1)\, dx_1\,. \end{aligned}$$

Therefore

$$
\begin{aligned}
\int_0^1 \mu^s |\partial_\mu \Psi^\epsilon(x,\mu)|\, d\mu \leq{}& \int_0^1 \frac{z_{_L}^\epsilon(x)}{\mu^{2-s}} \exp\Bigl(-\frac{z_{_L}^\epsilon(x)}{\mu}\Bigr)\, d\mu\, \|F_{_L}\|_{L^\infty} \\
&+ \Bigl\| \exp\Bigl(-\frac{z_{_L}^\epsilon(x)}{\mu}\Bigr)\Bigr\|_{L^\infty} \int_0^1 \mu^s |\partial_\mu F_{_L}(\mu)|\, d\mu \\
&+ \int_{x_L}^x K_s\bigl(z_{_L}^\epsilon(x) - z_{_L}^\epsilon(x_1)\bigr)\sigma^T(x_1)\, dx_1 \Bigl\|\frac{S^\epsilon}{\sigma^T}\Bigr\|_{L^\infty},
\end{aligned}
\tag{A.5}
$$

where for $a > 0$ we define

$$
K_s(a) \equiv \int_0^1 \mu^s \Bigl| \frac{a}{\mu^3} - \frac{1}{\mu^2} \Bigr| \exp\Bigl(-\frac{a}{\mu}\Bigr)\, d\mu\,.
$$

By making the change of variables $\nu = a/\mu$ in the integral above, one obtains the equivalent form

$$
K_s(a) = a^{s-1} \int_a^\infty \nu^{-s} |\nu - 1| e^{-\nu}\, d\nu\,.
$$

The factors in (A.5) can then be bounded as

$$
\begin{aligned}
\int_0^1 \frac{z_{_L}^\epsilon(x)}{\mu^{2-s}} \exp\Bigl(-\frac{z_{_L}^\epsilon(x)}{\mu}\Bigr)\, d\mu &\leq \int_0^1 \frac{z_{_L}^\epsilon(x)}{\mu^2} \exp\Bigl(-\frac{z_{_L}^\epsilon(x)}{\mu}\Bigr)\, d\mu = \exp\bigl(-z_{_L}^\epsilon(x)\bigr) \leq 1\,, \\
\Bigl\| \exp\Bigl(-\frac{z_{_L}^\epsilon(x)}{\mu}\Bigr)\Bigr\|_{L^\infty} &= \exp\bigl(-z_{_L}^\epsilon(x)\bigr) \leq 1\,, \\
\int_{x_L}^x K_s\bigl(z_{_L}^\epsilon(x) - z_{_L}^\epsilon(x_1)\bigr)\sigma^T(x_1)\, dx_1 &= \epsilon \int_0^\tau K_s(a)\, da = \epsilon \int_0^\tau a^{s-1} \int_a^\infty \nu^{-s}|\nu-1|e^{-\nu}\, d\nu\, da \\
&= \frac{\epsilon}{s}\Bigl(\tau^s \int_\tau^\infty \nu^{-s}|\nu-1|e^{-\nu}\, d\nu + \int_0^\tau |a-1|e^{-a}\, da\Bigr) \\
&\leq \frac{\epsilon}{s}\int_0^\infty |a-1|e^{-a}\, da = \frac{2\epsilon}{es}\,,
\end{aligned}
$$

where $\tau \equiv z_{_L}^\epsilon(x)$. Moreover, because of the maximum principle, from (A.2) one has

$$
\epsilon \Bigl\|\frac{S^\epsilon}{\sigma^T}\Bigr\|_{L^\infty} \leq \max\Bigl\{ \|F\|_{L^\infty}, \Bigl\|\frac{Q}{\sigma^A}\Bigr\|_{L^\infty} \Bigr\}\,.
$$

When the above bounds and definition (1.19) of $\|F_{_L}\|_{W^{1,1}}$ are combined in (A.5), one finds

$$
\int_0^1 \mu^s |\partial_\mu \Psi^\epsilon(x,\mu)|\, d\mu \leq \|F_{_L}\|_{W^{1,1}} + \frac{2}{es}\max\Bigl\{ \|F\|_{L^\infty}, \Bigl\|\frac{Q}{\sigma^A}\Bigr\|_{L^\infty} \Bigr\}\,.
$$

By averaging this bound with the corresponding one for $\int_{-1}^0$, the conclusion is (A.1). //

As we said above, the regularity estimate provided by Theorem A.1 is not enough to control the error associated to a general quadrature set by a direct application of (1.14). One can however use the elementary interpolation argument given below.

