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BOUNDARY CONTROL OF VLASOV–FOKKER–PLANCK EQUATIONS

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Abstract. We introduce a novel Lyapunov function for stabilization of linear Vlasov–Fokker–Planck type equations with stiff source term. Contrary to existing results relying on transport properties to obtain stabilization, we present results based on hypocoercivity analysis for the Fokker–Planck operator. The existing estimates are extended to derive suitable feedback boundary control to guarantee the exponential stabilization. Further, we study the associated macroscopic limit and derive conditions on the feedback boundary control such that in the formal limit no boundary layer exists.

1 Introduction. We are interested in stabilization of kinetic partial differential equations by boundary feedback laws and its associated macroscopic equations. As a prototype we consider the Vlasov–Fokker–Planck (VFP) equation with small Knudsen number and we derive suitable conditions on the boundary control to obtain damping of perturbations of the steady state exponentially fast in time. Kinetic partial differential equations belong to the class of hyperbolic balance laws and enjoy the particular feature of linearity in transport direction. However, the studied class usually has a (stiff) source term described by a small Knudsen number. As the Knudsen number tends to zero, the solution to the VFP equation converges to a solution to (macroscopic) partial differential equations for mass and momentum [28]. We are interested in stabilization results for *all* ranges of Knudsen numbers.

The theoretical and numerical discussion of stabilization properties of hyperbolic balance laws has recently been of interest in the mathematical and engineering community and we refer to [3, 5] for a survey and more references. Typically,

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its application has been on the stabilization of flows (on networks) governed by shallow–water or isothermal equations [16, 2, 14, 19, 4, 20, 17, 26, 10, 18]. The *core underlying tool* for the study of these problems are Lyapunov functions for deviation from steady states in suitable norms, e.g. L_x^2, H_x^2. A major breakthrough has been the design of suitable weighted Lyapunov functions that allow for an exponential decay (in time) of the such deviations. Exponential decay of a continuous Lyapunov function under the so-called *dissipative* boundary conditions has been proven in [8, 11, 6, 7]. Comparisons to other stability concepts are presented in [9]. Stability with respect to the H_x^2-norm yields stability of the nonlinear system [7, 3]. Recently, similar results for numerical schemes have been established [1, 30] and the theory has been extended to balance laws with mixed source terms [21]. In all cases, the exponential decay is obtained as *interplay* of the weights of the Lyapunov function and the linear transport property of the underlying linearized hyperbolic systems. In addition, most analytical results do not state explicitly decay rates and the influence of the source term is assumed to be *small* [8]. In particular, for kinetic equations with stiff forcing term the latter assumption is in general not fulfilled and it has been shown in [22] that a direct extension of the Lyapunov function to a kinetic equation will *not* yield exponential decay. Therefore, we propose here a *novel* Lyapunov function E_h for a particular class of VFP equations. This function has been introduced as energy norm for nonlinear Vlasov-Poisson-Fokker-Planck equation on the whole space without stiff source term in [23] and we generalize the results to our setting. This function does not include particular weights, but mixed derivatives and the corresponding decay estimate are used to yield hypocoercivity estimates of the underlying differential operator. The advantage of the novel Lyapunov function is the treatment of the case of small Knudsen numbers and even the limit of vanishing Knudsen number. We give conditions on boundary feedback stabilization in terms of the Knudsen number including the limit. A particular difficulty for the boundary problem of kinetic equations with small Knudsen number is the treatment of possible boundary layers. The stabilization in the interior of the domain follows from hypercoercivity estimates. Related work on similar Lyapunov functionals have been used also for results on Uncertainty Quantification problem for Fokker-Planck related equation with small Knudsen number *but* only in the case of periodic boundary conditions and without stabilizing feedback conditions [24, 25].

This paper is organized as follows. We review the Vlasov-Fokker-Planck equation with feedback control in Section 2.1, and present our main results in Section 2.2. The proof of the main results are given in Section 3. The outline of the proof is given in Section 3.1. The proof is divided into three parts. In Section 3.3, we give a sufficient condition such that the feedback control will not yield a boundary layer as the Knudsen number goes to zero. Then we derive an energy estimation for the Lyapunov functional in Section 3.4. Based on the above two sections, in Section 3.5 we give sufficient condition on the boundary such that any perturbation will exponentially decay .

