A LOCAL SENSITIVITY ANALYSIS IN LANDAU DAMPING FOR THE KINETIC KURAMOTO EQUATION WITH RANDOM INPUTS

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ABSTRACT. We present a local sensitivity analysis in Landau damping for the kinetic Kuramoto equation with random inputs. The kinetic Kuramoto equation governs the temporal-phase dynamics of the one-oscillator distribution function for an infinite ensemble of Kuramoto oscillators. When random inputs are absent in the coupling strength and initial data, it is well known that the incoherent state is nonlinearly stable in a subscritical regime where the coupling strength is below the critical coupling strength which is determined by the geometric shape of the distribution function for natural frequency. More precisely, Kuramoto order parameter measuring the fluctuations around the incoherent state tends to zero asymptotically and its decay mode depends on the regularity(smoothness) of natural frequency distribution function and initial datum. This phenomenon is called as Landau damping in the Kuramoto model in analogy with Landau damping arising from plasma physics. Our analytical results show that Landau damping is structurally robust with respect to random inputs at least in subscritical regime. As in the deterministic setting, the decay mode for the derivatives of the order parameter in random space can be either algebraic or exponential depending on the regularities of the initial datum and natural frequency distribution, respectively, and the smoothness for the order parameter in random space is determined by the smoothness of the coupling strength

1. INTRODUCTION

Synchronization of a weakly coupled oscillators is ubiquitous in classical and quantum oscillatory systems, for example, flashing of fireflies, beating of cardiac pacemaker cells, array of Josephson junctions, etc. (see [1, 3, 8, 17, 36, 39, 43, 44]). Recently, due to possible applications in the decentralized control of drones, robots, collective behavior of multi-agent systems has received lots of attention from diverse scientific and engineering disciplines. After Winfree and Kuramoto's seminal works in [35, 44], several models for synchronization were proposed in literature [1, 36]. Among others, our main interest lies on the Kuramoto model and its kinetic model with random inputs [23, 24]. Next, we briefly introduce our governing models with random inputs. Let z be a random input taking a value in $\Omega \subset \mathbb{R}^d$. Since the dimension of random space is irrelevent in the following local sensitivity analysis, for simplicity we assume that it is one-dimensional, although the dimension of the random parameters can cause computational challenges, so called "curse of dimension". Let $\theta_i = \theta_i(t, z)$ be a phase process of the i-th Kuramoto oscillator with

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uncertainty z. Then, the dynamics of the phase process is governed by the following Cauchy problem [23, 35] for the random Kuramoto model (in short RKM):

$$
\begin{cases}
\partial_t \theta_i(t,z) = \nu_i + \dfrac{\kappa(z)}{N}\displaystyle\sum_{j=1}^{N} \sin(\theta_j(t,z) - \theta_i(t,z)), \quad t>0,\ i=1,\cdots,N, \\
\theta_i(0,z) = \theta_i^{in}(z),
\end{cases} \tag{1.1}
$$

where ν_i is the natural (intrinsic) natural frequency of the i-th oscillator whose probability density function is given by $g = g(\nu)$, and $\kappa = \kappa(z)$ is a uniform coupling strength between oscillators with random inputs z. Now we consider a large ensemble with $N \gg 1$. In this case, it is well known from the kinetic theory [1, 30] that the dynamics of the large ensemble can be effectively described by the corresponding mean-field kinetic equation. More precisely, let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and $F = F(t, \theta, \nu, z)$ be a one-oscillator distribution function at phase θ, with the natural frequency ν at time t. Then, the dynamics of F is governed by the Cauchy problem to the random kinetic Kuramoto equation (in short RKKE) [24]:

$$
\begin{cases}
\partial_t F + \partial_\theta(V[F]F) = 0, \qquad (\theta,\nu,z) \in \mathbb{T}\times\mathbb{R}\times\Omega,\ t>0, \\
V[F](t,\theta,\nu,z) = \nu - \kappa(z)\displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta-\theta_*)F(t,\theta_*,\nu_*,z)d\nu_* d\theta_*, \\
F(0,\theta,\nu,z) = F^{in}(\theta,\nu,z), \quad F^{in}(\theta,\nu,z) = F^{in}(\theta+2\pi,\nu,z) \quad \text{and} \\
\displaystyle\int_{\mathbb{T}} F^{in}(\theta,\nu,z)d\theta = g(\nu).
\end{cases} \tag{1.2}
$$

Note that in the absence of random inputs(parameters), global well-posedness and asymptotic dynamic of (1.1) and (1.2) have been extensively studied in literature [7, 9, 12, 13, 14, 15, 16, 18, 19, 20, 25, 30, 31, 32, 33, 41, 42]. Moreover, the local sensitivity analysis for the Kuramoto models (1.1) and (1.2) with random input were also introduced in [23, 24] and the asymptotic dynamics of the z-derivatives of F has been studied in a *sufficiently large coupling regime* $\kappa(z) \gg 1$ where the complete synchronization estimates can be guaranteed for a generic initial data [25].

In this paper, we are interested in the dynamic features of the random kinetic Kuramoto equation (1.2) in a *small coupling regime* such as the random input effect on the nonlinear stability of the coherent solution $F_e = \frac{g(\nu)}{2\pi}$, which corresponds to the uniform distribution on the unit circle. In the absence of random inputs, for a subcritical coupling strength $\kappa < \kappa_c$ (where κ_c is the threshold coupling strength depending on g (see [1])), the order parameter $R(t)$ defined in (1.7) measuring the fluctuations of the incoherent state tends to zero, as t goes to infinity, whereas it becomes unstable for a super-scritical coupling strength $\kappa > \kappa_c$. More precisely, under the assumption that $g = g(\nu)$ is even and non-increasing, Mirollo and Strogatz [33] studied the linear stability of the incoherent solution. Their linear stability results showed that the incoherent solution is unstable for $\kappa > \kappa_c = \frac{2}{\pi g(0)}$, but neutrally stable for $\kappa < \kappa_c$. Later, in [31], they made complete analysis of the spectrum of the linearized evolution equation for the fixed states and proved that the fully locked special positive states are linearly stable, however the partially locked special positive states are only neutrally stable. This phenomenon was pioneered by Strogatz-Mirollo-Mattews [40] and it is coined as Landau damping for the Kuramoto model in analogy with Landau damping [4, 5, 34, 38] in plasma physics, and it becomes a recent hot topic in applied PDE community [6, 11, 18, 14, 26, 27]. Now, we return to the random kinetic Kuramoto equation. For the simplicity of presentation, it is more convenient to work with a conditional probability density function $f = f(t, \theta, \nu, z)$ defined by the relation:

$$F(t, \theta, \nu, z) = f(t, \theta, \nu, z)g(\nu). \tag{1.3}$$

Then, we substitute the ansatz (1.3) into $(1.2)_1$ to get an equation for f:

$$\begin{cases} \partial_t f + \partial_\theta(V[f]f) = 0, \qquad (\theta, \nu, z) \in \mathbb{T} \times \mathbb{R} \times \Omega,\ t > 0, \\ V[f](t, \theta, \nu, z) := \nu - \kappa(z) \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta - \theta_*) f(t, \theta_*, \nu_*, z) g(\nu_*) d\nu_* d\theta_*, \\ f(0, \theta, \nu, z) = f^{in}(\theta, \nu, z), \quad f^{in}(\theta, \nu, z) = f^{in}(\theta + 2\pi, \nu, z), \quad \displaystyle\int_{\mathbb{T}} f^{in}(\theta, \nu, z) d\theta = 1. \end{cases} \tag{1.4}$$

Since we are mainly interested in the asymptotic stability of $f_e := \frac{1}{2\pi}$, we introduce a perturbation ρ:

$$f(t, \theta, \nu, z) := \frac{1}{2\pi} + \rho(t, \theta, \nu, z), \quad \forall\ (\theta, \nu, z) \in \mathbb{T} \times \mathbb{R} \times \Omega. \tag{1.5}$$

Again, we substitute the ansatz (1.5) into (1.4) to obtain the equation for ρ:

$$\begin{cases} \partial_t \rho + \partial_\theta \Big[V[\rho] \Big(\dfrac{1}{2\pi} + \rho \Big) \Big] = 0, \qquad (\theta, \nu, z) \in \mathbb{T} \times \mathbb{R} \times \Omega,\ t > 0, \\ V[\rho](t, \theta, \nu, z) = \nu - \kappa(z) \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta - \theta_*) \rho(t, \theta_*, \nu_*, z) g(\nu_*) d\nu_* d\theta_*, \\ \rho(0, \theta, \nu, z) = \rho^{in}(\theta, \nu, z), \quad \displaystyle\int_{\mathbb{T}} \rho^{in}(\theta, \nu, z) d\theta = 0. \end{cases} \tag{1.6}$$

For the local sensitivity analysis for the Landau damping in a small coupling regime, we introduce local order parameters with random inputs as quantities of interest(QoI) (see [37]):

$$R(t, z) = \int_{\mathbb{T}^1\times\mathbb{R}} e^{\mathrm{i}\theta} f(t, \theta, \nu, z) g(\nu) d\theta d\nu, \tag{1.7}$$

where f is a solution to (1.6).

For a brief introduction to the local sensitivity analysis, we refer to survey papers by the third author [28, 29], and some related works [2, 10, 21, 22] on the synthesis of the collective dynamics and uncertainty quantification(UQ). Up to now, the existing literature for the Kuramoto models deals with mostly deterministic case except [23, 24]. However, as one can easily conceive, the presence of uncertainties are unavoidable in the realistic modeling. Uncertainties can result from diverse sources, e.g., measuring errors of the domain, data and transport coefficients etc. Therefore, it is important to see the sensitivity of QoI with respect to the random input parameters, and to study the uncertainty propagation and how it affects the evolution of the solution asymptotically, etc. This kind of uncertainly quantification (UQ) analysis will help to improve and calibrate the physical models. In this paper, we consider the uncertainty from the initial data $\rho^{in}(\theta, \nu, z)$ and the coupling strength $\kappa(z)$.

The main novelty of this paper is two-fold. First, we consider the asymptotic decay of z-variations of $R(t, z)$ in (1.7) as $t \to \infty$ in a smooth framework where the coupling strength, the natural frequency distribution and initial datum are sufficiently smooth in θ, ν and z-variables. More precisely, if the coupling strength κ is in $\mathcal{C}^M$, and natural frequency distribution g, initial datum ρ^{in} are in $\mathcal{C}^n$ with $n \geq 4$, then the z-variations of R decay to zero *algebraically* fast with the same decay rate t^{-n} (see Theorem 2.2):

$$|\partial_z^m R(t,z)| = \mathcal{O}(t^{-n}), \quad \text{as } t \to \infty \text{ for each } z \in \Omega \text{ and } 0 \leq m \leq M,$$

where M is the highest regularity of the coupling strength in z-variable. As a direct application of the first result, we also derive a scattering type estimate, which means that the ρ along the particle path tends to some stationary profile in lower Sobolev norm H^{n-2} (see Corollary 2.2).

Second, we revisit a local sensitivity analysis for the random kinetic Kuramoto equation in analytical framework where the coupling strength, natural frequency distribution and initial datum are *analytic*. In this case, we show that the z-variations of R decay to zero *exponentially* fast (see Theorem 2.4): there exists a positive constant Λ such that

$$|\partial_z^m R(t,z)| = \mathcal{O}(e^{-\Lambda t}), \quad \text{as } t \to \infty \text{ for each } z \in \Omega \text{ and and } 0 \leq m \leq M,$$

where M is the highest differentiability of the coupling strength. We also obtain a scattering type estimate as well.

The rest of this paper is organized as follows. In Section 2, we briefly discuss the basic properties of the RKKE, and discuss two frameworks. In each framework, we briefly summarize our main results on the asymptotic decay of $\partial_z^m R$ in two different frameworks (*smooth* and *analytic* frameworks). In Section 3, we present a local sensitivity analysis for the RKKE in a smooth framework where the natural frequency distribution and initial data are sufficiently smooth. In Section 4, we also present a local sensitivity analysis for the RKKE in an analytical framework where the natural frequency distribution and initial data are analytical. Finally Section 5 is devoted to a brief summary of our main results and discussion on remaining issues in connection with UQ for the RKKE.

2. Frameworks and main results

In this section, we briefly present two main frameworks. In each framework, we review the previous results for the deterministic case, and state our main local sensitivity estimate without proofs. The detailed proofs will be given in the subsequent sections.

Before we discuss main frameworks, we first study a conservation law and order parameter measuring the degree of collective behavior of the Kuramoto ensemble.

Lemma 2.1. *For a given $T \in (0,\infty]$, let $F = F(t,\theta,\nu,z)$ be a 2π-periodic in θ, nonnegative classical solution to* (1.2) *in the time-interval $[0,T)$ with a nonnegative initial datum F^{in}:*

$$\int_{\mathbb{T}} F^{in}(\theta,\nu,z)d\theta = g(\nu),$$

where $(\theta,\nu,z) \in \mathbb{T}\times\mathbb{R}\times\Omega$. Then, we have

$$\int_{\mathbb{T}} F(t,\theta,\nu,z)d\theta = g(\nu), \quad (\theta,\nu,z) \in \mathbb{T}\times\mathbb{R}\times\Omega, \ t \geq 0.$$

Proof. We use the 2π-periodicity of F in the θ-variable and (1.2) to get

$$\partial_t \int_{\mathbb{T}} F(t,\theta,\nu,z)d\theta = \int_{\mathbb{T}} \partial_t F(t,\theta,\nu,z)d\theta = -\int_{\mathbb{T}} \partial_\theta(\tilde{V}[F]F)(t,\theta,\nu,z)d\theta = 0,$$

which yields the desired result. $\square$

Remark 2.1. *Note that since*

$$F(t,\theta,\nu,z)=\frac{g(\nu)}{2\pi}+\rho(t,\theta,\nu,z)g(\nu), \tag{2.1}$$

Lemma 2.1 and $(1.6)_3$ *yield*

$$\int_{\mathbb{T}}\rho(t,\theta,\nu,z)d\theta=\int_{\mathbb{T}}\rho^{in}(\theta,\nu,z)d\theta=0.$$

Next, we introduce an order parameter measuring the collective behaviors of an infinite Kuramoto ensemble.

Definition 2.1. *Let F be a solution to (1.2). Then, the complex order parameter R is defined by the following relation: for $(t,z)\in\mathbb{R}_+\times\Omega$,*

$$R(t,z):=\int_{\mathbb{T}\times\mathbb{R}}e^{\mathrm{i}\theta_*}F(t,\theta_*,\nu_*,z)d\nu_*d\theta_*. \tag{2.2}$$

Remark 2.2. *1. For a Kuramoto ensemble $\{\theta_i\}_{i=1}^N$ with finite size, the complex order parameter r is also defined by the relation:*

$$r(t,z)=\frac{1}{N}\sum_{i=1}^N e^{\mathrm{i}\theta_i(t,z)}.$$

2. We subsitute the ansatz (2.1) into (2.2) to get

$$R(t,z)=\int_{\mathbb{T}\times\mathbb{R}}e^{\mathrm{i}\theta_*}\rho(t,\theta_*,\nu_*,z)g(\nu_*)d\nu_*d\theta_*. \tag{2.3}$$

3. The nonlocal velocity $V[\rho]$ in (1.6) can be rewritten in terms of $|R|$ and average phase ϕ i.e., ($R=|R|e^{\mathrm{i}\phi}$). For this, we first divide the relation (2.3) by $e^{\mathrm{i}\theta}$ and take an imaginary part to get

$$|R(t,z)|\sin(\phi(t,z)-\theta)=\int_{\mathbb{T}\times\mathbb{R}}\sin(\theta_*-\theta)\rho(t,\theta_*,\nu_*,z)g(\nu_*,z)d\nu_*d\theta_*.$$

Hence, the nonlocal velocity $V[\rho]$ becomes

$$V[\rho](t,\theta,\nu,z)=\nu(z)-\kappa(z)|R(t,z)|\sin(\theta-\phi(t,z)).$$

Next, we discuss two frameworks for our local sensitivity analysis.