Theorem A.2. Given $\mathcal{Y}_s$ as defined in (1.20) for some $s \in (0,1)$, then for every nonnegative $\Phi \in \mathcal{Y}_s$ and every $\eta \in (0,1)$ one has

$$\left|\overline{\Phi} - \overline{\mathcal{R}\Phi}\right| \leq \eta K \|\Phi\|_{L^\infty} + \frac{\delta}{\eta^s}\frac{1}{2}\int_{-1}^{1} |\mu|^s |\partial_\mu \Phi(\mu)|\, d\mu\,, \tag{A.6}$$

where K was defined in (1.8). In particular, upon setting

$$\eta = \left(\frac{\delta}{K}\right)^{\frac{1}{1+s}},$$

one has estimate (1.17) with $q = 1/(1+s)$ and $C_q = K^{s/(1+s)}$.

Proof: Because Φ is nonnegative, for each $m = 1, \cdots, M$ we have the crude estimate

$$\left| \frac{1}{2}\int_{-\mu_{m+\frac{1}{2}}}^{\mu_{m+\frac{1}{2}}} \Phi(\mu)\, d\mu - \frac{1}{2}\sum_{0<|k|\leq m} \Phi(\mu_k)\, \alpha_k \right| \leq \mu_{m+\frac{1}{2}} \|\Phi\|_{L^\infty} \leq \mu_m K \|\Phi\|_{L^\infty}\,. \tag{A.7}$$

If $\mu_M \leq \eta \leq 1$ then estimate (A.6) follows directly from (A.7) with $m = M$. Next, we observe that for each $m = 0, \cdots, M-1$ we have the identity

$$\frac{1}{2}\int_{\mu_{m+\frac{1}{2}}\leq|\mu|\leq 1} \Phi(\mu)\, d\mu - \frac{1}{2}\sum_{m<|k|\leq M} \Phi(\mu_k)\, \alpha_k = \frac{1}{2}\int_{\mu_{m+\frac{1}{2}}\leq|\mu|\leq 1} R(\mu)\, \partial_\mu \Phi(\mu)\, d\mu\,, \tag{A.8}$$

where $R(\mu)$ is the sawtoothed function that was defined in (1.13b). The right side above satisfies

$$\begin{aligned} \left| \frac{1}{2}\int_{\mu_{m+\frac{1}{2}}\leq|\mu|\leq 1} R(\mu)\, \partial_\mu \Phi(\mu)\, d\mu \right| &\leq \max_{\mu_{m+\frac{1}{2}}\leq\mu\leq 1} \left\{ \frac{R(\mu)}{\mu^s} \right\} \frac{1}{2}\int_{-1}^{1} |\mu|^s |\partial_\mu \Phi(\mu)|\, d\mu \\ &\leq \frac{\delta}{\mu_{m+1}^s}\frac{1}{2}\int_{-1}^{1} |\mu|^s |\partial_\mu \Phi(\mu)|\, d\mu\,. \end{aligned} \tag{A.9}$$

If $0 \leq \eta \leq \mu_1$ then estimate (A.6) follows directly from (A.8) and (A.9) with $m = 0$. Finally, if $\mu_m \leq \eta \leq \mu_{m+1}$ for some $m = 1, ..., M-1$ then estimate (A.6) follows from by combining (A.7) with (A.8) and (A.9). //

Remark: The convergence rate of $\delta^{1/(1+s)}$ obtained through Theorem A.2 is somewhat disappointing in that Gauss quadrature can have an arbitrary order of accuracy. The reason is simply the lack of regularity in Ψ at $\mu = 0$ manifest in (A.1).