Gallery of Notations 1.1. We introduce the following notation frequently used within the manuscript. Let $t \geq 0$ denote the temporal variable, $\Omega = [0,1] \times \mathbb{R}$ be the phase space of space x and velocity v, respectively. We denote by $H_x^{-1}, L_x^2, H_x^1, \dots$ the Lebesgue- and Sobolev spaces in x variable and analogously in v. Further, we define the following norms on the phase space Ω : For suitable integrable functions $g = g(x,v)$ and $\sigma = \sigma(x)$ we introduce their corresponding L^2- and higher–order norms:

$$
\begin{aligned}
&\|g\|^2 := \|g\|^2_{L^2_{x,v}} = \int_\Omega g^2\, dxdv,\ \|\sigma\|^2 := \|\sigma\|^2_{L^2_x} = \int_0^1 \sigma^2 dx,\\
&\|g\|^2_\omega := \|g\|^2 + \|\partial_v g\|^2 + \|vg\|^2,\\
&\|g\|^2_V := \|g\|^2 + \|\partial_x g\|^2 \text{ and } \|g\|^2_{V,\omega} := \|g\|^2_\omega + \|\partial_x g\|^2_\omega .
\end{aligned}
$$

We denote by $v \to M(v)$ the global Maxwellian on $\mathbb{R}$

$$
M(v) := \frac{1}{\sqrt{2\pi}} e^{-\frac{|v|^2}{2}}.
$$

The linearized Fokker–Planck collision operator is given by $\mathcal{L} : H^1_{x,v} \to H^{-1}_{x,v}$,

$$
\langle \mathcal{L}g, h\rangle = -\int_\Omega M \partial_v \left(\frac{g}{\sqrt{M}}\right) \partial_v \left(\frac{h}{\sqrt{M}}\right) dxdv. \tag{1.1}
$$

The operator $\mathcal{L}$ satisfies the local coercivity property [15, 24] for $g = g(x,v) \in H^1_{x,v}$ and $\|g\|_\omega < \infty$ and a constant $\lambda = \frac{1}{4}$:

$$
-\langle \mathcal{L}g, g\rangle \geq \lambda \|(1-\Pi)g\|^2_\omega . \tag{1.2}
$$

$$
-\langle \mathcal{L}g, g\rangle \geq \lambda \left(\|\partial_v(1-\Pi)g\|^2 + \|v(1-\Pi)g\|^2 - \|(1-\Pi)g\|^2\right). \tag{1.3}
$$

Here, we denote by $\langle \cdot, \cdot \rangle$ the L^2-scalar product in x and v and by $\Pi : L^2 \to \mathcal{N}(\mathcal{L})$ a weighted projection operator onto the null space of $\mathcal{L}$, i.e., for $g = g(x,v) \in L^2_{x,v}$,

$$
\Pi g := \left(\int_\mathbb{R} g\sqrt{M}\, dv\right)\sqrt{M}. \tag{1.4}
$$

For all $\sigma \in H^1_x$, by the Poincare inequality, there always exists constant $C \geq 1$ such that

$$
C\|\partial_x \sigma\|^2 \geq \left\|\sigma - \int_0^1 \sigma dx\right\|^2.
$$

If one adds $\|\partial_x\sigma\|^2$ to both sides of the above inequality, then one has

$$
(C+1)\|\partial_x\sigma\|^2 \geq \left\|\sigma - \int_0^1 \sigma dx\right\|^2_V \implies \|\partial_x\sigma\|^2 \geq \frac{C_s}{2}\left\|\sigma - \int_0^1 \sigma dx\right\|^2_V, \tag{1.5}
$$

for a constant $0 < C_s \leq 1$.
We define by $h(t,v,x)$ the weighted kinetic distribution function

$$
h(t,v,x) := \frac{f(t,v,x)}{\sqrt{M(v)}}. \tag{1.6}
$$

Denote by $A(t), B(t), A_x(t), B_x(t)$ non-negative functions of time t related to the boundary control of $h = h(t, x, v)$ at $x \in \{0, 1\}$

$$\begin{aligned} A(t) &:= -\int_{-\infty}^{0} \frac{v}{2} h^2(t,0,v)dv, \quad B(t) := \int_0^{\infty} \frac{v}{2} h^2(t,1,v)dv, \\ A_x(t) &:= -\int_{-\infty}^{0} \frac{v}{2} (\partial_x h)^2(t,0,v)dv, \quad B_x(t) := \int_0^{\infty} \frac{v}{2} (\partial_x h)^2(t,1,v)dv. \end{aligned} \tag{1.7}$$