2.1. **A smooth framework.** Before we depict our smooth $\mathcal{C}^n$-framework, we recall the Fourier transform and the Sobolev norm.

Definition 2.2. *(1) Let $u=u(x)$ be a real-valued L^1-function defined on $\mathbb{R}$. Then, its Fourier transform is defined as follows.*

$$\hat{u}(\tau):=\int_{\mathbb{R}}u(x)e^{-\mathrm{i}\tau x}dx,\quad \textit{for } \tau\in\mathbb{R}.$$

Moreover, if $u=u(x)\in\mathcal{C}^n(\mathbb{R})$, its weighted Sobolev norm $||\cdot||_{H^n}$ is defined as follows.

$$||u||_{H^n}^2:=\sum_{k=0}^n||\langle x\rangle u^{(k)}||_{L^2(\mathbb{R})}^2,$$

where $\langle x\rangle:=\sqrt{1+|x|^2}$ and $u^{(k)}$ denotes the k-th derivative of u.

(2) Let $u = u(\theta, \nu)$ be an L^1-function defined on the infinite cylinder $\mathbb{T}^1 \times \mathbb{R}$. Then, its Fourier transform is defined as follows.

$$\hat{u}_k(\tau) := \int_{\mathbb{T}^1\times\mathbb{R}} u(\theta,\nu)e^{-\mathrm{i}(\theta,\nu)\cdot(k,\tau)}d\theta d\nu, \quad \text{for } (k,\tau)\in\mathbb{Z}\times\mathbb{R}. \tag{2.4}$$

Moreover, if $u \in \mathcal{C}^n(\mathbb{T}^1 \times \mathbb{R})$, its weighted Sobolev norm $||\cdot||_{H^n}$ is also defined by

$$||u||_{H^n}^2 := \sum_{k_\theta,k_\nu\geqslant 0,\, k_\theta+k_\nu\leqslant n} ||\langle\nu\rangle\partial_\theta^{k_\theta}\partial_\nu^{k_\nu}u||^2_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{2.5}$$

2.1.1. *The deterministic setting.* In this part, we review previous results in [18] on the Landau damping for the deterministic kinetic Kuramoto equation:

$$\begin{cases} \partial_t\rho + \partial_\theta\Big(V[\rho](\rho+\dfrac{1}{2\pi})\Big) = 0, \quad (\theta,\nu)\in\mathbb{T}\times\mathbb{R},\ t>0, \\ V[\rho](t,\theta,\nu) = \nu - \kappa\displaystyle\int_{\mathbb{T}\times\mathbb{R}}\sin(\theta-\theta_*)\rho(t,\theta_*,\nu_*)g(\nu_*)d\nu_*d\theta_*, \\ \rho(0,\theta,\nu) = \rho^{in}(\theta,\nu), \quad \displaystyle\int_{\mathbb{T}}\rho^{in}(\theta,\nu)d\theta = 0. \end{cases} \tag{2.6}$$

For a smooth initial datum $\rho^{in} \in \mathcal{C}^n(\mathbb{T}^1\times\mathbb{R})$, it is easy to show that Cauchy problem (2.6) has a unique global solution $\rho \in \mathcal{C}^n(\mathbb{R}^+\times\mathbb{T}^1\times\mathbb{R})$ by the standard method of characteristics. Now, we discuss the Landau damping for (2.6), which corresponds to the decay of the order parameter R.

For this, we set

$$\Pi^- := \{\xi = x + iy : x\in\mathbb{R},\ y\in\mathbb{R}^-\}$$

to be the lower half complex plane consisting of complex numbers with negative imaginary part. Then, for any L^1-function $f = f(t)$,

$$\int_{\mathbb{R}^+} f(t)e^{-\mathrm{i}\xi t}dt \quad \text{is finite for every } \xi\in\Pi^-.$$

Now, we are ready to recall a Landau damping type result in a smooth framework.

Theorem 2.1. [18] *Suppose that the distribution function $g = g(\nu)$ and its Fourier transform satisfy the following conditions: for some $n \geqslant 4$,*

$$g\in\mathcal{C}^n(\mathbb{R}), \quad ||g||_{H^n}<\infty, \quad \hat{g}\in L^1(\mathbb{R}_+) \quad \text{and} \quad \int_{\mathbb{R}^+}\tau^n|\hat{g}(\tau)|d\tau<\infty. \tag{2.7}$$

Then, for every $\kappa \geqslant 0$ such that

$$1-\frac{\kappa}{2}\int_{\mathbb{R}^+}\hat{g}(\tau)e^{-\mathrm{i}\xi\tau}dt\neq 0, \quad \forall\ \xi\in\Pi^-, \tag{2.8}$$

there exists $\varepsilon_\kappa > 0$ such that for any initial datum ρ^{in}:

$$\rho^{in}\in\mathcal{C}^n(\mathbb{T}^1\times\mathbb{R}) \quad \text{and} \quad ||\rho^{in}g||_{H^n}\leqslant\varepsilon_\kappa,$$

then the order parameter decays to zero algebraically fast:

$$R(t) = \mathcal{O}(t^{-n}), \quad \text{as } t\to\infty.$$

Remark 2.3. *The decay rate n of R is equal to the maximal regularity of the distribution function g.*

As a direct application of Theorem 2.1, one has the following scattering type result for ρ.

Corollary 2.1. [18] *Suppose that* (2.7) *and* (2.8) *in Theorem 2.1 hold, and initial datum* ρ^{in} *satisfies*

$$||\rho^{in} \cdot g||_{H^n} < \varepsilon_\kappa.$$

Then, there exists a function ρ_∞ *defined on the cylinder with* $||\rho_\infty \cdot g||_{H^{n-2}} < \infty$ *such that*

$$\rho(t, \theta + t\nu, \nu)g(\nu) \to \rho_\infty(\theta, \nu)g \quad \text{in } H^{n-2} \text{ as } t \to \infty.$$

Remark 2.4. *Note that condition* (2.8) *is a stability criterion which was discussed in Section 3.2 of* [18].

For a later use in Section 3, we state some estimates, without proof, on the solution of the integral equation:

$$U(t) = S(t) + \int_0^t G(t-s)U(s)ds, \quad \text{for all } t \in \mathbb{R}^+, \tag{2.9}$$

where G is a complex-valued kernel.

Proposition 2.1. [18] *For* $n \in \mathbb{N}$, *let* $G \in (L^1 \cap L^\infty)(\mathbb{R}^+)$ *be a complex-valued function satisfying*

$$\int_{\mathbb{R}^+} t^4 |G(t)|^2 dt < \infty, \quad \int_{\mathbb{R}^+} t^n |G(t)| dt < \infty \quad \text{and} \quad D_G\Big|_{\Pi^-} \neq 0,$$

where

$$D_G := 1 - \int_{\mathbb{R}^+} G(t) e^{-\mathrm{i}\xi t} dt, \quad \text{for all } \xi \in \mathbb{R} \cup \Pi^-.$$

Then, there exists a positive constant $C_{n,G} \in \mathbb{R}^+$ *such that for every complex-valued function* F *defined on* $\mathbb{R}^+$, *the solution of the Volterra equation* (2.9) *satisfies the following estimate: for any* $T > 0$,

$$\sup_{0 \le t \le T} (1+t)^n |U(t)| \leqslant C_{n,G} \sup_{0 \le t \le T} (1+t)^n |S(t)|, \quad \text{for all } t \in \mathbb{R}^+.$$

Proof. A proof can be found in Proposition 4.1 of [18]. □

2.1.2. *The random uncertainty setting.* In this subsection, we consider the UQ setting, namely, random parameters z are incorporated into the initial data and the coupling strength:

$$\rho^{in} = \rho^{in}(\theta, \nu, z), \quad \kappa = \kappa(z).$$

In this setting, ρ is governed by the random kinetic Kuramoto equation:

$$\begin{cases} \partial_t \rho + \partial_\theta \Big[V[\rho] \Big(\rho + \dfrac{1}{2\pi} \Big) \Big] = 0, \quad (\theta, \nu, z) \in \mathbb{T} \times \mathbb{R} \times \Omega, \ t > 0, \\ V[\rho](t, \theta, \nu, z) = \nu - \kappa(z) \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta - \theta_*) \rho(t, \theta_*, \nu_*, z) g(\nu_*) d\nu_* d\theta_*, \\ \rho(0, \theta, \nu, z) = \rho^{in}(\theta, \nu, z), \quad \displaystyle\int_{\mathbb{T}} \rho^{in}(\theta, \nu, z) d\theta = 0. \end{cases} \tag{2.10}$$

Our first result says that the deterministic Landau damping is robust with respect to random inputs z.

Theorem 2.2. *Let $n \geq 4$ and $M \geq 1$ be positive integers, and for each $z \in \Omega$, we assume that κ and g satisfy conditions* (2.7) *and* (2.8)*, and the extra conditions:*

$$
\begin{aligned}
&(i)\ \max_{0\leqslant m\leqslant M} |\partial_z^m \kappa(z)| \leq C, \quad \text{for all } 0 \leqslant m \leqslant M. \\
&(ii)\ g \in \mathcal{C}^n(\mathbb{R}), \quad ||g||_{H^n} < \infty, \quad \widehat{g} \in L^1(\mathbb{R}^+), \quad \int_{\mathbb{R}^+} \tau^n |\widehat{g}(\tau)| d\tau < \infty.
\end{aligned} \tag{2.11}
$$

Then, there exists $\varepsilon_C > 0$ such that for any initial datum $\rho^{in} \in \mathcal{C}^n(\mathbb{T}^1 \times \mathbb{R} \times \Omega)$ such that $||\partial_z^m(\rho^{in} g)||_{H^n} \leqslant \varepsilon_C$ for all $0 \leqslant m \leqslant M$, the order parameter associated with (2.10) *satisfies*

$$
|\partial_z^m R(t,z)| = \mathcal{O}(t^{-n}), \quad \text{as } t \to \infty \quad \text{for all } 0 \leqslant m \leqslant M \text{ and } z \in \Omega.
$$

Remark 2.5. *Note that n is the maximal regularities of g and ρ^{in}, whereas M is the maximal regularity of κ in z.*

As a direct application of Theorem 2.2, we have the following asymptotic behavior of ρ.

Corollary 2.2. *Under the same conditions* (2.11) *of Theorem 2.2, for any initial datum ρ^{in} with $||\partial_z^m(\rho^{in} g)||_{H^n} \leqslant \varepsilon_C$ for all $0 \leqslant m \leqslant M$, there exists a stationary profile $\rho_\infty = \rho_\infty(\theta,\nu,z)$ in the cylinder with $||\partial_z^m(\rho_\infty g)||_{H^{n-2}} < \infty$ for all $0 \leqslant m \leqslant M$ such that the solution $\rho(t)$ to equation* (2.10) *satisfies a scattering type result:*

$$
\rho(t,\theta+t\nu,\nu,z)g(\nu) \quad \to \quad \rho_\infty(\theta,\nu,z))g(\nu) \quad \text{in } H^{n-2}\text{-norm} \qquad \text{as } t \to \infty.
$$

Remark 2.6. *1. Note that if we have higher regularity of z in initial datum and natural frequency, then we can see that the same regularity of z propagates in the solution and its order parameter's derivative in z-variable will also decay to zero algebraically with the same decay rate as in the deterministic case.*

2. In the proofs of Theorem 2.2 and Corolllary 2.2 in next section, we will see that, when $|\partial_z^m \kappa(z)|$ becomes small (i.e., upper bound C becomes small), ε_C can become large, and it turns out if we fix ε and we can find $\kappa(z)$ and corresponding bound C_ε to make the results still valid.

2.2. **The analytic framework.** In this part, we consider the kinetic Kuramoto equation in the analytic framework. As before, we begin with the notations:

$$
\langle t \rangle := (1+t^2)^{\frac{1}{2}} \quad \text{and} \quad \langle k,\tau \rangle := (1+k^2+\tau^2)^{\frac{1}{2}}.
$$

Note that $\langle \cdot \rangle$ satisfies the triangular inequality:

$$
\langle k_1+k_2, \tau_1+\tau_2 \rangle \leqslant \langle k_1,\tau_1 \rangle + \langle k_2,\tau_2 \rangle.
$$

For an analytic function u defined on $\mathbb{T}^1 \times \mathbb{R}$, its Fourier transform can be defined as in Definition 2.2. Next, we introduce a weight and a norm:

$$
A_k^{\lambda,p}(\tau) := e^{\lambda\langle k,\tau\rangle}\langle k,\tau\rangle^p, \quad ||u||_{\lambda,p} := \sup_{k\in\mathbb{Z},\tau\in\mathbb{R}} A_k^{\lambda,p}(\tau)|\widehat{u}_k(\tau)|, \quad \mathcal{X}_{\lambda,p} := \{u : \ ||u||_{\lambda,p} < \infty\}.
$$

Then, the space $\mathcal{X}_{\lambda,p}$ is a complete metric space. We also define a quantity to control the real order parameter $|R(t)|$:

$$
R_{\lambda,p}(t) := e^{\lambda t}\langle t \rangle^p |R(t)|. \tag{2.12}
$$

Finally, we define an important norm to be frequently used in the sequel.

Let $\lambda_0 > 0$ and a be positive constants such that

$$0 < a < \frac{2\lambda_0}{\pi},$$

and for $t \geqslant 0$ and $\lambda < \lambda_0$, we define the weight β_a and three analytic norms with $\gamma \geqslant 3$:

$$\begin{aligned} &\beta_a(t,\lambda) := \lambda_0 - \lambda - a \arctan t, \qquad |||h|||_{a,p} := \sup_{\lambda, t: \beta_a(t,\lambda) > 0} \beta_a^{1/2}(t,\lambda) ||h||_{\lambda,p}, \\ &|||h|||_a := |||h|||_{a,1} + |||h(\cdot)/\langle \cdot \rangle|||_{a,\gamma}, \qquad ||||R||||_a := \sup_{\lambda, t: \beta_a(t,\lambda) > 0} R_{\lambda,\gamma}(t), \end{aligned} \tag{2.13}$$

where the first two norms in (2.13) apply to functions of three variable $u = u(t, \theta, \nu)$ that are analytic with $(\theta, \nu) \in \mathbb{T}^1 \times \mathbb{R}$ and continuous with $t \in \mathbb{R}^+$. The third norm is applied to $R(t) \in C(\mathbb{R}^+; \mathbb{C})$, which is mainly used to control the order parameter $|R(t)|$. We denote the Banach spaces generated by the norms $|||h|||_{a,p} < \infty$ and $|||h|||_a < \infty$ by $\mathcal{B}_{a,p}$ and $\mathcal{B}_a$, respectively.

2.2.1. *The deterministic setting.* In this subsection, we review the previous results in [6] to be extended to the uncertain setting. In next theorem, we recall the previous result relevant to our second main result in analytical framework.

Theorem 2.3. [6] *Suppose that for $\lambda_0 > 0$ and $\gamma \geqslant 3$, the initial data and coupling strength satisfy*

$$||\rho^{in} g||_{\lambda_0,\gamma} < \infty \quad \textit{and} \quad 0 < \kappa \ll 1,$$

and let ρ be a solution to (2.6) with initial datum ρ^{in}. Then, there exits a positive constant C such that

(i) $|||\rho(t, \theta + t\nu, \nu) g(\nu)|||_a < C$ *and* $|||R|||_a < C$.
(ii) $R(t) \to 0$ *exponentially fast.*
(iii) $\exists\ \rho_\infty(\theta, \nu)$ *satisfying* $||\rho_\infty g||_{\bar\lambda,\gamma} < \infty$, *for some $\bar\lambda > 0$, such that* $\rho(t, \theta + t\nu, \nu) \to \rho_\infty(\theta, \nu)$ *exponentially fast.*

Remark 2.7. *Note that the main difference of this theorem compared with the smooth case is that the initial data and natural frequency are analytic so that we can get exponential decay and convergence.*

2.2.2. *The random setting.* Next, we return to our kinetic Kuramoto equation with random inputs (2.10).