Appendix B: Boundary Layer Correctors. As we have seen, the construction of the boundary layer correctors plays a crucial role in the proof of uniform convergence of the discrete-ordinate method in diffusive regimes. The analog of this construction for the transport equation has been considered by many authors with explicit representations of the solutions (cf. [6]). The case of the transport equation with general absorption/scattering cross-sections has been treated by Bensoussan-Lions-Papanicolaou [3] and Bardos-Santos-Sentis [2]. Whereas the convergence proof in [3] relies on stochastic methods, that in [2] is based an energy method. Here a similar energy method is adapted to the discrete-ordinate method.

The existence of the γ_L^ϵ and γ_R^ϵ asserted in Theorem 3.2 is a direct consequence of Theorem B.1 below.

Theorem B.1: For any boundary data $\{g_m : m = 1, \cdots, M\}$, there exists a unique classical solution $\gamma = \gamma_m(z)$ within the space $L^\infty((0,\infty) \times \mathcal{M})$ of the constant coefficient homogeneous discrete-ordinate equation

$$\mu_m \partial_z \gamma_m + \gamma_m - \overline{\gamma} = 0 \,, \tag{B.1a}$$

over $(0,\infty) \times \mathcal{M}$ with the boundary condition

$$\gamma_m(0) = g_m \,, \qquad \text{for } m > 0 \,. \tag{B.1b}$$

This solution satisfies the maximum principle

$$\sup_{z,m} \left\{ |\gamma_m(z)| \right\} \le \sup_m \left\{ |g_m| \right\} . \tag{B.2}$$

Moreover, the solution decays exponentially to the constant value γ^∞ given by

$$\gamma^\infty = \sum_{m=1}^{M} g_m w_m \,, \tag{B.3}$$

where the w_m are defined by (3.2b), and satisfies the pointwise bounds

$$|\overline{\gamma}(z) - \gamma^\infty| \le 2A\sqrt{K} \exp\big(-\tfrac{1}{2} z \big) \,, \qquad \text{over } [0,\infty) \,, \tag{B.4a}$$

$$|\gamma_m(z) - \gamma^\infty| \le \frac{4A\sqrt{K}}{2 - \mu_m} \exp\big(-\tfrac{1}{2} z \big) \,, \qquad \text{over } [0,\infty) \times \mathcal{M} \,, \tag{B.4b}$$

where $K > 1$ was introduced in (1.8) and A is any constant such that

$$\sup_{1 \le m \le M} \left\{ |g_m| \right\} \le A \,. \tag{B.4c}$$

Remark: The solution asserted in the theorem can be given explicitly. Indeed, it can be shown [6,9] that equation (B.1a) has the general bounded solution

$$\gamma_m(z) = a_0 + \sum_{n=1}^{M-1} a_n \frac{\exp\big(-\xi_{n+\frac{1}{2}} z \big)}{1 - \xi_{n+\frac{1}{2}} \mu_m} \,, \tag{B.5}$$

where the $\xi_{n+\frac{1}{2}}$ are determined by (3.3). This fact may be verified by direct substitution upon observing that (3.3) implies

$$\overline{\gamma}(z) = a_0 + \sum_{n=1}^{M-1} a_n \exp\big(-\xi_{n+\frac{1}{2}} z\big)\,. \tag{B.6}$$

Clearly, this solution decays exponentially to the constant value a_0. The a_n are then determined uniquely by the boundary condition (B.1b) to be the solution of the linear system

$$g_m = a_0 + \sum_{n=1}^{M-1} a_n \frac{1}{1-\xi_{n+\frac{1}{2}}\mu_m}\,. \tag{B.7}$$

The above coefficients form a classical Cauchy matrix [21], which allows the Cramer determinants to be evaluated and relatively simple expressions for the a_n to be found [9]. In particular, one finds $a_0 = \gamma^\infty$ as given by (B.3) with the w_m given by (3.2b).