The function $C_B(t)$ is defined by

$$C_B(t) := \frac{\left(\sqrt{A(t)} - \sqrt{B(t)}\right)^2 + \left(\sqrt{A_x(t)} - \sqrt{B_x(t)}\right)^2}{2(\sqrt{A(t)} + \sqrt{B(t)})(\sqrt{A_x(t)} + \sqrt{B_x(t)})}. \tag{1.8}$$

The energy $E_h(t)$ at time t and for some positive small parameter $\epsilon > 0$ is defined by

$$E_h(t) := \frac{1}{2} \|h(t,\cdot,\cdot)\|_V^2 + \epsilon a \langle u(t,\cdot), \partial_x \sigma(t,\cdot) \rangle, \tag{1.9}$$

for any constant $a > 0$, suitable functions h, u, σ and where $\langle \cdot, \cdot \rangle$ denotes the L^2-scalar product in x. We will establish the estimate

$$|\langle u, \partial_x \sigma \rangle| \le \frac{1}{2} \left(\|u\|^2 + \|\partial_x \sigma\|^2 \right) \le \frac{1}{2} \|h(t,\cdot,\cdot)\|_V^2. \tag{1.10}$$

Further, we establish that for each fixed time t, $E_h(t)$ and $\|h(t,\cdot,\cdot)\|_V$ are two equivalent norms. This follows from the inequalities

$$\frac{1-a\epsilon}{2} \|h(t,\cdot,\cdot)\|_V^2 \le E_h(t) \le \frac{1+a\epsilon}{2} \|h(t,\cdot,\cdot)\|_V^2. \tag{1.11}$$

2 The linear Vlasov-Fokker-Planck (VFP) equation and the main result. We are interested in boundary feeback stabilization of equation (2.1) in the sense of Definition 2.1 and its associated macroscopic limit at $\epsilon \to 0$. Hence, we consider the case where the dynamics given by (2.1) is stabilized at equilibrium $f^* = f^*(x, v)$ by suitable boundary controls (2.2). In order to derive stabilization results we consider a linear perturbation $\tilde{f} = f + f^*$ of an equilibrium state f^*, i.e., we assume that $\partial_t f^*(t,\cdot,\cdot) = 0$.

Note that $\tilde{f}$ is supposed to be a kinetic probability density and therefore

$$\int_\Omega \tilde{f}(t,x,v)dxdv = 1$$

for all $t \ge 0$ holds true. This implies that perturbation f is required to fulfill $\int f(t,x,v)dxdv = 0$. Also, the perturbation f fulfills (2.1)-(2.2) due to the linearity of the Fokker–Planck operator and boundary conditions.

Hence, we will focus our discussion on the stabilization of (2.1) for a perturbation denoted by $f = f(t,x,v)$. Note that the same model (2.1) also appears as formal first–order approximation to nonlinear VFP equations. However, we do not discuss the stabilization of nonlinear VFP equations in the following.

2.1 Definition of the problem. Consider the linear Vlasov-Fokker-Planck equation for $f = f(t,x,v)$

$$\epsilon \partial_t f + v \partial_x f - E \partial_v f = \frac{1}{\epsilon} \mathcal{F} f, \quad (x,v) \in \Omega, \tag{2.1}$$

and initial data $f(0,x,v) = f_0(x,v)$. The linear Fokker Planck operator $\mathcal{F}$ is given by

$$\mathcal{F}f = \partial_v \left(M \partial_v \left(\frac{f}{M} \right) \right),$$

where M is the global Maxwellian. The given bounded function $(t,x) \to E(t,x)$ models the electric field. The mean free path (or Knudsen number) is $0 < \epsilon \leq 1$. The equation is accompanied by (feedback) boundary conditions specified by the matrix $K(\epsilon)$ (2.3):

$$\begin{cases} f(t,0,v) = k_{00}(\epsilon) f(t,0,-v) + k_{10}(\epsilon) f(t,1,v), & v > 0, \\ f(t,1,v) = k_{01}(\epsilon) f(t,0,v) + k_{11}(\epsilon) f(t,1,-v), & v < 0, \end{cases} \tag{2.2}$$