Theorem 2.4. *Let $M \geq 1$ be a positive integer, and suppose that for $\lambda_0 > 0$ and $\gamma \geqslant 3$, the initial datum, natural frequency distribution and the coupling strength satisfy*

$$||\partial_z^m (\rho^{in} g)||_{\lambda_0,\gamma} < \infty \quad \textit{and} \quad |\partial_z^m \kappa(z)| \leqslant C \ll 1 \quad \textit{for all } 0 \leqslant m \leqslant M, \tag{2.14}$$

and let $\rho = \rho(t, \theta, \nu, z)$ be a solution to (2.10) with initial datum ρ^{in}. Then, for $0 \leqslant m \leqslant M$, there exists a positive constant $\overline{C}$ such that

(i) $|||\partial_z^m(\rho(t, \theta + t\nu, \nu, z) g(\nu))|||_a < \overline{C}$ *and* $|||\partial_z^m R|||_a < \overline{C}$.
(ii) $|\partial_z^m R(t)| \to 0$ *exponentially fast.*
(iii) $\exists\ \rho_\infty(\theta, \nu, z)$ *such that* $||\partial_z^m(\rho_\infty g)||_{\bar\lambda,\gamma} < \infty$, *for some $\bar\lambda > 0$ such that* $\partial_z^m \rho(t, \theta + t\nu, \nu, z) \to \partial_z^m \rho_\infty(\theta, \nu, z)$ *exponentially fast.*

In the following two sections, we will provide proofs for Theorem 2.2 and Theorem 2.4 respectively.

3. A local sensitivity analysis in the smooth framework

In this section, we provide a proof of Theorem 2.2 on the local sensitivity analysis for Landau damping in a smooth framework.

3.1. An integral equation for the scaled order parameter.

In this subsection, we derive a Volterra integral equation (2.9) for the rescaled order parameter, and using this integral equation, we find a upper bound estimate for $\sup_{0\le t\le T}(1+t)^n|R(t)|$ using Proposition 3.1.

From Theorem 2.2, we need the condition that $||\partial_z^m(\rho^{in}g)||_{H^n} \leqslant \varepsilon_C$, but we want to normalize this and see ε in the equation. Therefore, we first introduce rescaled quantities (u, U): For a given $\varepsilon > 0$,

$$u^\varepsilon(t,\theta,\nu,z) := \frac{\rho(t,\theta,\nu,z)}{\varepsilon}, \qquad U^\varepsilon(t,z) := \frac{R(t,z)}{\varepsilon}.$$

In the sequel, as long as there is no confusion, we suppress ε dependence in u^ε and U^ε:

$$u = u^\varepsilon, \quad U = U^\varepsilon.$$

Then, it follows from (1.5) and (2.2) that or all $t \in \mathbb{R}^+$,

$$\begin{aligned} f(t,\theta,\nu,z) &= \frac{1}{2\pi} + \varepsilon u(t,\theta,\nu,z), \\ |U(t,z)|e^{\mathrm{i}\phi(t,z)} &= \int_{\mathbb{T}^1\times\mathbb{R}} e^{\mathrm{i}\theta} u(t,\theta,\nu,z)g(\nu)d\theta d\nu. \end{aligned}$$

From now on, we call $U(t,z)$ as the rescaled (Kuramoto) order parameter. Recall that our immediate goal is to show that the quantity $\partial_z^m U(t,z)$ decays to zero algebraically fast for $0 \leqslant m \leqslant M$. For this, we introduce a rescaled Lagrangian density evaluated along the free path:

$$p(t,\theta,\nu,z) := u(t,\theta+t\nu,\nu,z)g(\nu).$$

Now, substitute the above ansatz into (1.4) to derive an equation for p: for all $(\theta,\nu,z) \in \mathbb{T}^1\times\mathbb{R}\times\Omega$ and $t > 0$,

$$\begin{cases} \partial_t p(t,\theta,\nu,z) + \varepsilon\partial_\theta p(t,\theta,\nu,z)W[p](\theta+t\nu,z) \\ \qquad\qquad + \left(\dfrac{g(\nu)}{2\pi} + \varepsilon p(t,\theta,\nu,z)\right)\partial_\theta W[p](\theta+t\nu,z) = 0, \\ W[p](t,\theta,z) = \kappa(z)\displaystyle\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta'+t\nu-\theta)p(t,\theta',\nu,z)d\theta' d\nu. \end{cases} \tag{3.1}$$

Then, it is easy to see from (2.4) and $\int_{\mathbb{T}^1}\rho(t,\theta,\nu,z)d\theta = 0$ that

$$U(t,z) = \widehat{p}_1(t,t,z) \quad \text{and} \quad \widehat{p}_0(t,\tau,z) = 0.$$

Multiplying both sides of (3.1) by $e^{k\theta+\tau\nu}$ and replacing $\sin(\theta'+t\nu-\theta)$ by $e^{\theta'+t\nu-\theta}$, then doing integration, one gets

$$\begin{cases} \partial_t\widehat{p}_k(t,\tau,z) + \dfrac{k\kappa(z)}{2}\Big[\overline{U(t,z)}(\widehat{g}(\tau+t)\delta_{k,-1} + \varepsilon\widehat{p}_{k+1}(t,\tau+t,z)) \\ \quad -U(t,z)(\widehat{g}(\tau-t)\delta_{k,1} + \varepsilon\widehat{p}_{k-1}(t,\tau-t,z))\Big] = 0, \ (k,\tau,z) \in \mathbb{Z}\times\mathbb{R}\times\Omega, \ t > 0, \\ \widehat{p}_k(0,\tau,z) = \widehat{(u(0)g)}_k(\tau,z) = \dfrac{\widehat{f(0)}_k(\tau,z)}{\varepsilon}, \end{cases} \tag{3.2}$$

where $\delta_{k,j}$ is the Kronecker delta function. (See formula (4.2) in [18] for detailed derivation of (3.2)).

Let $k=1$ and $\tau=t$, we can get a desired equation for U:

$$\begin{cases} U(t,z) - \dfrac{\kappa(z)}{2}\displaystyle\int_0^t \widehat{g}(t-s)U(s,z)ds = F(t,z), \quad \forall t\in\mathbb{R}^+, \\ F(t,z) = \widehat{p}_1(0,t,z) - \dfrac{\varepsilon\kappa(z)}{2}\displaystyle\int_0^t \widehat{p}_2(s,t+s,z)\overline{U(s,z)}ds. \end{cases} \tag{3.3}$$

If we regard the term $F(t,z)$ as an autonomous input signal, equation (3.3) appears to be a Volterra equation of the second kind, and one can apply Proposition 2.1 on it if $\widehat{g}$ satisfies the conditions in Theorem 2.1, which is the main method [18] used to prove Theorem 2.1.

Remark 3.1. *It follows from Corollary 4.2 in* [18] *that, if g and κ satisfy the conditions of Theorem 2.1, then $\frac{\kappa}{2}\hat{g}(\tau)$ satisfies the condition of Proposition 2.1.*

3.2. **Proof of Theorem 2.2.** In this subsection, we first present a key ingredient without proof and then by using this ingredient, we provide a proof of Theorem 2.2. First, we apply a differential operator ∂_z^m to both sides of equations (3.1) and (3.3) to get

$$\begin{aligned} &\partial_t\partial_z^m p(t,\theta,\nu,z) + \frac{g(\nu)}{2\pi}\partial_\theta\partial_z^m W[p](t,\theta+t\nu,z) \\ &\quad + \varepsilon\sum_{j=0}^m \binom{m}{j}\Big[\partial_\theta\partial_z^j p(t,\theta,\nu,z)\partial_z^{m-j}W[p](t,\theta+t\nu,z) \\ &\quad + \partial_z^j p(t,\theta,\nu,z)\partial_\theta\partial_z^{m-j}W[p](t,\theta+t\nu,z)\Big] = 0, \end{aligned} \tag{3.4}$$

and

$$\begin{cases} \partial_z^m U(t,z) - \displaystyle\sum_{m'=0}^m \binom{m}{m'}\frac{\partial_z^{m-m'}\kappa(z)}{2}\int_0^t \widehat{g}(t-s)\partial_z^{m'}U(s,z)ds = \partial_z^m F(t,z), \\ \partial_z^m F(t,z) = \partial_z^m\widehat{p}_1(0,t,z) \\ \qquad - \varepsilon\displaystyle\sum_{m'=0}^m \binom{m}{m'}\frac{\partial_z^{m-m'}\kappa(z)}{2}\int_0^t\sum_{j=0}^{m'}\binom{m'}{j}\partial_z^j\widehat{p}_2(s,t+s,z)\partial_z^{m'-j}\overline{U(s,z)}ds. \end{cases} \tag{3.5}$$

As in [18], given $n\in\mathbb{N}$, a solution $\partial_z^m p$ of equation (3.4) and $T>0$, we consider the quantity $Q_{n,T}^{(m)}(p)$ for $0\leqslant m\leqslant M$:

$$\begin{aligned} &Q_{n,T}^{(m)}(p,z) \\ &:= \max\left\{\sup_{t\in[0,T]}(1+t)^n|\partial_z^m U(t,z)|, \sup_{t\in[0,T]}\frac{||\partial_z^m p(t,z)||_{H^n}}{1+t}, \sup_{t\in[0,T]}||\partial_z^m p(t,z)||_{H^{n-2}}\right\}. \end{aligned} \tag{3.6}$$

Next, we provide a key ingredient for the proof of Theorem 2.2.

Proposition 3.1. *Let $n\geq 4$ and $M\geq 1$ be positive integers, and suppose that $\kappa=\kappa(z)$ and $g=g(\nu)$ satisfy the following conditions* (2.11). *Then there exists $Q^*>0$, and for every $Q\geqslant Q^*$, there exists $\varepsilon_{C,Q}$ such that for every $\varepsilon\in(0,\varepsilon_{C,Q})$, if $p=p(t,\theta,\nu,z)$ is a solution to* (3.1) *with initial datum $p^{in}(z)$ satisfying*

$$||\partial_z^m p^{in}(z)||_{H^n}\leqslant 1 \quad \textit{for all } 0\leqslant m\leqslant M,$$

and for every $T > 0$,

$$Q^{(m)}_{n,T}(p) \leqslant Q, \quad \textit{for all } 0 \leqslant m \leqslant M,$$

Then, we have

$$Q^{(m)}_{n,T}(p) \leqslant \frac{Q}{2} \quad \textit{for all } 0 \leqslant m \leqslant M.$$

Proof. Since the proof is quite lengthy, we complete it in the following two subsections for the cases of $M = 1$ and $M > 1$ respectively. □

Proof of Theorem 2.2: In the sequel, we provide a proof of our first main result using Proposition 3.1. First note that initial condition

$$||\partial^m_z p^{in}(z)||_{H^n} = ||\partial^m_z \left(u^{in}(z)\right) \cdot g||_{H^n} \leqslant 1,$$

Then, by (2.5) and (3.1) we can estimate

$$|\partial^m_z U(0,z)| \leqslant \int_{\mathbb{T}\times\mathbb{R}} |\partial^m_z p^{in}(z) e^{-i\theta}| d\theta d\nu \leqslant || \langle v \rangle \, \partial^m_z p^{in}(z)||_{L^2(\mathbb{T}\times\mathbb{R})} || \, \langle v \rangle^{-1} \, ||_{L^2(\mathbb{T}\times\mathbb{R})} \leqslant \pi\sqrt{2}.$$

This implies

$$Q^{(m)}_{n,0}(p) \leqslant \max\{|\partial^m_z U(0,z)|, 1\} \leqslant \pi\sqrt{2}, \quad \text{for all } 0 \leqslant m \leqslant M,$$

where the second inequality follows from Cauchy-Schwarz inequality. Let $Q > \max\{Q^*, \pi\sqrt{2}\}$ and for any $\varepsilon \in (0, \varepsilon_{C,Q})$ and $z \in \Omega$, by the standard existence theorem $\partial^m_z p(t,\theta,\nu,z)$ is continuous as a function of $t \to C^n(\mathbb{T}^1 \times \mathbb{R})$. Then $Q^{(m)}_{n,T}(p)$ is continuous in T. By the initial condition $Q^{(m)}_{n,0}(p) \leqslant Q$, we use Proposition 3.1 to get

$$Q^{(m)}_{n,0}(p) \leqslant \frac{Q}{2} < Q.$$

As a result, we can conclude for all $T > 0$,

$$Q^{(m)}_{n,T}(p) \leqslant \frac{Q}{2}.$$

If not, we can define

$$T^* := \sup_{T>0}\{Q^{(m)}_{n,T}(p) < Q\}. \tag{3.7}$$

If $T^* < \infty$, then by continuity, we have $Q^{(m)}_{n,T^*}(p) \leqslant Q$. Thus, it follows from Proposition 3.1 that

$$Q^{(m)}_{n,T^*}(p) \leqslant \frac{Q}{2} < Q.$$

Then again by continuity, there must exists a $T^{**} > T^*$ such that $Q^{(m)}_{n,T^{**}}(p) < Q$, which contradicts to definition of T^* in (3.7). This implies that for any $\rho^{in}(\theta,\nu,z)$ and $g(\nu)$ satisfying

$$||\partial^m_z(\rho^{in}(z) \cdot g(\nu))||_{H^n} = ||\partial^m_z p^{in}(z)||_{H^n} \leqslant \varepsilon \quad \text{for all } 0 \leqslant m \leqslant M,$$

we have

$$(1+t)^n |\partial^m_z R(t,z)| = \varepsilon(1+t)^n |\partial^m_z U(t,z)| < \varepsilon Q \quad \text{for all } t > 0 \text{ and } 0 \leqslant m \leqslant M.$$

□

Remark 3.2. *We can see in this proposition, Q^* is uniform. In the following proof, we like to choose different Q, and it turns out that if we can prove that proposition is true for M, then Q_M, ε_M can be used to make some conditions and conclusion automatically satisfied for $m \leqslant M$ in the case of $M+1$, we only need to estimate $M+1$ term by induction.*

3.3. **Proof of Proposition 3.1 for $M = 1$.** In this subsection, we provide a proof of Proposition 3.1 with $M = 1$ following the presentation of Proposition 5.1 in [18] which corresponds to the case $M = m = 0$. We use a similar method to prove that assertion in Proposition 3.1 holds for $M = 1$, and then by induction we can prove it for all $M > 0$. In the remaining part of this subsection, the constant C denotes the constant from Theorem 2.2. Now, we estimate $\sup_{t\in[0,T]}(1+t)^n|\partial_z U(t)|$ via several technical lemmas.