Remark: Solutions of (B.1) in $L^\infty((0,\infty)\times\mathcal{M})$ are formulated within the class of mild solutions, namely, those satisfying

$$\gamma_m(z) = \begin{cases} \exp\Big(-\dfrac{z}{\mu_m}\Big) g_m + \displaystyle\int_0^z \exp\Big(-\frac{z-s}{\mu_m}\Big)\frac{\overline{\gamma}(s)}{\mu_m}\,ds\,, & \text{for } m>0\,;\\ \displaystyle\int_z^\infty \exp\Big(-\frac{s-z}{|\mu_m|}\Big)\frac{\overline{\gamma}(s)}{|\mu_m|}\,ds\,, & \text{for } m<0\,. \end{cases} \tag{B.8}$$

The maximum principle (B.2) asserted in the theorem follows immediately from this formulation using standard arguments and the uniqueness then follows from (B.2).

Remark: As the preceding remarks make clear, the main point of Theorem B.1 is the decay estimates (B.4a) and (B.4b), where the parameters A and K are uniformly bounded over a converging family of quadrature sets. Such estimates are not readily derivable from the explicit representation of the solution (B.5), but they do arise naturally from the energy method used in the proof below. We know that the estimates in (B.4) are not optimal, but they suffice for our purposes.

Proof: The strategy of the proof is to directly estimate the mild formulation (B.8) after subtracting γ^∞ from both sides to obtain

(B.9)

$$\gamma_m(z) - \gamma^\infty = \begin{cases} \exp\Big(-\dfrac{z}{\mu_m}\Big)(g_m-\gamma^\infty) + \displaystyle\int_0^z \exp\Big(-\frac{z-s}{\mu_m}\Big)\frac{\overline{\gamma}(s)-\gamma^\infty}{\mu_m}\,ds\,, & \text{for } m>0\,;\\ \displaystyle\int_z^\infty \exp\Big(-\frac{s-z}{|\mu_m|}\Big)\frac{\overline{\gamma}(s)-\gamma^\infty}{|\mu_m|}\,ds\,, & \text{for } m<0\,. \end{cases}$$

The proof rests on two basic estimates. The first is the elementary uniform bound

$$\big|\gamma_m(z)-\gamma^\infty\big| \le A\,, \tag{B.10}$$

which follows from (B.9) by applications of the triangle and Gronwall inequalities. The second is the exponential decay estimate

$$\overline{\mu^2(\gamma-\gamma^\infty)^2} \leq A^2 \exp(-2z)\,, \tag{B.11}$$

the proof of which will be deferred.

Supposing (B.11) holds, we show that $\overline{\gamma}(z)-\gamma^\infty$ satisfies the exponential decay estimate (B.4a). Let $\eta \in (0,1)$ be an arbitrary cut-off parameter. Applications of (B.10), the Cauchy-Schwarz inequality and (B.11) provide the estimate

$$\begin{aligned}
\left|\overline{\gamma}-\gamma^\infty\right| &\leq \frac{1}{2}\sum_{|\mu_m|<\eta}\alpha_m|\gamma_m-\gamma^\infty| + \frac{1}{2}\sum_{|\mu_m|\geq\eta}\alpha_m|\gamma_m-\gamma^\infty| \\
&\leq \frac{A}{2}\sum_{|\mu_m|<\eta}\alpha_m + \frac{1}{\eta}\overline{|\mu||\gamma-\gamma^\infty|} \\
&\leq \frac{A}{2}\sum_{|\mu_m|<\eta}\alpha_m + \frac{1}{\eta}\Big(\overline{\mu^2(\gamma-\gamma^\infty)^2}\Big)^{\frac{1}{2}} \\
&\leq AK\eta + \frac{1}{\eta}A\exp(-z)\,,
\end{aligned} \tag{B.12}$$

where K was introduced in (1.8). The choice $\eta = \exp(-\frac{1}{2}z)/\sqrt{K}$ optimizes the above estimate, giving (B.4a).