The matrix $K(\epsilon)$ is given by

$$K(\epsilon) = \begin{bmatrix} k_{00}(\epsilon) & k_{10}(\epsilon) \\ k_{01}(\epsilon) & k_{11}(\epsilon) \end{bmatrix}. \tag{2.3}$$

We denote the limit of the coefficients K^0 as ϵ tends to zero by

$$K^0 := \begin{bmatrix} k^0_{00} & k^0_{10} \\ k^0_{01} & k^0_{11} \end{bmatrix} := \lim_{\epsilon \to 0} \begin{bmatrix} k_{00}(\epsilon) & k_{10}(\epsilon) \\ k_{01}(\epsilon) & k_{11}(\epsilon) \end{bmatrix}. \tag{2.4}$$

The existence of the solution to the VFP equation in the whole space has been studied for example in [13, 27]. However, results on bounded domains do not exist so far. In this paper, we assume there exists a solution to the VFP equation (2.1), (2.2) in distributional sense and we assume in the following that f has at least regularity $f(t,\cdot,\cdot) \in H^1_{x,v}$.

Definition 2.1. *For fixed $\epsilon > 0$, we call the system (2.1), (2.2) exponentially stabilizable at equlibrium $f^* \equiv 0$, if there exists a matrix $K(\epsilon)$ for any initial data $f_0 \in V$, $\int_\Omega f_0 dxdv = 0$ and $\|f_0\|_V < \infty$, such that*

$$\|f(t,\cdot,\cdot)\|_V \leq C_0 \|f_0\|_V \exp(-C_1 t),\ t \geq 0,$$

for some non–negative constants C_0 and C_1.

We briefly comment on the notion of stabilization that is an extension of the L^2-stabilization for hyperbolic balance laws [8]. The boundary condition (2.2) is a state feedback determined by $K(\epsilon)$. The equilibrium $f^* = 0$ is a solution to the stationary Vlasov–Fokker-Planck equation for all ϵ and stabilization ensures that $f \to f^*$ exponentially fast in time in the norm $\|\cdot\|_V$. The energy E_h defined by equation (1.9) plays the role of the Lyapunov function in [8]. Note that contrary to the Lyapunov function, **no** exponential weights are required here. The dissipation is due to the hypercoercivity of the Fokker–Planck operators as shown in the main result of Theorem 2.4. As indicated in the definition the choice of K may depend on the value of ϵ and so may C_0 and C_1, too. An interesting question we discuss below is if the decay rate C_1 deteriorates as ϵ becomes smaller. We show that in fact, C_1 can be chosen *independently* of ϵ provided that the coefficients of $K(\epsilon)$ fulfill additional conditions, see Section 3.2.

In the following section we establish conditions on the operators and prove the stabilization property in Theorem 2.4. For the analysis it turns out to be advantageous to discuss the properties of equation (2.1) in terms of the weighted kinetic distribution (1.6), i.e.,

$$h(t,x,v) = \frac{f(t,x,v)}{\sqrt{M(v)}}, \quad \forall (x,v) \in \Omega, t \geq 0. \tag{2.5}$$

Provided that f fulfills (2.1) in distributional sense, a simple formal computation shows that $h = h(t,x,v)$ fulfills in this sense

$$\epsilon\partial_t h + v\partial_x h - \frac{1}{\epsilon}\mathcal{L}h = E\left(\partial_v - \frac{v}{2}\right)h, \quad (x,v) \in \Omega, t \geq 0. \tag{2.6}$$