Lemma 3.1. *Let $n \geq 4$ be a positive integer, and suppose that $\kappa = \kappa(z)$ and $g = g(\nu)$ satisfy conditions (2.11) with $M = 1$. Then, there exists $Q^* > 0$, and for every $Q \geqslant Q^*$, there exists $\varepsilon_{1,C,Q}$ such that for every $\varepsilon \in (0, \varepsilon_{1,C,Q})$, if $p = p(t,\theta,\nu,z)$ is a solution to (3.1) with initial datum $p^{in}(z)$ satisfying*

$$||\partial_z^m p^{in}(z)||_{H^n} \leqslant 1 \quad \textit{for all } 0 \leqslant m \leqslant 1,$$

and for every $T > 0$,

$$Q^{(m)}_{n,T}(p) \leqslant Q, \quad \textit{for all } 0 \leqslant m \leqslant 1. \tag{3.8}$$

Then, we have

$$\sup_{t\in[0,T]} (1+t)^n|\partial_z^m U(t,z)| \leqslant \frac{Q}{2} \quad \textit{for all } 0 \leqslant m \leqslant 1. \tag{3.9}$$

Proof. To estimate $|U(t,z)|$, we first deal with the deterministic case and relate $||p(s)||_{H^j}$ to $\tau^j|\widehat{p}_k(s,\tau)|$. By definition of $\widehat{p}_k(s,\tau)$, we can see that $\tau^j\widehat{p}_k(s,\tau) = (-\mathrm{i})^j\widehat{\partial_\nu^j p_k}(s,\tau)$, which is defined by

$$\widehat{\partial_\nu^j p_k}(s,\tau) = \int_{\mathbb{T}^1\times\mathbb{R}} \partial_\nu^j p(s,\theta,\nu)e^{-i(k\theta+\tau\nu)}d\theta d\nu,\ \forall (k,\tau) \in \mathbb{Z}\times\mathbb{R}.$$

We use definition of H^n-norm and the Cauchy-Schwarz inequality to see

$$\begin{aligned}\int_{\mathbb{T}^1\times\mathbb{R}} |\partial_\nu^j p e^{-i(k\theta+\tau\nu)}|d\theta d\nu &\leqslant \int_{\mathbb{T}^1\times\mathbb{R}} \langle\nu\rangle^{-1}\langle\nu\rangle|\partial_\nu^j p|d\theta d\nu \\ &\leqslant \left(\int_{\mathbb{T}^1\times\mathbb{R}} \langle\nu\rangle^{-2}d\theta d\nu\right)^{1/2}\left(\int_{\mathbb{T}^1\times\mathbb{R}} |\langle\nu\rangle\partial_\nu^j p|^2 d\theta d\nu\right)^{1/2} \leqslant \sqrt{2}\pi||p||_{H^j}.\end{aligned}$$

Therefore, we can easily get the following estimate: for all $k \in \mathbb{Z}$ and $s,\tau \in \mathbb{R}^+$,

$$\tau^j|\widehat{p}_k(s,\tau)| \leqslant \sqrt{2}\pi||p(s)||_{H^j}, \quad \sup_{\tau\in\mathbb{R}^+}(1+\tau)^n|\widehat{p}_k(s,\tau)| \leqslant 2^n\sqrt{2}\pi||p(s)||_{H^j}.$$

For the uncertain case, similarly, we have

$$\begin{aligned}&\tau^j|\partial_z\widehat{p}_k(s,\tau,z)| \leqslant \sqrt{2}\pi||\partial_z p(s,z)||_{H^j}, \\ &\sup_{\tau\in\mathbb{R}^+}(1+\tau)^n|\partial_z\widehat{p}_k(s,\tau,z)| \leqslant 2^n\sqrt{2}\pi||\partial_z p(s,z)||_{H^j}.\end{aligned} \tag{3.10}$$

It follows from Proposition 5.1 [18] that we can find Q_0, ε_0 such that for every $\varepsilon \in (0, \varepsilon_0)$ and every initial condition $p^{in}(z)$ satisfying $||p^{in}(z)||_{H^n} \leqslant 1$, we have

$$Q_{n,T}(p) \leqslant \frac{Q_0}{2}. \tag{3.11}$$

Now we will see how to choose Q_1 and ε_1 to obtain the result. Suppose that Q and ε satisfy the condition:

$$Q \geqslant Q_0, \quad \varepsilon < \varepsilon_0 \quad \text{and} \quad Q^{(1)}_{n,T}(p) \leqslant Q.$$

Consider equation (3.5) with $m = 1$:

$$\begin{cases} \partial_z U(t,z) - \dfrac{\kappa(z)}{2} \displaystyle\int_0^t \widehat{g}(t-s)\partial_z U(s,z)ds = F_1(t,z) + F_2(t,z), \\ F_1(t,z) = \partial_z \widehat{p}_1(0,t,z) \\ \qquad\qquad -\dfrac{\varepsilon\kappa(z)}{2} \displaystyle\int_0^t \Big(\partial_z \widehat{p}_2(s,t+s,z)\overline{U(s,z)} + \widehat{p}_2(s,t+s,z)\partial_z \overline{U(s,z)}\Big)ds, \\ F_2(t,z) = \dfrac{\partial_z\kappa(z)}{2} \displaystyle\int_0^t \widehat{g}(t-s)U(s,z)ds - \varepsilon\dfrac{\partial_z\kappa(z)}{2} \displaystyle\int_0^t \widehat{p}_2(s,t+s,z)\overline{U(s,z)}ds. \end{cases} \tag{3.12}$$

By Proposition 2.1 and Remark 3.1, we have

$$\sup_{t\in[0,T]} (1+t)^n |\partial_z U(t,z)| \leqslant D_{n,C} \sup_{t\in[0,T]} (1+t)^n \Big| F_1(t,z) + F_2(t,z)\Big|, \tag{3.13}$$

where $D_{n,C}$ is a constant depending on n, C from Theorem 2.2. Next, we estimate the R.H.S. of (3.13).

• (First term of R.H.S. of (3.13)): To deal with the term involving $|F_1(t,z)|$, we use (3.12) to consider the two terms separately:

$$|(1+t)^n \partial_z \widehat{p}_1^{in}(t,z)| \leqslant 2^n\sqrt{2}\pi ||\partial_z p^{in}(z)||_{H^n} \leqslant 2^n\sqrt{2}\pi. \tag{3.14}$$

By conditions of Lemma 3.1 and (3.11), we have

$$\begin{aligned} (1+t)^n \int_0^t |\partial_z \widehat{p}_2(s,t+s,z)\overline{U(s,z)}|ds &\leqslant \frac{Q_0}{2}(1+t)^n \int_0^t \frac{|\partial_z \widehat{p}_2(s,t+s,z)|}{(1+s)^n}ds \\ &\leqslant \frac{2^n\sqrt{2}\pi Q_0}{2}(1+t)^n \int_0^t \frac{||\partial_z p(s,z)||_{H^n}}{(1+t+s)^n(1+s)^n}ds \leqslant \frac{C'Q_0Q}{2}\int_0^t \frac{1}{(1+s)^{n-1}}ds \\ &\leqslant \frac{C''Q_0Q}{2(n-2)}, \end{aligned} \tag{3.15}$$

where the second inequality is due to (3.10) and first and third equalities are due to definition of Q_0 and (3.8)). Similar estimate can be applied to obtain

$$(1+t)^n \int_0^t |\widehat{p}_2(s,t+s,z)\partial_z \overline{U(s,z)}|ds \leqslant \frac{C''Q_0Q}{2(n-2)}. \tag{3.16}$$

• (Second term of R.H.S. of (3.13)): To estimate the term involving $|F_2(t,z)|$ in (3.12), we use almost the same method as in (3.15) and (3.16) to get

$$\begin{aligned}
&\int_0^t |\widehat{g}(t-s)U(s,z)|ds \leqslant C'Q_0 \int_0^\infty |(1+s)^n \widehat{g}(s)|ds \leqslant C''Q_0, \\
&\int_0^t |\widehat{p}_2(s,t+s,z)\overline{U(s,z)}|ds \leqslant \frac{C''Q_0^2}{2(n-2)}.
\end{aligned} \tag{3.17}$$

Finally, we combine all estimates (3.14), (3.15), (3.16), (3.17) and use (3.13)

$$|\partial_z^m \kappa(z)| \leqslant C \quad \text{for } 0 \leqslant m \leqslant 1$$

to obtain

$$\sup_{t\in[0,T]} (1+t)^n |\partial_z U(t,z)| \leqslant D'_{n,C}\Big(2^n\sqrt{2}\pi + \varepsilon\frac{Q_0Q+Q_0^2}{(n-2)} + Q_0\Big), \tag{3.18}$$

where $D'_{n,C}$ is a constant depending on n, C from Theorem 2.2 and (3.13). Therefore, it is obvious that one can first make Q larger such that

$$\sup_{t\in[0,T]} (1+t)^n |\partial_z U(t,z)| \leqslant D'_{n,C}\varepsilon\frac{2Q_0Q}{(n-2)},$$

and then choose ε enough small to make it less than $\frac{Q}{2}$.

Define

$$Q_1 := \max\{Q_0, Q\} \quad \text{and} \quad \varepsilon_{1,C,Q} := \min\{\varepsilon_0, \varepsilon\},$$

with Q and ε satisfying the above condition. Then we can get that for every $\varepsilon' \in (0, \varepsilon_{1,C,Q})$ and every initial condition $p^{in}(z)$ satisfying

$$||\partial_z^m p^{in}(z)||_{H^n} \leqslant 1 \quad \text{for } m = 0, 1,$$

one has

$$Q_{n,T}(p) \leqslant \frac{Q_0}{2} < \frac{Q_1}{2}.$$

This implies the condition (3.8) and the further estimation (3.9) for $m = 0$ are automatically satisfied. By the above estimate, if we further have $Q^{(1)}_{n,T}(p) \leqslant Q_1$, we can get

$$\sup_{t\in[0,T]} (1+t)^n |\partial_z U(t,z)| \leqslant \frac{Q_1}{2}.$$

□

Remark 3.3. *By looking carefully into Lemma 3.1 and its proof, one sees that there is no need to pick two different Q_0, Q_1, but it is more convenient and clear to make them different. Actually, it's not hard to see that the different case is equivalent to the identical case, that's why in Proposition 3.1, we have only one Q^*.*

Lemma 3.2. *Given $l \geqslant 1$, there exists a constant $C_l' > 0$ such that for all $t > 0$*

$$\begin{aligned} \frac{d||\partial_z p||_{H^l}}{dt} &\leqslant C_l' \Big[\sum_{m=0}^{1} \sum_{m'=0}^{m} \binom{1}{m} \binom{m}{m'} |\partial_z^{m'} U(t,z)||\partial_z^{m-m'}\kappa(z)| \\ &\times \Big((1+t)^l (||g||_{H^l}\delta_{m,1} + \varepsilon ||\partial_z^{1-m} p||_{H^0}) + \varepsilon(1+t) \sum_{j=1}^{l} t^{l-j} ||\partial_z^{1-m} p||_{H^j} \Big) \Big]. \end{aligned} \tag{3.19}$$

Proof. Since the proof is similar to that of Lemma 5.3 in [18], we omit some details here. By definition of $|| \cdot ||_{H^l}$, we need to control

$$\frac{d}{dt} ||\langle \nu \rangle \partial_\theta^{k_\theta} \partial_\nu^{k_\nu} \partial_z p||_{L^2}^2 = 2 \int_{\mathbb{T}^1 \times \mathbb{R}} \langle \nu \rangle^2 (\partial_t \partial_\theta^{k_\theta} \partial_\nu^{k_\nu} \partial_z p)(\partial_\theta^{k_\theta} \partial_\nu^{k_\nu} \partial_z p) d\theta d\nu. \tag{3.20}$$

Now, we apply $\partial_\theta^{k_\theta} \partial_\nu^{k_\nu}$ for equation (3.4) with $m = 1$. Then, we need to consider three terms:

$$\begin{aligned} &\partial_\theta^{k_\theta} \partial_\nu^{k_\nu} \left(\frac{g(\nu)}{2\pi} \partial_\theta \partial_z^1 W[p](t, \theta + t\nu, z) \right), \\ &\sum_{j=0}^{1} \binom{1}{j} \partial_\theta^{k_\theta} \partial_\nu^{k_\nu} (\partial_z^j p(\theta, \nu, z) \partial_\theta \partial_z^{1-j} W[p](t, \theta + t\nu, z))), \\ &\sum_{j=0}^{1} \binom{1}{j} \partial_\theta^{k_\theta} \partial_\nu^{k_\nu} (\partial_\theta \partial_z^j p(t, \theta, \nu, z) \partial_z^{1-j} W[p](t, \theta + t\nu, z)). \end{aligned} \tag{3.21}$$

We use definition of $W[p](t, \theta, z)$ (3.1) to find

$$\begin{aligned} &\partial_\theta^{j_\theta} \partial_\nu^{j_\nu} \partial_z^J W[p](t, \theta + t\nu, z) \\ &= \sum_{j}^{J} \binom{J}{j} \frac{-\mathrm{i}\partial_z^{J-j}\kappa(z) t^{j_\nu}}{2} \left((-\mathrm{i})^{j_\theta + j_\nu} e^{-\mathrm{i}(\theta + t\nu)} \overline{\partial_z^j U(t,z)} - (\mathrm{i})^{j_\theta + j_\nu} e^{\mathrm{i}(\theta + t\nu)} \partial_z^j U(t,z) \right). \end{aligned} \tag{3.22}$$

• (Estimate of the first term in (3.21)): We substitute this into the first term with ∂_z on W to obtain

$$\begin{aligned} \mathcal{I}_{11} &= \int_{\mathbb{T}^1 \times \mathbb{R}} \langle \nu \rangle^2 \partial_\nu^{k_\nu} \Big(\frac{g(\nu)}{\pi} \partial_\theta^{k_\theta + 1} \partial_z W[p](t, \theta + t\nu, z) \Big) \partial_\nu^{k_\nu} \partial_\theta^{k_\theta} \partial_z p d\theta d\nu \\ &= \frac{1}{\pi} \int_{\mathbb{T}^1 \times \mathbb{R}} \langle \nu \rangle^2 \partial_\nu^{k_\nu} \partial_\theta^{k_\theta} \partial_z p \Big(\sum_{j_\nu = 0}^{k_\nu} \binom{k_\nu}{j_\nu} \Big[\sum_{j=0}^{1} \frac{-\mathrm{i}\partial_z^{1-j}\kappa(z) t^{j_\nu}}{2} \\ &\times \Big((-\mathrm{i})^{k_\theta + j_\nu} e^{-\mathrm{i}(\theta + t\nu)} \overline{\partial_z^j U(t,z)} - (\mathrm{i})^{k_\theta + j_\nu} e^{i(\theta + t\nu)} \partial_z^j U(t,z) \Big) \Big] \partial_\nu^{k_\nu - j_\nu} g \Big) d\theta d\nu. \end{aligned}$$

We use

$$\Big| \Big((-\mathrm{i})^{k_\theta + j_\nu} e^{-i(\theta + t\nu)} \overline{\partial_z^j U(t,z)} - (\mathrm{i})^{k_\theta + j_\nu} e^{i(\theta + t\nu)} \partial_z^j U(t,z) \Big) \Big| \leqslant 2 |\partial_z^j U(t,z)|,$$

and Cauchy Schwarz inequality to estimate

$$
\begin{aligned}
|\mathcal{I}_{11}| &\leqslant C' \sum_{j=0}^{1} \frac{|\partial_z^{1-j}\kappa(z)|}{\pi} |\partial_z^j U(t,z)| ||\langle\nu\rangle \partial_\theta^{k_\theta}\partial_\nu^{k_\nu}\partial_z p||_{L^2} \sum_{j_\nu=0}^{k_\nu} \binom{k_\nu}{j_\nu} t^{j_\nu} ||\langle\nu\rangle\partial_\nu^{k_\nu-j_\nu} g||_{L^2} \\
&\leqslant C''(1+t)^l \sum_{j=0}^{1} |\partial_z^{1-j}\kappa(z)| \cdot |\partial_z^j U(t,z)| \cdot ||g||_{H^l} \cdot ||\partial_z p||_{H^l}.
\end{aligned} \tag{3.23}
$$

• (Estimate of the second and third terms in (3.21)): Then, we use a similar argument as in the first term and proof of Lemma 5.3 in [18] to obtain the upper bound estimate of the second and third terms:

(3.24)

$$
\begin{aligned}
|\mathcal{I}_{12}| &= \left| \int_{\mathbb{T}^1\times\mathbb{R}} \langle\nu\rangle^2 \sum_{j=0}^{1}\binom{1}{j} \partial_\theta^{k_\theta}\partial_\nu^{k_\nu}\left(\partial_z^j p(\theta,\nu,z)\partial_\theta\partial_z^{1-j} W[p](t,\theta+t\nu,z)\right)\partial_\nu^{k_\nu}\partial_\theta^{k_\theta}\partial_z p d\theta d\nu \right| \\
&\leqslant C'||\partial_z p||_{H^l} \sum_{j=0}^{1}\sum_{j'=0}^{1-j} \binom{1}{j}\binom{1-j}{j'} |\partial_z^{j'} U(t,z)||\partial_z^{1-j-j'}\kappa(z)| \\
&\times \left(||\partial_z^j p||_{H^l} + \sum_{k=1}^{l} t^k ||\partial_z^j p||_{H^{l-k+1}} \right),
\end{aligned}
$$

and

(3.25)

$$
\begin{aligned}
|\mathcal{I}_{13}| &= \left| \int_{\mathbb{T}^1\times\mathbb{R}} \langle\nu\rangle^2 \sum_{j=0}^{1}\binom{1}{j} \partial_\theta^{k_\theta}\partial_\nu^{k_\nu}(\partial_\theta\partial_z^j p(\theta,\nu,z)\partial_z^{1-j} W[p](t,\theta+t\nu,z))\partial_\nu^{k_\nu}\partial_\theta^{k_\theta}\partial_z p d\theta d\nu \right| \\
&\leqslant C'||\partial_z p||_{H^l} \sum_{j=0}^{1}\sum_{j'=0}^{1-j} \binom{1}{j}\binom{1-j}{j'} |\partial_z^{j'} U(t,z)||\partial_z^{1-j-j'}\kappa(z)| \sum_{k=1}^{l} t^k ||\partial_z^j p||_{H^{l-k}}.
\end{aligned}
$$

Finally, combining estimates (3.23), (3.24), (3.25) and the relation

$$
\frac{d||\partial_z p||_{H^l}^2}{dt} = 2||\partial_z p||_{H^l}\frac{d||\partial_z p||_{H^l}}{dt}
$$

give the desired estimate. □

Remark 3.4. *In this proof, many formulas have $\sum$ that we did not expand, and this kind of formula is very similar for $m > 0$.*

Now, with the above lemma, we can give an estimate for the last two terms in $Q_{n,T}^{(m)}(p)$ and finish the proof of Proposition 3.1 for the case $M = 1$.

Lemma 3.3. *Let $n \geq 4$ be a positive integer, and suppose that $\kappa = \kappa(z)$ and $g = g(\nu)$ satisfy the conditions (2.11) with $M = 1$. Then, there exists $Q^* > 0$, and for every $Q \geqslant Q^*$, there exists $\varepsilon_{1,C,Q}$ such that for every $\varepsilon \in (0, \varepsilon_{1,C,Q})$, if $p = p(t,\theta,\nu,z)$ is a solution to (3.1) with initial datum $p^{in}(z)$ satisfying*

$$
||\partial_z^m p^{in}(z)||_{H^n} \leqslant 1 \quad \textit{for all } 0 \leqslant m \leqslant 1,
$$

and for every $T > 0$,

$$Q_{n,T}^{(m)}(p) \leqslant Q, \quad \textit{for all } 0 \leqslant m \leqslant 1,$$

then

$$\sup_{t\in[0,T]} \frac{||\partial_z^m p(t,z)||_{H^n}}{1+t} \leqslant \frac{Q}{2}, \quad \sup_{t\in[0,T]} ||\partial_z^m p(t,z)||_{H^{n-2}} \leqslant \frac{Q}{2}, \quad \textit{for all } 0 \leqslant m \leqslant 1. \tag{3.26}$$

Proof. As in the proof of Lemma 3.1, we use the similar argument in Proposition 5.1 [18], to find Q_0, ε_0 such that for every $\varepsilon \in (0, \varepsilon_0)$ and every initial condition $p^{in}(z)$ satisfying $||p^{in}(z)||_{H^n} \leqslant 1$, we have

$$Q_{n,T}(p) \leqslant \frac{Q_0}{2}. \tag{3.27}$$

This implies

$$\sup_{t\in[0,T]} \frac{||p(t,z)||_{H^n}}{1+t} \leqslant \frac{Q_0}{2} \quad \text{and} \quad \sup_{t\in[0,T]} ||p(t,z)||_{H^{n-2}} \leqslant \frac{Q_0}{2}.$$

Now, we discuss how to choose Q_1 and ε_1 to obtain the result. If Q and ε satisfy the conditions

$$Q \geqslant Q_0, \quad \varepsilon < \varepsilon_0 \quad \text{and} \quad Q_{n,T}^{(1)}(p) \leqslant Q, \tag{3.28}$$

we use (3.19) with $l = n$ to get

$$\begin{aligned}
&\frac{d||\partial_z p||_{H^n}}{dt} \\
&\leqslant C' \Big[\sum_{j=0}^{1} \binom{1}{j} |\partial_z^{1-j}\kappa(z)||\partial_z^j U(t,z)| \\
&\times \Big((1+t)^n (||g||_{H^l} + \varepsilon ||p||_{H^0}) + \varepsilon(1+t) \sum_{j=1}^{n} t^{n-j} ||p||_{H^j} \Big) \Big] \\
&+ C'\kappa(z) \Big[|U(t,z)| \Big((1+t)^n (\varepsilon ||\partial_z p||_{H^0}) + \varepsilon(1+t) \sum_{j=1}^{n} t^{n-j} ||\partial_z p||_{H^j} \Big) \Big] \\
&=: \mathcal{I}_{21} + \mathcal{I}_{22}.
\end{aligned} \tag{3.29}$$

Below, we estimate the terms $\mathcal{I}_{2i}$ separately.

• (Estimate of $\mathcal{I}_{21}$): By direct estimate, we have

$$\begin{aligned}
|\mathcal{I}_{21}| &\leqslant C' \left[(1+t)^n \sum_{j=0}^{1} \binom{1}{j} |\partial_z^{1-j}\kappa(z)||\partial_z^j U(t,z)| \left(||g||_{H^l} + \varepsilon ||p||_{H^0} + \varepsilon \sum_{j=1}^{n} \frac{||p||_{H^j}}{(1+t)^{j-1}} \right) \right] \\
&\leqslant C'' \Big(2^n \sqrt{2}\pi + \varepsilon \frac{Q_0 Q + Q_0^2}{(n-2)} + Q_0 \Big) \Big(||g||_{H^l} + \varepsilon(n+1) Q_0 \Big),
\end{aligned}$$

where the second inequality is by (2.11), (3.18), (3.27) and (3.28).

• (Estimate of $\mathcal{I}_{22}$): Similarly, we have

$$
\begin{aligned}
|\mathcal{I}_{22}| &\leqslant C'\left[(1+t)^n|U(t,z)|\left(\varepsilon||\partial_z p||_{H^0}+\varepsilon\sum_{j=1}^{n}\frac{||\partial_z p||_{H^j}}{(1+t)^{j-1}}\right)\right] \\
&\leqslant C''Q_0\left(\varepsilon(n-1)Q+2\varepsilon\frac{||\partial_z p||_{H^n}}{(1+t)^{n-2}}\right),
\end{aligned}
\tag{3.30}
$$

where we used $||\partial_z p||_{H^{n-1}}\leqslant||\partial_z p||_{H^n}$ and (2.11), (3.18), (3.27), (3.28).

Therefore, we can get the estimate for $\frac{||\partial_z p||_{H^n}}{1+t}$ as follows.

$$
\begin{aligned}
\frac{||\partial_z p||_{H^n}}{1+t} &\leqslant \max\{1, A'+\epsilon B'Q\}+2C'''\varepsilon\int_0^t\frac{||\partial_z p||_{H^n}}{(1+t)(1+s)^{n-2}}ds \\
&\leqslant \max\{1, A'+\epsilon B'Q\}e^{2C'''\varepsilon/(n-3)},
\end{aligned}
$$

where A', B' and C''' depend on g, Q_0, n, C.

As before, one can choose Q large enough, and then choose ε small enough to get

$$
\frac{||\partial_z p||_{H^n}}{1+t}\leqslant\frac{Q}{2}.
$$

Now, we consider the case $l=n-2$ with s similar procedure to get

$$
\begin{aligned}
\frac{d||\partial_z p||_{H^{n-2}}}{dt} &\leqslant C'\left(2^n\sqrt{2}\pi+\varepsilon\frac{Q_0Q+Q_0^2}{(n-2)}+Q_0\right)\left(||g||_{H^{n-2}}+\varepsilon(n-1)Q_0\right)\frac{1}{(1+t)^2} \\
&\quad + C'Q_0\left(\varepsilon(n-1)Q\right)\frac{1}{(1+t)^2} \\
&\leqslant (A''+\varepsilon B''Q)\frac{1}{(1+t)^2},
\end{aligned}
\tag{3.31}
$$

where A'', B'' depend on g, Q_0, n, C. Next, we integrate (3.31) to obtain

$$
||\partial_z p||_{H^{n-2}}\leqslant 1+A''+\varepsilon B''Q.
$$

Then, as before, we choose Q large enough and then choose ε enough to obtain

$$
||\partial_z p||_{H^{n-2}}\leqslant\frac{Q}{2}.
$$

For Q, ε satisfying the above conditions, we define

$$
Q_1 := \max\{Q_0, Q\} \quad \text{and} \quad \varepsilon_{1,C,Q} := \min\{\varepsilon_0,\ \varepsilon\}.
$$

Then we can get for every $\varepsilon'\in(0,\varepsilon_{1,C,Q})$ and every initial condition $p^{in}(z)$ satisfying

$$
||\partial_z^m p(0,z)||_{H^n}\leqslant 1 \quad \text{for } m=0,1,
$$

we have

$$
Q_{n,T}(p)\leqslant\frac{Q_0}{2}<\frac{Q_1}{2}.
$$

This implies (3.26) for $m=0$ is satisfied automatically. On the other hand, if we have $Q^{(1)}_{n,T}(p)\leqslant Q_1$, we have

$$
\sup_{t\in[0,T]}\frac{||\partial_z p(t)||_{H^n}}{1+t}\leqslant\frac{Q_1}{2}, \quad \sup_{t\in[0,T]}||\partial_z p(t)||_{H^{n-2}}\leqslant Q/2\leqslant\frac{Q_1}{2}.
$$

□

3.4. **Proof of Proposition 3.1 for** $M > 1$**.** In the previous subsection, we have provided a proof of Proposition 3.1 for $M = 1$ which corresponds to the initial step in a mathematical induction.

Suppose that the results for Proposition 3.1 hold for $M = N - 1 > 0$, and we need to prove the proposition holds for $M = N$. We use the same argument as before, and most of details in the proof are similar, because we can exchange ∂_z with $\partial_\theta^{j_\theta}\partial_\nu^{j_\nu}$ or integral.

Lemma 3.4. *Let $n \geq 4$ be a positive integer, and suppose that $\kappa = \kappa(z)$ and $g = g(\nu)$ satisfy conditions (2.11) with $M = 1$. Then, there exists $Q^* > 0$, and for every $Q \geqslant Q^*$, there exists $\varepsilon_{1,C,Q}$ such that for every $\varepsilon \in (0, \varepsilon_{1,C,Q})$, if $p = p(t, \theta, \nu, z)$ is a solution to (3.1) with initial datum $p^{in}(z)$ satisfying*

$$||\partial_z^m p^{in}(z)||_{H^n} \leqslant 1 \quad \textit{for all } 0 \leqslant m \leqslant M,$$

and for every $T > 0$,

$$Q_{n,T}^{(m)}(p) \leqslant Q, \quad \textit{for all } 0 \leqslant m \leqslant M.$$

Then, we have

$$\sup_{t\in[0,T]} (1+t)^n |\partial_z^m U(t,z)| \leqslant \frac{Q}{2} \quad \textit{for all } 0 \leqslant m \leqslant M.$$

Proof. Recall that we need to show

$$\sup_{t\in[0,T]} (1+t)^n |\partial_z^M U(t,z)| \leqslant Q/2.$$

From the assumption, we can find $Q_{M-1}, \varepsilon_{M-1}$ such that for every $\varepsilon \in (0, \varepsilon_{M-1})$ and every initial condition $p^{in}(z)$ that satisfies $||\partial_z^m p^{in}(z)||_{H^n} \leqslant 1$ for all $0 \leqslant m \leqslant M-1$, we have

$$Q_{n,T}^m(p) \leqslant \frac{Q_{M-1}}{2} \quad \text{for all } 0 \leqslant m \leqslant M-1.$$

Now we will see how to choose Q_M and ε_M to obtain the result. Suppose that Q and ε satisfy the condition

$$Q \geqslant Q_{M-1}, \quad \varepsilon < \varepsilon_{M-1} \quad \text{and} \quad Q_{n,T}^{(M)}(p) \leqslant Q.$$

Next, we consider equation (3.3) for $m = M$:

$$\begin{cases} \partial_z^M U(t,z) - \dfrac{\kappa(z)}{2}\displaystyle\int_0^t \widehat{g}(t-s)\partial_z^M U(s,z)ds = F_1(t,z) + F_2(t.z), \\ F_1(t,z) = \partial_z^M \widehat{p}_1(0,t) - \dfrac{\varepsilon\kappa(z)}{2}\displaystyle\sum_{j=0}^{M}\binom{M}{j}\int_0^t \partial_z^j \widehat{p}_2(s,t+s)\partial_z^{M-j}\overline{U(s,z)}ds, \\ F_2(t,z) = \displaystyle\sum_{m=0}^{M-1}\binom{M}{m}\frac{\partial_z^{M-m}\kappa(z)}{2}\int_0^t \widehat{g}(t-s)\partial_z^m U(s,z)ds \\ \qquad\qquad -\varepsilon\displaystyle\sum_{m=0}^{M-1}\sum_{j=0}^{m}\binom{M}{m}\binom{m}{j}\partial_z^{M-m}\kappa(z)\int_0^t \partial_z^j\widehat{p}_2(s,t+s,z)\partial_z^{m-j}\overline{U(s,z)}ds. \end{cases}$$

Then, it follows from Proposition 2.1 and Remark 3.1 that

$$\sup_{t\in[0,T]}(1+t)^n|\partial_z^M U(t,z)| \leqslant C_{n,K}\sup_{t\in[0,T]}(1+t)^n\left|F_1(t,z)+F_2(t,z)\right|. \tag{3.32}$$

We follow a similar procedure as in Lemma 3.1 with the estimation on the terms in the right hand side of (3.32).