To show that $\gamma_m(z)-\gamma^\infty$ satisfies the exponential decay estimate (B.4b), we use estimate (B.4a) in the mild formulation (B.9). One sees that for $m>0$

$$\begin{aligned}
\left|\gamma_m(z)-\gamma^\infty\right| &\leq A\exp\Big(-\frac{z}{\mu_m}\Big) + \frac{2A\sqrt{K}}{\mu_m}\int_0^z \exp\Big(-\frac{z-s}{\mu_m}\Big)\exp\big(-\tfrac{1}{2}s\big)\,ds \\
&= A\exp\Big(-\frac{z}{\mu_m}\Big) + \frac{4A\sqrt{K}}{2-\mu_m}\Big(\exp\big(-\tfrac{1}{2}z\big) - \exp\Big(-\frac{z}{\mu_m}\Big)\Big) \\
&\leq \frac{4A\sqrt{K}}{2-\mu_m}\exp\big(-\tfrac{1}{2}z\big)\,,
\end{aligned} \tag{B.13}$$

while for $m<0$

$$\begin{aligned}
\left|\gamma_m(z)-\gamma^\infty\right| &\leq \frac{2A\sqrt{K}}{|\mu_m|}\int_z^\infty \exp\Big(-\frac{s-z}{|\mu_m|}\Big)\exp\big(-\tfrac{1}{2}s\big)\,ds \\
&= \frac{4A\sqrt{K}}{2+|\mu_m|}\exp\big(-\tfrac{1}{2}z\big)\,.
\end{aligned} \tag{B.14}$$

Combining (B.13) and (B.14) yields (B.4b). Hence, the theorem will be established upon proving the validity of (B.11).

To prove (B.11), we begin by taking the angular average of equation (B.1a) and of equation (B.1a) multiplied by μ_m to find

$$\partial_z\big(\overline{\mu\gamma}\big) = 0\,, \qquad \partial_z\big(\,\overline{\mu^2\gamma}\,\big) + \overline{\mu\gamma} = 0\,. \tag{B.15}$$

These relations show that $\overline{\mu\gamma}$ must be constant in z while $\overline{\mu^2\gamma}$ must grow linearly like $-\overline{\mu\gamma}z$. However, using the explicit solution (B.5) it follows that

$$\overline{\mu\gamma} = 0\,, \qquad \overline{\mu^2\gamma} = \tfrac{1}{3}\gamma^\infty\,, \tag{B.16}$$

because first $\overline{\mu\gamma}$ and then $\overline{\mu^2\gamma}$ can be evaluated as $z \to \infty$.

Next we show that $\overline{\mu\gamma^2}$ is a nonnegative function that decays both monotonically and exponentially to zero as $z \to \infty$. Multiplying (B.1a) by γ_m and taking the angular average yields

$$\partial_z\big(\tfrac{1}{2}\overline{\mu\gamma^2}\big) + \overline{(\gamma-\overline{\gamma})^2} = 0\,, \tag{B.17}$$

where we have used the identity

$$\overline{\gamma\big(\gamma-\overline{\gamma}\big)} = \overline{\gamma^2} - \overline{\gamma}^2 = \overline{(\gamma-\overline{\gamma})^2}\,. \tag{B.18}$$

Hence, $\overline{\mu\gamma^2}(z)$ is nonincreasing with respect to z. The explicit solution (B.5) shows that $\overline{\mu\gamma^2}(z) \to 0$ as $z \to \infty$, so that $\overline{\mu\gamma^2}$ must be nonnegative too. Using (B.16) it is easy to verify the identity

$$\overline{\mu\gamma^2} = \overline{\mu(\gamma-\overline{\gamma})^2}\,. \tag{B.19}$$

Multiplying (B.17) by $\exp(2z)$ and using the above identity gives

$$\begin{aligned}\partial_z\big(\tfrac{1}{2}\exp(2z)\overline{\mu\gamma^2}\big) &= \exp(2z)\overline{\mu\gamma^2} - \exp(2z)\overline{(\gamma-\overline{\gamma})^2}\\ &= -\exp(2z)\overline{(1-\mu)(\gamma-\overline{\gamma})^2} \le 0\,.\end{aligned} \tag{B.20}$$

Integrating this differential inequality yields the exponential bound

$$\overline{\mu\gamma^2}(z) \le \overline{\mu\gamma^2}(0)\exp(-2z)\,. \tag{B.21}$$

In fact, (B.20) shows that $\exp(2z)\overline{\mu\gamma^2}(z)$ is a nonincreasing function of z.