The initial conditions and boundary conditions are

$$h(0,x,v) = \frac{f_0(x,v)}{\sqrt{M(v)}} \ \forall (x,v) \in \Omega \tag{2.7}$$

and for all $t \geq 0$ and for all ϵ

$$\begin{cases} h(t,0,v) = k_{00}(\epsilon)h(t,0,-v) + k_{10}(\epsilon)h(t,1,v), & v > 0; \\ h(t,1,v) = k_{01}(\epsilon)h(t,0,v) + k_{11}(\epsilon)h(t,1,-v), & v < 0, \end{cases} \tag{2.8}$$

respectively. The operator $\mathcal{L}$ is defined by equation (1.1). The set of equations (2.6) – (2.8) are referred to the *microscopic equations* or kinetic equations, since they describe the evolution of the density function h. For given function $h(t,x,v)$ the density $\sigma = \sigma(t,x)$ and flux $u = u(t,x)$ are defined by

$$\sigma(t,x) = \int_{\mathbb{R}} h(t,x,v)\sqrt{M}\,dv = \int_{\mathbb{R}} f(t,x,v)dv, \tag{2.9}$$

$$u(t,x) = \int_{\mathbb{R}} h(t,x,v)v\sqrt{M}\,dv = \int_{\mathbb{R}} vf(t,x,v)dv. \tag{2.10}$$

The *macroscopic equations* describe the evolution of the density and flux. Upon multiplication of the microscopic equation by $\sqrt{M}$ and $v\sqrt{M}$ and integration on v, we formally obtain macroscopic equations as evolution for σ and u respectively. In particular, we have

$$\epsilon\partial_t\sigma + \partial_x u = 0, \tag{2.11}$$

and for the flux we obtain

$$\epsilon\partial_t u + \partial_x\sigma + \int v^2\sqrt{M}(1-\Pi)\partial_x h dv + \frac{1}{\epsilon}u = -E\sigma. \tag{2.12}$$

The latter equation is obtain by the following computation

$$\begin{aligned} 0 = & \int_{\mathbb{R}} \epsilon\partial_t hv\sqrt{M} + v^2\sqrt{M}\partial_x h dv + \frac{1}{\epsilon}\int_{\mathbb{R}} M\partial_v(v)\partial_v\left(\frac{h}{\sqrt{M}}\right)dv - E\int_{\mathbb{R}} v\partial_v(h\sqrt{M})dv \\ = & \epsilon\partial_t u + \int v^2\sqrt{M}\partial_x h dv + \int_{\mathbb{R}}\frac{1}{\epsilon}\frac{h}{\sqrt{M}}\partial_v M dv + E\int_{\mathbb{R}} h\sqrt{M}dv \\ = & \epsilon\partial_t u + \int v^2\sqrt{M}\partial_x h dv + \frac{1}{\epsilon}\int_{\mathbb{R}} h\sqrt{M}v dv + E\sigma \\ = & \epsilon\partial_t u + \int v^2\sqrt{M}\partial_x(1-\Pi)h + \partial_x\sigma + \frac{1}{\epsilon}u + E\sigma. \end{aligned} \tag{2.13}$$

The last line follows due to the definition of Π in equation (1.4), i.e.,

$$\Pi h = \int h\sqrt{M}dv\sqrt{M} = \sigma\sqrt{M} \tag{2.14}$$

and

$$\int_{\mathbb{R}} v^2 \sqrt{M} \partial_x \sigma \sqrt{M} dv = \partial_x \sigma. \tag{2.15}$$

Since by definition of Π and σ we obtain that the operators Π and $1-\Pi$ are perpendicular to each other under $L^2_{x,v}$, that is,

$$\|h\|^2 = \|\Pi h\|^2 + \|(1-\Pi)h\|^2 = \|\sigma\|^2 + \|(1-\Pi)h\|^2 . \tag{2.16}$$

The first equality holds true due to

$$\int_{\Omega} (\Pi h)(1-\Pi) h dx dv = \int_{\Omega} \sigma \sqrt{M} (h - \sigma \sqrt{M}) dx dv = 0, \tag{2.17}$$

$$\int_{\Omega} (\Pi h) dx dv = \int \sigma^2 \left(\int_{\mathbb{R}} M dv \right) dx = \|\sigma\| . \tag{2.18}$$

2.2 Theoretical results. In this section we present conditions on the matrix $K(\epsilon)$ and the external field E, such that the dynamics given by equation (2.1) is stabilizable in the sense of Definition 2.1.