• (Estimate on the first term): For the estimate of $(1+t)^n|F_1(t,z)|$, we get

$$\begin{aligned}
&|(1+t)^n\partial_z^M\widehat{p}_1(0,t)| \leqslant 2^n\pi\sqrt{2}||\partial_z^M p(0,z)||_{H^n} \leqslant 2^n\pi\sqrt{2},\\
&\left|(1+t)^n\int_0^t \partial_z^j\widehat{p}_2(s,t+s)\partial_z^{M-j}\overline{U(s,z)}ds\right|\\
&\qquad\leqslant \frac{Q_{M-1}}{2}(1+t)^n\int_0^t\frac{|\partial_z^j\widehat{p}_2(s,t+s,z)|}{(1+s)^n}ds\\
&\qquad\leqslant \frac{C'Q_{M-1}}{2}(1+t)^n\int_0^t\frac{||\partial_z^j p(s,z)||_{H^n}}{(1+t+s)^n(1+s)^n}ds\\
&\qquad\leqslant \frac{C'Q_{M-1}^2}{2}\int_0^t\frac{ds}{(1+s)^{n-1}}=\frac{C''Q_{M-1}^2}{2(n-2)},\quad\text{for } 0<j<M.
\end{aligned}\tag{3.33}$$

Similarly, for $j=0$ and $j=M$

$$\left|(1+t)^n\int_0^t \partial_z^j\widehat{p}_2(s,t+s)\partial_z^{M-j}\overline{U(s,z)}ds\right| \leqslant \frac{C'Q_{M-1}Q}{2(n-2)}. \tag{3.34}$$

• (Estimate on the second term): For the estimate of $(1+t)^n|F_2(t)|$, we use a similar method in (3.17), (3.33) and (3.34) to see

$$(1+t)^n|F_2(t)| \leqslant \varepsilon\frac{C''Q_{M-1}^2}{2(n-2)}+C''Q_{M-1}. \tag{3.35}$$

In (3.32), we combine all estimates (3.33), (3.34) and (3.35) to obtain

$$\sup_{t\in[0,T]}(1+t)^n|\partial_z^M U(t,z)| \leqslant C'''\Big[Q_{M-1}+2^n\pi\sqrt{2}+\frac{\varepsilon}{2}\Big(Q_{M-1}Q+Q_{M-1}^2\Big)\Big], \tag{3.36}$$

where C''' depends on n,g,C. Therefore, we can first make Q larger such that

$$\sup_{t\in[0,T]}(1+t)^n|\partial_z^M U(t,z)| \leqslant \varepsilon C'''Q_{M-1}Q,$$

and then make $\varepsilon \leq \frac{Q}{2}$. Then, the same argument gives the desired estimate. □

Lemma 3.5. *Given $l\geqslant 1$, there exists a constant $C_l'>0$ such that for all $t>0$*

$$\begin{aligned}
\frac{d||\partial_z^M p||_{H^l}}{dt} &\leqslant C_l'\Big[\sum_{m=0}^{M}\sum_{m'=0}^{m}\binom{M}{m}\binom{m}{m'}|\partial_z^{m'}U(t)||\partial_z^{m-m'}\kappa(z)|\\
&\quad\times\Big((1+t)^l(||g||_{H^l}\delta_{M,m}+\varepsilon||\partial_z^{M-m}p||_{H^0})+\varepsilon(1+t)\sum_{j=1}^{l}t^{l-j}||\partial_z^{M-m}p||_{H^j}\Big)\Big].
\end{aligned}\tag{3.37}$$

Proof. This proof is exactly the same as in the proof of Lemma 3.2, because we can exchange ∂_z and $\partial_\theta^{j_\theta}\partial_\nu^{j_\nu}$ or integral. Therefore, we omit its details. □

Lemma 3.6. *Let $n \geq 4$ be a positive integer, and suppose that $\kappa = \kappa(z)$ and $g = g(\nu)$ satisfy the conditions* (2.11) *with $M = 1$. Then, there exists $Q^* > 0$, and for every $Q \geqslant Q^*$, there exists $\varepsilon_{1,C,Q}$ such that for every $\varepsilon \in (0, \varepsilon_{1,C,Q})$, if $p = p(t, \theta, \nu, z)$ is a solution to* (3.1) *with initial datum $p^{in}(z)$ satisfying*

$$||\partial_z^m p^{in}(z)||_{H^n} \leqslant 1 \quad \textit{for all } 0 \leqslant m \leqslant M,$$

and for every $T > 0$,

$$Q_{n,T}^{(m)}(p) \leqslant Q, \quad \textit{for all } 0 \leqslant m \leqslant M,$$

then

$$\sup_{t\in[0,T]} \frac{||\partial_z^m p(t,z)||_{H^n}}{1+t} \leqslant \frac{Q}{2}, \quad \sup_{t\in[0,T]} ||\partial_z^m p(t,z)||_{H^{n-2}} \leqslant \frac{Q}{2}, \quad \textit{for all } 0 \leqslant m \leqslant M.$$

Proof. By induction hypothesis. we can find $Q_{M-1}, \varepsilon_{M-1}$ such that for every $\varepsilon \in (0, \varepsilon_{M-1})$ and every initial datum $p^{in}(z)$ satisfying $||\partial_z^m p^{in}(z)||_{H^n} \leqslant 1$ for all $0 \leqslant m \leqslant M-1$, we have

$$Q_{n,T}^m(p) \leqslant \frac{Q_{M-1}}{2} \quad \text{for all } 0 \leqslant m \leqslant M-1.$$

This implies that for all $0 \leqslant m \leqslant M-1$,

$$\sup_{t\in[0,T]} \frac{||\partial_z^m p(t)||_{H^n}}{1+t} \leqslant \frac{Q_{M-1}}{2}, \quad \sup_{t\in[0,T]} ||\partial_z^m p(t)||_{H^{n-2}} \leqslant \frac{Q_{M-1}}{2}.$$

Next, we discuss how to choose Q_M and ε_M to obtain the desired result. Suppose that Q and ε satisfy the conditions:

$$Q \geqslant Q_{M-1}, \quad \varepsilon < \varepsilon_{M-1} \quad \text{and} \quad Q_{n,T}^{(M)}(p) \leqslant Q.$$

We use (3.37) with $l = n$ to obtain

$$\begin{aligned}
&\frac{d||\partial_z^M p||_{H^n}}{dt} \\
&\leqslant C'\Big[\sum_{m=0}^{M}\sum_{m'=0}^{m}\binom{M}{m}\binom{m}{m'}|\partial_z^{m'}U(t,z)||\partial_z^{m-m'}\kappa(z)| \\
&\times\left((1+t)^n(||g||_{H^n}\delta_{M,m}+\varepsilon||\partial_z^{M-m}p||_{H^0})+\varepsilon(1+t)\sum_{j=1}^{n}t^{n-j}||\partial_z^{M-m}p||_{H^j}\right)\Big] \\
&\leqslant C''\sum_{m+m'\leqslant M}\left[|\partial_z^{m'}U(t,z)|\left((1+t)^n(||g||_{H^n}\delta_{m,0}+\varepsilon||\partial_z^m p||_{H^0})+\varepsilon(1+t)\sum_{j=1}^{n}t^{n-j}||\partial_z^m p||_{H^j}\right)\right] \\
&\leqslant C''\left(Q_{M-1}(\varepsilon(n+1)Q_{M-1}+1)+(A+\varepsilon BQ)\left(||g||_{H^n}+\varepsilon(n+1)Q_{M-1}\right)\right) \\
&+C''Q_{M-1}\left(\varepsilon(n-1)Q+2\varepsilon\frac{||\partial_z^M p||_{H^n}}{(1+t)^{n-2}}\right),
\end{aligned}$$

where A, B are from (3.36) and the first term is for $m, m' < M$, the second term is for $m' = M$, and the third term is for $m = M$.

Therefore, we can get a final estimate for $\frac{||\partial_z^M p||_{H^n}}{1+t}$ as

$$\begin{aligned}\frac{||\partial_z^M p||_{H^n}}{1+t} &\leqslant \max\{1, A' + \varepsilon B'Q\} + 2C'''\varepsilon \int_0^t \frac{||\partial_z^M p||_{H^n}}{(1+t)(1+s)^{n-2}} ds \\ &\leqslant \max\{1, A' + \epsilon B'Q\} e^{2C'''\varepsilon/(n-3)},\end{aligned}$$

where A', B', C''' depend on g, Q_{M-1}, M, n, C.

Again, we apply (3.19) with $l = n-2$ to get

$$\begin{aligned}&\frac{d||\partial_z^M p||_{H^{n-2}}}{dt} \\ &\leqslant C'' \sum_{m+m' \leqslant M} \Big[|\partial_z^{m'} U(t,z)| \\ &\times \Big((1+t)^{n-2} (||g||_{H^n} \delta_{m,0} + \varepsilon ||\partial_z^m p||_{H^0}) + \varepsilon (1+t) \sum_{j=1}^{n-2} t^{n-2-j} ||\partial_z^m p||_{H^j} \Big) \Big] \\ &\leqslant \frac{C''}{(1+t)^2} \Big[(\varepsilon(n-1) Q_{M-1} + 1) Q_{M-1} + (A + \varepsilon BQ) \left(||g||_{H^{n-2}} + \varepsilon(n-1) Q_{M-1} \right) \\ &+ \varepsilon (n+1) Q_{M-1} Q \Big],\end{aligned}$$

where A, B is from (3.36) and the first term is for $m, m' < M$, the second term is for $m' = M$, ans the third term is for $m = M$.

Therefore, we can get the final estimate for $\frac{||\partial_z^M p||_{H^n}}{1+t}$:

$$||\partial_z^M p||_{H^{n-2}} \leqslant 1 + A'' + \varepsilon B'' Q,$$

where A'', B'' depend on g, Q_{M-1}, M, n, C. As before, we choose Q large enough, and then choose ε small enough to get

$$||\partial_z^M p||_{H^{n-2}} \leqslant \frac{Q}{2}.$$

□

3.5. **Proof of Corollary 2.2.** Suppose that p^{in}, g and κ satisfy conditions (2.11) of Theorem 2.2, and let $p(t,z)$ be a solution to (3.1) with initial datum $p^{in}(z)$. Then, it suffices to show $\sup_t ||\partial_z^m p(t)||_{H^{n-2}} < \infty$ and

$$\begin{aligned}p^{in}(\theta,\nu,z) + \int_{\mathbb{R}^+} \Big[\varepsilon \sum_{j=0}^m \binom{m}{j} \Big(\partial_\theta \partial_z^j p(t,\theta,\nu,z) \partial_z^{m-j} W[p](t,\theta+t\nu,z) \\ + \partial_z^j p(\theta,\nu,z) \partial_\theta \partial_z^{m-j} W[p](t,\theta+t\nu,z) \Big) + \frac{g(\nu)}{2\pi} \partial_\theta \partial_z^m W[p](t,\theta+t\nu,z) \Big] dt\end{aligned}$$

is well-defined in H^{n-2}. This is equivalent to show that the following three terms

$$\mathcal{I}_{31}(T) := \int_T^\infty \frac{g(\nu)}{2\pi}\partial_\theta\partial_z^m W[p](t,\theta+t\nu,z)dt,$$

$$\mathcal{I}_{32}(T) := \int_T^\infty \sum_{j=0}^m \binom{m}{j}\partial_z^j p(t,\theta,\nu,z)\partial_\theta\partial_z^{m-j}W[p](t,\theta+t\nu,z)dt,$$

$$\mathcal{I}_{33}(T) := \int_T^\infty \sum_{j=0}^m \binom{m}{j}\partial_\theta\partial_z^j p(t,\theta,\nu,z)\partial_z^{m-j}W[p](t,\theta+t\nu,p,z)dt$$

converge to zero in H^{n-2}, as $T \to \infty$.

• (Zero convergence of $\mathcal{I}_{31}(T)$): We use $||g||_{H^{n-2}} \leqslant ||g||_{H^n} < \infty$, (3.22), Proposition 3.1 and $n \geq 4$ to see

$$\begin{aligned}||\mathcal{I}_{31}(T)||_{H^{n-2}} &\leqslant C'\int_T^\infty \sum_{j=0}^m |\partial_z^{m-j}\kappa(z)||\partial_z^j U(t,z)|||g||_{H^{n-2}}dt \\ &\leqslant \frac{C''Q}{2}\int_T^\infty \frac{1}{(1+t)^n}dt \quad \to 0, \quad \text{as } T\to\infty,\end{aligned} \tag{3.38}$$

where C', C'' depend on g, m, C.

• (Zero convergence of $\mathcal{I}_{32}(T)$ and $\mathcal{I}_{32}(T)$): Note that Proposition 4.1 yields

$$||\partial_z^j p||_{H^{n-2}} < \frac{Q}{2}, \quad \frac{||\partial_z^j p||_{H^n}}{1+t} < \frac{Q}{2}.$$

This and $n \geq 4$ yield

$$\begin{aligned}||\mathcal{I}_{32}(T)||_{H^{n-2}} &\leqslant C'\int_T^\infty \sum_{j=0}^m\sum_{j'=0}^{m-j} ||\partial_z^j p||_{H^{n-2}}|\partial_z^{j'}\kappa(z)||\partial_z^{m-j-j'}U(t,z)|dt \\ &\leqslant \frac{C''Q^2}{2}\int_T^\infty \frac{1}{(1+t)^n}dt \quad \to 0, \quad \text{as } T\to\infty, \\ ||\mathcal{I}_{33}(T)||_{H^{n-2}} &\leqslant C'\int_T^\infty \sum_{j=0}^m\sum_{j'=0}^{m-j} ||\partial_z^j p||_{H^{n-1}}|\partial_z^{j'}\kappa(z)||\partial_z^{m-j-j'}U(t,z)|dt \\ &\leqslant \frac{C''Q^2}{2}\int_T^\infty \frac{1}{(1+t)^{n-1}}dt \quad \to 0, \quad \text{as } T\to\infty,\end{aligned} \tag{3.39}$$

where C', C'' depend on m, C. Finally, we collect all estimates (3.38) and (3.39) to complete the proof.

□

4. A local sensitivity analysis in an analytic framework

In this section, we provide the proof of Theorem 2.4 on the local sensitivity for Landau damping in the analytic framework.

4.1. **Transformation of the equation.** First, we introduce a Lagrangian density along the free path:

$$h(t,\theta,\nu,z) := f(t,\theta+t\nu,\nu,z)g(\nu) = \frac{g(\nu)}{2\pi} + \rho(t,\theta+t\nu,\nu)g(\nu).$$

By this definiton, because of $\widehat{g}_k(\tau)=0$ for any k,τ, we have

$$||h(t,\theta,\nu,z)||_{\lambda,p} = ||\rho(t,\theta+t\nu,\nu)g(\nu)||_{\lambda,p} \tag{4.1}$$

for all $\lambda>0, p>0$. Then, it is easy to see that h satisfies

$$\begin{cases} \partial_t h(t,\theta,\nu,z) + \partial_\theta\Big(h(t,\theta,\nu,z)V[h](t,\theta+t\nu,z)\Big) = 0, \\ h(t,\theta,\nu,z) \geqslant 0, \quad \displaystyle\int_{\mathbb{T}} h(t,\theta,\nu,z)d\theta = g(\nu), \\ V[h](t,\theta,z) := \kappa(z)\int_{\mathbb{T}\times\mathbb{R}} \sin(\theta'+t\nu-\theta)h(t,\theta',\nu,z)d\theta' d\nu, \end{cases} \tag{4.2}$$

and the complex order parameter can be represented by:

$$R(t,z) = \int_{\mathbb{T}\times\mathbb{R}} e^{\mathrm{i}(\theta+t\nu)}h(t,\theta,\nu,z)d\theta d\nu.$$

Therefore, with this h, the condition in (2.14) and (4.1) could give us $||\partial_z^m h||_{\lambda_0,r}<\infty$ for all $0\leqslant m\leqslant M$, and it suffices to prove for all $0\leqslant m\leqslant M$

(i) $|||\partial_z^m h(t,\theta,\nu,z)|||_a < \overline{C}$ and $|||\partial_z^m R|||_a < \overline{C}$.
(ii) $|\partial_z^m R(t)| \to 0$ exponentially fast.
(iii) $\exists\ h_\infty(\theta,\nu,z)$ such that $||\partial_z^m h_\infty||_{\bar\lambda,\gamma}<\infty$, for some $\bar\lambda>0$ such that $\partial_z^m h(t,\theta,\nu,z) \to \partial_z^m h_\infty(\theta,\nu,z)$ exponentially fast.

For $M=0$, this has been proven by Theorem 2.3.