We now show that $\overline{\mu^2(\gamma-\gamma^\infty)^2}(z)$ decays both monotonically and exponentially to zero as $z \to \infty$. Multiplying (B.1a) by $\mu_m\gamma_m$ and taking the angular average (again using (B.16)) yields

$$\partial_z\big(\overline{\mu^2\gamma^2}\big) + 2\overline{\mu\gamma^2} = 0\,. \tag{B.22}$$

Integrating this relation over the interval (z,∞), the first term can be evaluated as

$$\overline{\mu^2\gamma^2}\Big|_z^\infty = \overline{\mu^2}\gamma^{\infty\,2} - \overline{\mu^2\gamma^2}(z) = -\overline{\mu^2(\gamma-\gamma^\infty)^2}(z)\,, \tag{B.23}$$

while the second may be bound above using (B.21) as

$$\int_z^\infty 2\overline{\mu\gamma^2}(s)\,ds \le \int_z^\infty 2\overline{\mu\gamma^2}(0)\exp(-2s)\,ds = \overline{\mu\gamma^2}(0)\exp(-2z)\,. \tag{B.24}$$

Using (B.16) it is easy to verify the identity

$$\overline{\mu\gamma^2} = \overline{\mu(\gamma-\gamma^\infty)^2}\,. \tag{B.25}$$

Combining (B.22-25) leads to the exponential bound

$$\overline{\mu^2(\gamma-\gamma^\infty)^2}(z) \le \overline{\mu(\gamma-\gamma^\infty)^2}(0)\exp(-2z)\,. \tag{B.26}$$

Finally, estimating the right side of (B.26) with the uniform bound (B.10) leads to the exponential decay estimate (B.11), thereby completing the proof of Theorem B.1. //

The existence and uniqueness asserted in Theorem 3.2 for γ_L^ϵ and γ_L^ϵ respectively follows from that asserted for γ in Theorem B.1 by setting $g = g_L^\epsilon$ and $g = g_L^\epsilon$ as defined by (3.8). The fact that θ^ϵ satisfies the diffusion boundary condition (3.1b) ensures that $\gamma^\infty = 0$ by (B.3), whereby the bounds (3.10) follow directly from (B.4).

The existence and uniqueness asserted in the proof of Theorem 3.2 for β_L^ϵ and β_R^ϵ will be a consequence of the following result that shows the existence of a bounded solution of nonhomogeneous half-space transfer equations.

Theorem B.2: Given $S = S(z)$ that for some $G > 0$ satisfies the uniform bound

$$|S(z)| \le G\exp\big(-\tfrac{1}{2}z\big)\,, \tag{B.27}$$

there exists a unique classical solution $\beta = \beta_m(z)$ within $L^\infty([0,\infty)\times\mathcal{M})$ to the inhomogeneous half-space equation

$$\mu_m\partial_z\beta_m + \beta_m - \overline{\beta} = S\,, \tag{B.28a}$$

over $(0,\infty)\times\mathcal{M}$ with the homogeneous boundary condition

$$\beta_m(0) = 0\,, \qquad \text{for } m > 0\,. \tag{B.28b}$$

Moreover, it satisfies the bound

$$|\beta_m(z)| \le 22G\,, \qquad \text{over } [0,\infty)\times\mathcal{M}\,. \tag{B.29}$$