For the external electric field we assume sufficient regularity and growth conditions:

Assumption 2.2. *The electric field $E : \mathbb{R}^+_0 \times [0,1] \to \mathbb{R}$ is assumed to be sufficiently smooth, for all time $t \in \mathbb{R}^+_0$, and its derivatives are bounded by the following constants independent of t:*

$$\|E(t,\cdot)\|_{L^\infty_x}, \|\partial_x E(t,\cdot)\|_{L^\infty_x}, \left\|\partial_x^3 E(t,\cdot)\right\|_{L^\infty_x} \le \frac{C_E}{2} \le \frac{\lambda C_s}{16}, \tag{2.19}$$

where λ, C_s are constants defined in (1.2), (1.5). In addition, at the boundary $x \in \{0,1\}$, we assume that the electric field and its second derivative are periodic:

$$E(t,0) = E(t,1) = 0, \quad \partial_x^2 E(t,0) = \partial_x^2 E(t,1). \tag{2.20}$$

As discussed in Section 2 the perturbations f and f_0 of the steady state are required to have zero mean.

Assumption 2.3. *Assume for initial data $f_0(\cdot,\cdot) \in V$,*

$$\int f_0(x,v)\, dx\, dv = 0. \tag{2.21}$$

Next, we present exponential stability results in the case of small and large electric field governed by C_E as well as in the limit case $\epsilon = 0$. The latter corresponds to stabilization of the formal hydrodynamic limit. We refer to the remarks below the theorems for some discussion as well as to Lemma 3.4 for details on the imposed assumptions. The next Theorem states the exponential stability for solutions with small electric field.

Theorem 2.4. *Assumptions 2.2 and 2.3 hold true and let $\epsilon > 0$. Further, assume that the upper bound of the electric field C_E given by equation (2.19) and the coefficients of the matrix $K(\epsilon)$ given by equation (2.3) satisfy*

$$\begin{aligned} &C_E < \frac{3C_s}{2} - \frac{3C_s}{2 + C_B(t)}, \\ &0 \le k_{11}(\epsilon) = k_{00}(\epsilon) \le 1,\ k_{01}(\epsilon) = k_{10}(\epsilon) = 1 - k_{00}(\epsilon), \end{aligned} \tag{2.22}$$

Then, any weak solution $f \in C^0(\mathbb{R}^0_+; H^1_{x,v})$ to (2.1) with boundary condition (2.2) will decay exponentially in time to zero with rate given by

$$\left\| \frac{f(t)}{\sqrt{M}} \right\|_V^2 \leq \frac{5}{4} \left\| \frac{f(0)}{\sqrt{M}} \right\|_V^2 e^{-2\xi t}, \tag{2.23}$$

where

$$\xi = \min\left\{ \frac{\lambda - C_E - 4a}{\epsilon^2}, \frac{a(3C_s - 2C_E) - 4C_E}{8} \right\} > 0. \tag{2.24}$$

Here, $\frac{4C_E}{3C_s - 2C_E} < a < \min\left\{C_B(t), \frac{\lambda - C_E}{4}\right\}$ and the functions $C_B(t), C_s, \lambda$ are defined in equations (1.8), (1.5) and (1.2), respectively.

If the electric field is large, we observe that we have less degrees of freedom to chose a linear feedback boundary condition. The following theorem states sufficient conditions on the feedback matrix $K(\epsilon)$ for exponential stability and possibly large electric fields.