As before, we perform the Fourier transform on both sides of $(4.2)_1$ to get

$$\widehat{h}_k(t,\tau,z) = \widehat{h^{in}}_k(\tau,z) + k\kappa(z)\sum_{q\in\pm1}\frac{q}{2}\int_0^t \Gamma_q(s)\widehat{h}_{k-q}(s,\tau-qs,z)ds, \tag{4.3}$$

where we have

$$\Gamma_1(t,z) = \overline{R(t,z)}, \quad \Gamma_{-1}(t,z) = R(t,z) \quad \text{and} \quad R(t,z) = \widehat{h}_1(t,t,z).$$

Therefore, we set $k=1,\tau=t$ to see

$$\Gamma_1(t,z) = \widehat{h^{in}}_1(t,z) + k\kappa(z)\sum_{q\in\pm1}\frac{q}{2}\int_0^t \Gamma_q(s)\widehat{h}_{1-q}(s,t-qs,z)ds. \tag{4.4}$$

Then, we define an operator L_t acting on a function $u=u(t,\theta,\nu,z)$ such that

$$\widehat{L_t u}(t,k,\tau,z) = k\sum_{q\in\pm1}\frac{q}{2}\Gamma_q(t,z)\widehat{u}_{k-q}(t,\tau-qt,z).$$

This yields

$$\partial_t h = \kappa L_t h$$

Next, we apply ∂_z^m on both sides of equations (4.3) and (4.4):

$$\begin{aligned}\widehat{\partial_z^m h}_k(t,\tau,z) &= \widehat{\partial_z^m h^{in}}_k(\tau,z) + k\sum_{m'=0}^{m}\sum_{j=0}^{m'}\binom{m}{m'}\binom{m'}{j}\sum_{q\in\pm1}\frac{q}{2}\partial_z^{m-m'}\kappa(z)\\ &\quad\times\int_0^t \partial_z^j\Gamma_q(s,z)\partial_z^{m'-j}\widehat{h}_{k-q}(s,\tau-qs,z)ds,\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\partial_z^m\Gamma_1(t,z) &= \widehat{\partial_z^m h^{in}}_1(t,z) + k\sum_{m'=0}^{m}\sum_{j=0}^{m'}\binom{m}{m'}\binom{m'}{j}\sum_{q\in\pm1}\frac{q}{2}\partial_z^{m-m'}\kappa(z)\\ &\quad\times\int_0^t \partial_z^j\Gamma_q(s,z)\partial_z^{m'-j}\widehat{h}_{1-q}(s,t-qs,z)ds,\end{aligned} \tag{4.6}$$

where

$$\int_{\mathbb{T}^1}\partial_z^m h(t,\theta,\nu,z)d\theta = 0,\ \forall\nu\in\mathbb{R},\ z\in\Omega,\ t>0,\quad \text{for } m=1,\cdots,M. \tag{4.7}$$

4.2. **A priori estimates.** By definition of $(\partial_z^m R)_{\lambda,p}(t,z)$ in (2.12), it suffices to show that $(\partial_z^m R)_{\lambda,p}(t,z)$ is bounded for some λ,p and all t. As in [6], we have a similar estimate for $(\partial_z^m R)_{\lambda,p}(t,z)$. In the sequel, the constant C is from Theorem 2.4.

Proposition 4.1. *For $\lambda,p,m\geqslant 0$ and $z\in\Omega$, we have*

$$\begin{aligned}(\partial_z^m R)_{\lambda,p}(t,z) &\leqslant ||\partial_z^m h^{in}(z)||_{\lambda,p} + C'\sum_{m'=0}^{m}\sum_{j=0}^{m'}|\partial_z^{m-m'}\kappa(z)|||\partial_z^{m'-j}h^{in}(z)||_{\lambda,p}\\ &\quad\times\int_0^t(\partial_z^j R)_{\lambda,p}(s,z)\left(\frac{1}{\langle s\rangle^p}+\frac{1}{\langle t-s\rangle^p}\right)ds\\ &\quad + C'\sum_{m'=0}^{m}\sum_{j=0}^{m'}|\partial_z^{m-m'}\kappa(z)|\int_0^t(\partial_z^j R)_{\lambda,p}(s,z)\frac{||\partial_z^{m'-j}h(s,z)||_{\lambda,p}}{\langle s\rangle^p}ds,\\ h^{in}(\theta,\nu,z) &= \rho^{in}(\theta,\nu,z)\cdot g(\nu).\end{aligned} \tag{4.8}$$

Proof. We multiply $e^{\lambda t}\langle t\rangle^p$ on both sides of (4.6) to get

$$\begin{aligned}&e^{\lambda t}\langle t\rangle^p\widehat{\partial_z^m h}_k(t,\tau,z)\\ &= e^{\lambda t}\langle t\rangle^p\widehat{\partial_z^m h^{in}}_k(\tau,z) + e^{\lambda t}\langle t\rangle^p k\sum_{m'=0}^{m}\sum_{j=0}^{m'}\binom{m}{m'}\binom{m'}{j}\sum_{q\in\pm1}\frac{q}{2}\partial_z^{m-m'}\kappa(z)\\ &\quad\times\int_0^t\partial_z^j\Gamma_q(s,z)\partial_z^{m'-j}\widehat{h}_{k-q}(s,\tau-qs,z)ds\\ &=: \mathcal{I}_{41}+\mathcal{I}_{42}.\end{aligned} \tag{4.9}$$

Below, we estimate the terms $\mathcal{I}_{4i},\ i=1,2$ on the right hand sides of (4.9) as follows.

• (Estimate on $\mathcal{I}_{41}$): By direct estimate, one has

$$|e^{\lambda t}\langle t\rangle^p\widehat{\partial_z^m h^{in}}_1(t,z)| \leqslant e^{\lambda\langle 1,t\rangle}\langle 1,t\rangle^p|\widehat{\partial_z^m h^{in}}_1(t,z)| \leqslant ||\partial_z^m h^{in}(z)||_{\lambda,p}. \tag{4.10}$$

• (Estimate on $\mathcal{I}_{42}$): We consider the case of $q=1$ and use (4.7) to obtain

$$\widehat{\partial_z^{m'-j}h}_{1-q}(s,t-qs,z)=\widehat{\partial_z^{m'-j}h}_0(s,t-s,z)=\widehat{\partial_z^{m'-j}h}_0(0,t-s,z).$$

Therefore, for each m' and j, $\mathcal{I}_{42}$ can be bounded by

$$\begin{aligned}
& e^{\lambda t}\langle t\rangle^p\int_0^t|\partial_z^j\Gamma_1(s,z)\widehat{\partial_z^{m'-j}h}_0(s,t-s,z)|ds\\
&\leqslant e^{\lambda t}\langle t\rangle^p\int_0^t|\partial_z^j\Gamma_1(s,z)\widehat{\partial_z^{m'-j}h}_0(0,t-s,z)|ds\\
&\leqslant ||\partial_z^{m'-j}h^{in}(z)||_{\lambda,p}e^{\lambda t}\langle t\rangle^p\int_0^t e^{-\lambda\langle t-s\rangle}\langle t-s\rangle^{-p}|\partial_z^j\Gamma_1(s,z)|ds\\
&\leqslant ||\partial_z^{m'-j}h^{in}(z)||_{\lambda,p}\int_0^t(\partial_z^jR)_{\lambda,p}(s,z)\frac{\langle t\rangle^p}{\langle s\rangle^p\langle t-s\rangle^p}e^{\lambda(t-s-\langle t-s\rangle)}ds\\
&\leqslant C'||\partial_z^{m'-j}h^{in}(z)||_{\lambda,p}\int_0^t(\partial_z^jR)_{\lambda,p}(s,z)\left(\frac{1}{\langle s\rangle^p}+\frac{1}{\langle t-s\rangle^p}\right)ds,
\end{aligned}\tag{4.11}$$

where the last inequality is due to the following relations:

$$\langle t-s\rangle\geqslant(t-s)\quad\text{and}\quad\langle t\rangle^p\leqslant C(\langle s\rangle^p+\langle t-s\rangle^p).$$

For the case of $q=-1$, we use a similar method to get that for each j. The term can be bounded by

$$\begin{aligned}
& e^{\lambda t}\langle t\rangle^p|\partial_z^j\Gamma_{-1}(s,z)\widehat{\partial_z^{m'-j}h}_2(s,t+s,z)|ds\\
&\leqslant\int_0^t(\partial_z^jR)_{\lambda,p}(s,z)||\partial_z^{m'-j}h(s,z)||_{\lambda,p}e^{\lambda(t-s)-\lambda\langle t+s\rangle}\frac{\langle t\rangle^p}{\langle s\rangle^p\langle t+s\rangle^p}ds\\
&\leqslant C'\int_0^t(\partial_z^jR)_{\lambda,p}(s,z)\frac{||\partial_z^{m'-j}h(s,z)||_{\lambda,p}}{\langle s\rangle^p}ds.
\end{aligned}\tag{4.12}$$

In (4.9), by combining estimates (4.10), (4.11) and (4.12), one can get the desired estimate. □

Next, we provide a proposition to be used in the estimation of $||\partial_z^m h||_{\lambda,p}$.

Proposition 4.2. *Given $\Gamma_{\pm1}(t,z)$, for $\lambda,p,m\geqslant0$, we have the following assertions:*

$$\begin{aligned}
&||\partial_z^mL_th(t,z)||_{\lambda,p}\\
&\leqslant C'\sum_{j=0}^m\left[(\partial_z^jR)_{\lambda,0}(t,z)||\partial_z^{m-j}h(t,z)||_{\lambda,p+1}+(\partial_z^jR)_{\lambda,p}(t,z)||\partial_z^{m-j}h(t,z)||_{\lambda,1}\right],
\end{aligned}\tag{4.13}$$

and

$$||f||_{\lambda,p+1}\leqslant\frac{1}{\lambda'-\lambda}||f||_{\lambda',p},\tag{4.14}$$

for all $f\in\mathcal{X}_{\lambda,p}$ and $\lambda'>\lambda$

Proof. The second part and case $m=0$ in (i) follow from Proposition 2 [6]. On the other hand, we use a similar method to prove the first part. First, it follows from (4.1) that

$$\widehat{\partial_z^mL_th}(t,k,\tau,z)=k\sum_{j=0}^m\binom{m}{j}\sum_{q\in\pm1}\frac{q}{2}\partial_z^j\Gamma_q(t,z)\widehat{\partial_z^{m-j}h}_{k-q}(t,\tau-qt,z).$$

For each j, we set

$$\mathcal{J}_j := e^{\lambda\langle k,\tau\rangle}\langle k,\tau\rangle^p \sum_{q\in\pm1} \frac{q}{2}\partial_z^j\Gamma_q(t,z)\widehat{\partial_z^{m-j}h}_{k-q}(t,\tau-qt,z).$$

Then, we have

$$|\mathcal{J}_j| \leqslant \frac{|\partial_z^j R(t,z)||k|}{2}\sum_{q\in\pm1} e^{\lambda\langle k,\tau\rangle}\langle k,\tau\rangle^p\left|\widehat{\partial_z^{m-j}h}_{k-q}(t,\tau-qt,z)\right|. \tag{4.15}$$

On the other hand, recall some inequalities on $\langle k,\tau\rangle$:

$$\begin{aligned}&\langle k,\tau\rangle \leqslant \langle k-q,\tau-qt\rangle + \langle q,qt\rangle, \quad \langle q,qt\rangle \leqslant C'+t,\ |q|=1,\\ &\langle k,\tau\rangle^p \leqslant C'(\langle k-q,\tau-qt\rangle^p + \langle t\rangle^p),\ |q|=1.\end{aligned}$$

Now, we further estimate the term $|\mathcal{J}_j|$ as follows.

$$\begin{aligned}|\mathcal{J}_j| &\leqslant C'e^{\lambda t}|\partial_z^j R(t,z)|\sum_{q\in\pm1} e^{\lambda\langle k-q,\tau-qt\rangle}|k|\langle k-q,\tau-qt\rangle^p\left|\widehat{\partial_z^{m-j}h}_{k-q}(t,\tau-qt,z)\right|\\ &\quad + C'e^{\lambda t}|\partial_z^j R(t,z)|\sum_{q\in\pm1} e^{\lambda\langle k-q,\tau-qt\rangle}|k|\langle t\rangle^p\left|\widehat{\partial_z^{m-j}h}_{k-q}(t,\tau-qt,z)\right|.\end{aligned} \tag{4.16}$$

Since $|k| \leqslant 2\langle k-q,\tau-qt\rangle$, $|q|=1$, we can obtain

$$|\mathcal{J}_j| \leqslant C'\left[(\partial_z^j R)_{\lambda,0}(t)||\partial_z^{m-j}h(t)||_{\lambda,p+1} + (\partial_z^j R)_{\lambda,p}(t)||\partial_z^{m-j}h(t)||_{\lambda,1}\right]. \tag{4.17}$$

Finally, adding all $|\mathcal{J}_j|$ in (4.15), (4.16) and (4.17) yields the desired estimate. $\square$

From the above proposition, we can see that the operator L_t maps function in $\mathcal{X}_{\lambda,p}$ to a different space $\mathcal{X}_{\lambda,p+1}$, i.e., they are not the same space. This is why paper [6] introduces the nrom $|||\cdot|||_a$. Next, we provide an estimate for the control of $|||\partial_z^m h|||_a$. We fix

$$\gamma \geqslant 3, \quad \lambda_0 > \lambda > 0 \quad \text{and} \quad \frac{2\lambda_0}{\pi} > a > 0.$$

Then, we define the corresponding $\beta_a(t,\lambda)$ and norm $|||\cdot|||_a$ and $||||\cdot||||_a$.

Proposition 4.3. *For given $\Gamma_{\pm1}(t)$ with $|||\partial_z^j R|||_a < \infty$ for all $j \leqslant m$ and $m \geqslant 0$, let $h = h(t,\theta,\nu,z)$ be a solution to (4.3) and (4.5). Then, we have*

$$|||\partial_z^m h|||_a \leqslant C'||\partial_z^m h^{in}||_{\lambda_0,\gamma} + C'\sum_{m'=0}^{m}\sum_{j=0}^{m'}|\partial_z^{m-m'}\kappa(z)|||||\partial_z^j R||||_a|||\partial_z^{m'-j}h|||_a.$$

Proof. For the desired estimate, we need to use (4.5) and the estimate (4.13) in Proposition 4.2. First, we consider $||\partial_z^m h(t)||_{\lambda,1}$ using (4.13) in the case of $p=1$. We use $\gamma \geqslant 3$ to obtain

$$
\begin{aligned}
&||\partial_z^m h(t,z)||_{\lambda,1}\\
&\leqslant C'||\partial_z^m h^{in}(z)||_{\lambda,1}+C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\\
&\times\int_0^t\sum_{j=0}^{m'}\Big[(\partial_z^j R)_{\lambda,0}(s,z)||\partial_z^{m'-j}h(s,z)||_{\lambda,2}+(\partial_z^j R)_{\lambda,1}(s,z)||\partial_z^{m'-j}h(s,z)||_{\lambda,1}\Big]\,ds\\
&\leqslant C'||\partial_z^m h^{in}(z)||_{\lambda,1}+C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\\
&\times\int_0^t\sum_{j=0}^{m'}(\partial_z^j R)_{\lambda,\gamma}(s,z)\left(\frac{1}{\langle s\rangle^{\gamma-1}}\frac{||\partial_z^{m'-j}h(s,z)||_{\lambda,\gamma}}{\langle s\rangle}+\frac{1}{\langle s\rangle^{\gamma-1}}||\partial_z^{m'-j}h(s,z)||_{\lambda,1}\right)ds.
\end{aligned}
\tag{4.18}
$$