Proof: The proof uses the maximum principle to bound β with a positive solution υ of

$$\mu_m\partial_z\upsilon_m + \upsilon_m - \overline{\upsilon} = \exp\big(-\tfrac{1}{2}z\big)\,, \tag{B.30}$$

over $(0,\infty)\times\mathcal{M}$. In particular, the solution of (B.30) that we will use is

$$v_m(z) \equiv C_v\left(2 - \frac{2}{2-\mu_m}\exp\left(-\tfrac{1}{2}z\right)\right), \tag{B.31}$$

where the constant C_v is given by

$$C_v \equiv \left(\sum_{m=1}^{M}\frac{\mu_m^2}{4-\mu_m^2}\,\alpha_m\right)^{-1}. \tag{B.32}$$

Below we will prove the existence of a unique solution β of (B.28) within $L^\infty([0,\infty)\times\mathcal{M})$ that satisfies

$$\big|\beta_m(z)\big| \le G v_m(z)\,, \qquad \text{over } [0,\infty)\times\mathcal{M}\,. \tag{B.33}$$

The right side of (B.33) may be bounded above by observing from (B.31) that

$$v_m(z) \le 2C_v\,, \tag{B.34}$$

from (B.32), the Jensen inequality, and (1.2c) that

$$\frac{1}{C_v} = \sum_{m=1}^{M}\frac{\mu_m^2}{4-\mu_m^2}\,\alpha_m \ \ge\ \frac{\displaystyle\sum_{m=1}^{M}\mu_m^2\alpha_m}{4-\displaystyle\sum_{m=1}^{M}\mu_m^2\alpha_m} = \frac{\frac{1}{3}}{4-\frac{1}{3}} = \frac{1}{11}\,. \tag{B.35}$$

Hence, given the solution β of (B.28) satifying (B.33), it follows from (B.34) and (B.35) that the bound (B.29) holds.

All that remains to be done to complete the proof of Theorem B.2 is to construct a solution of (B.28) that satisfies the bound (B.33). The uniqueness of the solution follows from Theorem B.1. The construction is achieved by the following iteration procedure. Define $\beta^{(0)}$ by $\beta_m^{(0)}(z) \equiv -Gv_m(z)$ and $\beta^{(k+1)}$ in terms of $\beta^{(k)}$ by

$$\mu_m\partial_z\beta_m^{(k+1)} + \beta_m^{(k+1)} - \overline{\beta^{(k)}} = S\,, \tag{B.36a}$$

over $(0,\infty)\times\mathcal{M}$ with the homogeneous boundary condition

$$\beta_m^{(k+1)}(0) = 0\,, \qquad \text{for } m>0\,. \tag{B.36b}$$

Recast in its mild formulation, (B.36) becomes

$$\beta_m^{(k+1)}(z) = \begin{cases} \displaystyle\int_0^z \exp\left(-\frac{z-s}{\mu_m}\right)\frac{\overline{\beta^{(k)}}(s)+S(s)}{\mu_m}\,ds\,, & \text{for } m>0\,;\\[2ex] \displaystyle\int_z^\infty \exp\left(-\frac{s-z}{|\mu_m|}\right)\frac{\overline{\beta^{(k)}}(s)+S(s)}{|\mu_m|}\,ds\,, & \text{for } m<0\,. \end{cases} \tag{B.37}$$

It is clear from the maximum principle for (B.37) that

$$-Gv \leq \beta^{(1)} \leq \cdots \leq \beta^{(k)} \leq \beta^{(k+1)} \leq \cdots \leq Gv\,, \tag{B.38}$$

so that, passing to the limit in (B.36) while taking into account the above chain of inequalities proves Theorem B.2. //

The existence and uniqueness asserted in the proof of Theorem 3.2 for $\beta_{_L}^{\epsilon}$ and $\beta_{_R}^{\epsilon}$ respectively follows from that asserted for β in Theorem B.1 by setting $S = S_{_L}^{\epsilon}$ and $S = S_{_R}^{\epsilon}$ as defined in (3.31). That this S satisfies hypothesis (B.27) follows from (3.10a); with this G, bound (B.29) becomes bound (3.32).

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