Theorem 2.5. *Assume Assumptions 2.2 and 2.3 hold true. Further, assume that the upper bound of the electric field C_E given by equation (2.19) and the coefficients of the matrix $K(\epsilon)$ given by equation (2.3) satisfies,*

$$C_E \geq \frac{3C_s}{2} - \frac{3C_s}{2 + C_B(t)}, \; k_{11}(\epsilon) = k_{00}(\epsilon) = 0, \; k_{01}(\epsilon) = k_{10}(\epsilon) = 1. \tag{2.25}$$

Then, any weak solution $f \in C^0(\mathbb{R}^0_+; H^1_{x,v})$ to (2.1) with boundary condition (2.2) will decay exponentially in time to zero with rate given by

$$\left\| \frac{f(t)}{\sqrt{M}} \right\|_V^2 \leq \frac{5}{4} \left\| \frac{f(0)}{\sqrt{M}} \right\|_V^2 e^{-2\xi t}, \tag{2.26}$$

where for $\epsilon > 0$

$$\xi = \min\left\{ \frac{\lambda - C_E - 4a}{\epsilon^2}, \frac{a(3C_s - 2C_E) - 4C_E}{8} \right\} > 0,$$

where $\frac{4C_E}{3C_s - 2C_E} < a < \frac{\lambda - C_E}{4}$ and the values C_s, λ are defined in equations (1.5) and (1.2), respectively.

The formal hydrodynamic limit is obtained for $\epsilon \to 0$. In this case both theorems yield the same condition on the matrix $lim_{e \to 0} K(\epsilon) = K(0)$. Namely, in the limit we obtain

$$K(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This feedback matrix describes in fact periodic boundary conditions without additional damping. Hence, exponential decay of solutions is guaranteed independent of ϵ if periodic conditions are prescribed. However, the decay rate deteriorates. Hence, the previous results show that the only asymptotic preserving feedback boundary conditions are the periodic boundary conditions. The following lemma is a consequence of the previous theorems.

Lemma 2.6. *Let Assumptions 2.2 and 2.3 hold true and let $\epsilon = 0$. Assume that*

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and E fulfills assumption (2.19). Then, any weak solution $f \in C^0(\mathbb{R}^0_+; H^1_{x,v})$ to (2.1) with boundary condition (2.2) will decay exponentially in time to zero with rate given by

$$\left\| \frac{f(t)}{\sqrt{M}} \right\|_V^2 \leq \frac{5}{4} \left\| \frac{f(0)}{\sqrt{M}} \right\|_V^2 e^{-2\xi t}, \tag{2.27}$$

where

$$\xi = \frac{a(3C_s - 2C_E) - 4C_E}{8} > 0. \tag{2.28}$$

Here, $\frac{4C_E}{3C_s - 2C_E} < a < \min\left\{C_B(t), \frac{\lambda - C_E}{4}\right\}$ and the functions $C_B(t), C_s, \lambda$ are defined in equations (1.8), (1.5) and (1.2), respectively.

Theorems (2.4) and (2.5) will be proven in the forthcoming section. Lemma (2.6) is a simple consequence. Prior to proceeding, some remarks are in order.

- The result guarantees stabilization exponentially fast in time provided that the size of the electric field is suitable small. This case is covered by assumption (2.22). In case of potentially large fields we may only stabilize the dynamics by periodic boundary conditions as given by equation (2.25).
- Due to the linearity of the underlying equations we also obtain that the boundary condition (2.3) translates to a condition on $f^* + f$ where f^* is any equilibrium state.
- The question whether we obtain a boundary layer for small values of ϵ will be addressed in the beginning of Section 3.
- In all cases we have $\xi > 0$, which is guaranteed by $\frac{4C_E}{2C_s - 2C_E} < a < \frac{\lambda - C_E}{4}$ in both theorems. In Theorem 2.5, the smallness of $C_E \leq \frac{\lambda C_s}{8}$ in Assumption 2.2 guarantees that there always exists constants $a > 0$ in the range of $(\frac{4C_E}{2C_s - 2C_E}, \frac{\lambda - C_E}{4})$. In Theorem 2.4, we further require $\frac{4C_E}{2C_s - 2C_E} < a < C_B$, the existence of $a > 0$ can be guaranteed by the additional assumption in (2.22) that $C_E < \frac{3C_s}{2} - \frac{3C_s}{2 + C_B(t)}$.
- Stated in terms of the boundary feedback matrix $K(\epsilon)$ it is clear that condition (2.25) is a particular case of condition (2.22) and hence included in Theorem 2.4. However, for positive value of ϵ, sufficiently small electric field C_E and under the additional assumption on $0 < a \leq C_B(t)$ we have a weaker condition on the boundary feedback and thus a wider range of possible feedback matrices $K(\epsilon)$. The additional condition on $a \leq C_B(t)$ in the following lemma needs to be combined with the condition $\frac{4C_E}{3C_s - 2C_E} < a < \frac{\lambda - C_E}{4}$ of Lemma 3.9. This shows that only if $C_E < \frac{3C_s}{2} - \frac{3C_s}{2 + C_B(t)}$, the existence of $a > 0$ is granted. Therefore, we have differentiated the results into two cases depending on the size of the electric fields C_E. This is detailed in Lemma (3.4).
- In Section 3.2 we show that the conditions imposed on the feedback $K(\epsilon)$ can not be changed substantially provided exponential stability is obtained using the Lyapunov function E_h defined by (1.9).
- In the limit $\epsilon \to 0$ in Theorem 2.4, we observe that only for periodic boundary conditions exponential stability is obtained. The size C_E of the electric field is independent of this condition. Hence, only periodic boundary conditions are asymptotically preserving for stabilization of the considered Vlasov–Fokker–Planck dynamics. It is still an open question if for other Lyapunov- (or energy) functionals exponential stabilization results can be obtained.