Then, multiplying $\beta^{\frac{1}{2}}(t,\lambda)$ on both sides of (4.18) gives

$$
\begin{aligned}
&\beta^{\frac{1}{2}}(t,\lambda)||\partial_z^m h(t,z)||_{\lambda,1}\leqslant C'||\partial_z^m h^{in}(z)||_{\lambda,1}\\
&\quad+C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\sum_{j=0}^{m'}||||\partial_z^j R||||_a|||\partial_z^{m'-j}h|||_a\int_0^t\frac{1}{\langle s\rangle^2}\frac{\beta^{\frac{1}{2}}(t,\lambda)}{\beta^{\frac{1}{2}}(s,\lambda)}ds,
\end{aligned}
$$

where we used the inequality:

$$
\int_0^t\frac{1}{\langle s\rangle^2}\frac{\beta^{\frac{1}{2}}(t,\lambda)}{\beta^{\frac{1}{2}}(s,\lambda)}ds\leqslant\frac{\pi}{2}.
$$

Next, we need to estimate $||\partial_z^m h(t)||_{\lambda,\gamma}$ using (4.13) in the case of $p=\gamma$. We also use (4.14) for any $\lambda'(s)>\lambda$ with $\lambda_0-\lambda'(s)-a\arctan(s)>0$ to get

$$
\begin{aligned}
&||\partial_z^m h(t,z)||_{\lambda,\gamma}\\
&\leqslant C'||\partial_z^m h^{in}(z)||_{\lambda,1}+C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\\
&\times\int_0^t\sum_{j=0}^{m'}\Big[(\partial_z^j R)_{\lambda,0}(s,z)||\partial_z^{m'-j}h(s,z)||_{\lambda,\gamma+1}+(\partial_z^j R)_{\lambda,\gamma}(s,z)||\partial_z^{m'-j}h(s,z)||_{\lambda,1}\Big]\\
&\leqslant C'||\partial_z^m h^{in}(z)||_{\lambda,1}+C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\\
&\times\int_0^t\sum_{j=0}^{m'}(\partial_z^j R)_{\lambda,\gamma}(s,z)\Big(\frac{1}{\langle s\rangle^{\gamma}}\frac{||\partial_z^{m'-j}h(s,z)||_{\lambda,\gamma}}{\lambda'(s)-\lambda}+||\partial_z^{m'-j}h(s,z)||_{\lambda,1}\Big)ds.
\end{aligned}
\tag{4.19}
$$

We divide (4.19) by $\langle t\rangle$ and multiply the resulting relation by $\beta^{1/2}(t,\lambda)$ to obtain

$$\begin{aligned}\frac{\beta^{1/2}(t,\lambda)}{\langle t\rangle}||\partial_z^m h(t,z)||_{\lambda,\gamma} &\leqslant C'||\partial_z^m h^{in}(z)||_{\lambda,1}\\ &\quad + C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\sum_{j=0}^{m'}||||\partial_z^j R||||_a|||\partial_z^{m'-j}h|||_a(\mathcal{I}_{51}+\mathcal{I}_{52}),\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_{51} &:= \frac{\beta^{1/2}(t,\lambda)}{\langle t\rangle}\int_0^t \frac{ds}{\langle s\rangle^2\beta^{1/2}(s,\lambda)(\lambda'(s)-\lambda)},\\ \mathcal{I}_{52} &:= \frac{\beta^{1/2}(t,\lambda)}{\langle t\rangle}\int_0^t \frac{ds}{\beta^{1/2}(s,\lambda)}.\end{aligned}$$

Following Proposition 3 in [6], one can choose $\lambda'(s)$ properly to make $|\mathcal{I}_{51}|,|\mathcal{I}_{52}|$ less than a constant. Finally, for the very first term, we use $\lambda_0>\lambda$ to derive

$$||\partial_z^m h^{in}(z)||_{\lambda,1} \leqslant ||\partial_z^m h^{in}(z)||_{\lambda_0,1}.$$

□

Proposition 4.4. *Let h be a function such that*

$$|||\partial_z^j h|||_a < \infty \quad \textit{for all } j\leqslant m, \textit{ and } m\geqslant 0.$$

If $\Gamma_{\pm1}(t,z)$ solves (4.4) and (4.6), it satisfies

$$\begin{aligned}||||\partial_z^m R||||_a &\leqslant C'||\partial_z^m h^{in}||_{\lambda_0,\gamma} + C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\\ &\quad\times\sum_{j=0}^{m'}(||\partial_z^{m'-j}h(0)||_{\lambda_0,\gamma}||||\partial_z^j R||||_a + |||\partial_z^{m'-j}h|||_a||||\partial_z^j R||||_a).\end{aligned}$$

Proof. For our desired estimate, one only needs to use (4.8) for $p=\gamma$ and $\lambda_0>\lambda$. The first two terms in (4.8) yield

$$C'||\partial_z^m h^{in}(z)||_{\lambda_0,\gamma} + C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\sum_{j=0}^{m'}||\partial_z^{m'-j}h(0,z)||_{\lambda_0,\gamma}||||\partial_z^j R||||_a.$$

We use the last term and $\int_0^t \frac{1}{\langle s\rangle^2\beta^{1/2}(t,\lambda)}ds \leqslant \frac{2}{a}\lambda_0^{1/2}$ due to Proposition 4 [6] to find

$$C'\sum_{m'=0}^{m}|\partial_z^{m-m'}\kappa(z)|\sum_{j=0}^{m'}|||\partial_z^{m'-j}h|||_a||||\partial_z^j R||||_a.$$

□

4.3. **Proof of Theorem 2.4.** In this subsection, we provide a proof for our second main result. For this, we use Propositions 4.3 and 4.4.

• Case A ($M=1$): Suppose that for each $z\in\Omega$, let $h(t,\theta,\nu,z),\Gamma_{\pm1}(t,z),\partial_z h(t,\theta,\nu,z),\partial_z\Gamma_{\pm1}(t,z)$ be solutions to (4.3) and (4.4), respectively.

For $m = 1$, we can apply Propositions 4.3 and 4.4 to obtain

$$\begin{aligned} ||||\partial_z R||||_a &\leqslant C'||\partial_z h^{in}||_{\lambda_0,\gamma} \\ &\quad + C'C \sum_{j_1+j_2\leqslant 1} (||\partial_z^{j_1} h^{in}||_{\lambda_0,\gamma}||||\partial_z^{j_2} R||||_a + |||\partial_z^{j_1} h|||_a||||\partial_z^{j_2} R||||_a), \\ |||\partial_z h|||_a &\leqslant C'||\partial_z h^{in}||_{\lambda_0,\gamma} + C'C \sum_{j_1+j_2\leqslant 1} ||||\partial_z^{j_1} R||||_a|||\partial_z^{j_2} h|||_a, \end{aligned}$$

where C is a bound for $|\partial_z^l \kappa(z)|$. This implies

$$\begin{aligned} &(1 - C'C(||h^{in}||_{\lambda_0,\gamma} + |||h|||_a))||||\partial_z R||||_a \\ &\qquad \leqslant C'||\partial_z h^{in}||_{\lambda_0,\gamma} + C'C||||R||||_a\Big(\sum_{j=0}^{1} |||\partial_z^j h|||_a + |||\partial_z^j h^{in}|||_{\lambda_0,\gamma}\Big), \end{aligned} \tag{4.20}$$

and

$$(1 - C'C||||R||||_a)|||\partial_z h|||_a \leqslant C'||\partial_z h^{in}||_{\lambda_0,\gamma} + C'C|||h|||_a(||||\partial_z R||||_a + ||||R||||_a). \tag{4.21}$$

Substituting (4.21) into (4.20) gives

$$\begin{aligned} &\Big\{1 - C'C(||h^{in}(z)||_{\lambda_0,\gamma} + |||h|||_a) - \frac{C'^2C^2}{1 - C'C||||R||||_a}||||R||||_a|||h|||_a\Big\}||||\partial_z R||||_a \\ &\leqslant C'||\partial_z h^{in}(z)||_{\lambda_0,\gamma} + C'C||||R||||_a \\ &\times \Big(|||h|||_a + \frac{C'||\partial_z h^{in}(z)||_{\lambda_0,\gamma} + C'C|||h|||_a||||R||||_a}{1 - C'C||||R||||_a} + \sum_{j=0}^{1} |||\partial_z^j h^{in}(z)|||_{\lambda_0,\gamma}\Big). \end{aligned}$$

Choosing C sufficiently small such that

$$1 - C'C\Big(||h^{in}(z)||_{\lambda_0,\gamma} + |||h|||_a\Big) - \frac{C'^2C^2}{1 - CC||||R||||_a}||||R||||_a|||h|||_a < 1$$

then

$$\begin{aligned} &||||\partial_z R||||_a \\ &\leqslant \frac{C'||\partial_z h^{in}||_{\lambda_0,\gamma} + C'C||||R||||_a\Big(|||h|||_a + \frac{C'||\partial_z h^{in}||_{\lambda_0,\gamma}+C'C|||h|||_a||||R||||_a}{1-C'C||||R||||_a} + \sum_{j=0}^{1} |||\partial_z^j h^{in}|||_{\lambda_0,\gamma}\Big)}{1 - \{1 - CK(||h^{in}||_{\lambda_0,\gamma} + |||h|||_a) - \frac{C^2K^2}{1-CK||||R||||_a}||||R||||_a|||h|||_a\}}. \end{aligned}$$

Note that the bound is a constant, so is $|||\partial_z h|||_a = |||\partial_z^m (f(t, \theta + t\nu, \nu, z))|||_a$.

• Case B ($M > 1$): We basically use the method of induction on M. Suppose that the desired results hold for $M \leqslant N - 1$. Next, we verify the desired estimate for $M = N$. As before, for $m = M$, applying Propositions 4.3 and 4.4 gives

$$\begin{aligned} ||||\partial_z^M R||||_a &\leqslant C'||\partial_z^M h^{in}||_{\lambda_0,\gamma} \\ &\quad + C'C \sum_{j_1+j_2\leqslant M} (||\partial_z^{j_1} h^{in}||_{\lambda_0,\gamma}||||\partial_z^{j_2} R||||_a + |||\partial_z^{j_1} h|||_a||||\partial_z^{j_2} R||||_a), \\ |||\partial_z^M h|||_a &\leqslant C'||\partial_z^M h^{in}||_{\lambda_0,\gamma} + C'C \sum_{j_1+j_2\leqslant M} ||||\partial_z^{j_1} R||||_a|||\partial_z^{j_2} h|||_a. \end{aligned}$$

Since

$$||\partial_z^j h^{in}(z)||_{\lambda_0,\gamma} < \infty \quad \text{for } j \leqslant M \quad \text{and} \quad ||||\partial_z^j R||||_a < \infty, |||\partial_z^j h|||_a < \infty \quad \text{for } j < M,$$

by choosing C sufficiently small such that

$$1 - C'C(||h^{in}(z)||_{\lambda_0,\gamma} + |||h|||_a) - \frac{C'^2C^2}{1 - C'C||||R||||_a}||||R||||_a|||h|||_a < 1,$$

one has

$$||||\partial_z^M R||||_a, |||\partial_z^M h|||_a < C'',$$

where C'' is a constant.

We choose $0 < \lambda_1 < \lambda_2 < \lambda_0 - a\pi/2$ and use (4.13) for $\lambda = \lambda_1, p = \gamma$ to get

$$||\partial_z^m L_t h(t)||_{\lambda_1,\gamma} \leqslant C' \sum_{j=0}^{m} \left[(\partial_z^j R)_{\lambda_1,0}(t)||\partial_z^{m-j} h(t)||_{\lambda_1,\gamma+1} + (\partial_z^j R)_{\lambda_1,\gamma}(t)||\partial_z^{m-j} h(t)||_{\lambda_1,1}\right].$$

By definition of above elements and (4.14), one obtains

$$\begin{aligned}
&(\partial_z^j R)_{\lambda_1,0}(t,z) \leqslant C'(\partial_z^j R)_{\lambda_2,0}(t,z)e^{(\lambda_1-\lambda_2)t},\\
&||\partial_z^{m'-j} h(t,z)||_{\lambda_1,\gamma+1} \leqslant \frac{1}{\lambda_2 - \lambda_1}||\partial_z^{m'-j} h(t,z)||_{\lambda_2,\gamma},\\
&||\partial_z^{m'-j} h(t,z)||_{\lambda_1,1} \leqslant \frac{1}{\lambda_2 - \lambda_1}||\partial_z^{m'-j} h(t,z)||_{\lambda_2,0} \leqslant \frac{1}{\lambda_2 - \lambda_1}||\partial_z^{m'-j} h(t,z)||_{\lambda_2,\gamma}.
\end{aligned}$$

Therefore, the following inequality holds:

$$||\partial_z^{m'} L_t h(t,z)||_{\lambda_1,\gamma} \leqslant \frac{C'}{\lambda_2 - \lambda_1}\left(\sum_{j=0}^{m'} ||||\partial_z^j R||||_a|||\partial_z^{m'-j} h|||_a\right) e^{(\lambda_1-\lambda_2)t}.$$

This implies

$$\begin{aligned}
&||\partial_z^m h^{in}(z)||_{\lambda_1,\gamma} + \sum_{m'=0}^{m} \binom{m}{m'} |\partial_z^{m-m'}\kappa(z)| \int_{R^+} ||\partial_z^m L_t h(t,z)||_{\lambda_1,\gamma} dt\\
&\qquad \leqslant ||\partial_z^m h^{in}(z)||_{\lambda_0,\gamma} + \frac{C'}{(\lambda_2 - \lambda_1)^2}\left(\sum_{j_1+j_2\leqslant m} ||||\partial_z^{j_1} R||||_a|||\partial_z^{j_2} h|||_a\right) < \infty.
\end{aligned}$$

Now, we set

$$\partial_z^m h_\infty(\theta,\nu,z) := \partial_z^m h^{in} + \sum_{m'=0}^{m} \binom{m}{m'} |\partial_z^{m-m'}\kappa(z)| \int_{R^+} \partial_z^{m'} L_t h(t) dt.$$

Then, it is well-defined in $\mathcal{X}_{\lambda_1,\gamma}$ and $||\partial_z^m h_\infty||_{\lambda_1,\gamma} < \infty$ with

$$||\partial_z^m h_\infty - \partial_z^m h(t)||_{\lambda_1,\gamma} \leqslant \frac{C''}{(\lambda_2 - \lambda_1)^2} e^{(\lambda_1-\lambda_2)t},$$

which converges exponentially fast. This completes the proof.

Remark 4.1. *In the course of the proof, one might need to assume*

$$||||\partial_z^M R||||_a < \infty, \quad |||\partial_z^M h|||_a < \infty.$$

Then, one can apply Propositions 4.3 and 4.4 to get the uniform bound. By performing a similar iteration as in Theorem 1 in [6]*, one can get this condition. Since this is not our main goal, we omit this part.*

5. Conclusion

In this paper, we have conducted a local sensitivity analysis for Landau damping to the random kinetic Kuramoto equation in a small coupling strength regime. The kinetic Kuramoto equation can be derived from the random Kuramoto model via the mean-field limit. As aforementioned in Introduction, the realistic modelings of physical phenomena involve many uncertainties in , for examples, physical domain, boundary conditions, transport coefficients, etc. Thus, uncertainties in problem setting are ubiquitous and need to be quantified.

The stability of the incoherent state for the Kuramoto model has been addressed in Kuramoto's seminal work in a mean-field setting. In particular, he found that the incoherent state can be linearly stable in the subcritical coupling strength regime, whereas it is linearly unstable in a supercritical regime. The extension of this linear results to the original nonlinear equation is called the Kuramoto conjecture. Recently, there were lots of progress in this direction. In this paper, we revisit the nonlinear stability problem of the incoherent solution in the framework of "*uncertainty quantification:*, by conducting the local sensitivity analysis. Our finding shows that as long as the coupling strength and initial datum are sufficiently regular and small, the Landau damping is still true even with random uncertain perturbation, with the same decay in time as in the deterministic case.

There are several interesting issues that we have not addressed in this paper, for example the random effects on the nonlinear instability of the incoherent state in the supercritical coupling regime, and dynamic features in the vicinity of critical coupling strength, etc. We point out that these issues are also not clearly understood even in the deterministic setting. We leave these studies for a future work.

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