3 Proof of the main results. The proofs of the two theorems are similar and we therefore will focus the proof of Theorem 2.4 and point out the modifications for the proof of Theorem 2.5.

3.1 Outline of the proof and sufficient conditions on K for exponential decay. Prior to the detailed analysis of the proof, we discuss the dependence on the Knudsen number ϵ and derive necessary conditions on the boundary control matrix $K = K(\epsilon)$ to establish the exponential decay. The conditions will be summarized below together with some explanation of their origin.

We aim to find conditions for exponential stability possibly independent of the parameter ϵ. In Lemma 3.8 we formally derive the limiting equation of (2.1) for $\epsilon \to 0$ and we state a sufficient condition on K such that no boundary layer appears. Lemma 3.8 yields the following sufficient condition on K. To simplify the notation, we drop the dependence of the coefficients of K on ϵ if $\epsilon > 0$. The limiting case $K = K(0)$ is indicated in the coefficients as in equation (2.3) by a superscript zero.

Assumption 3.1. *We assume that the entries of $K = K(0)$ fulfill the following quadratic equations,*

$$\left(1 - k^0_{00}\right)\left(1 - k^0_{11}\right) = k^0_{10}k^0_{01}, \tag{3.1}$$

$$\left(1 + k^0_{00}\right)\left(1 + k^0_{11}\right) = k^0_{10}k^0_{01}. \tag{3.2}$$

The proof of Theorem 2.4 relies on estimates of the Lyapunov functional E_h defined in (1.9). The following energy estimate is proved in Lemma 3.9.

$$\begin{aligned}\partial_t E_h + \xi \left\|h\right\|_V^2 \le &- \int_{\mathbb{R}} \frac{v}{2}\left(h^2(t,1,v) - h^2(t,0,v)\right) dv \\ &- \int_{\mathbb{R}} \frac{v}{2}\left((\partial_x h)^2(t,1,v) - (\partial_x h)^2(t,0,v)\right) dv \\ &- a\left(u\partial_x u(t,1) - u\partial_x u(t,0)\right) + \frac{aC_s}{2\epsilon^2}\left(\int_0^t \left(u(s,1) - u(s,0)\right) ds\right)^2.\end{aligned} \tag{3.3}$$

In order to obtain exponential decay of $t \to \left\|h(t)\right\|^2$, we in particular require the right–hand side of the above equation to be non-positive. This in turn requires, in Lemma 3.10, a second condition on K making the last term of equation (3.39) vanish. This condition is as follows:

Assumption 3.2. *We assume that the entries of K fulfill the following conditions:*

$$k_{00} + k_{01} = 1, \quad k_{10} + k_{11} = 1.$$

Finally, we require, in Lemma 3.10, the first three terms of the RHS of equation (3.3) to be non–positive. This leads to the final conditions on K:

Assumption 3.3. *We assume that the entries of K fulfill the following condition, where A, B, A_x, B_x are functions of t defined in equation (1.7):*

$$\begin{aligned}&- 2k_{00}(1 - k_{00})\left(A + A_x\right) - 2k_{11}(1 - k_{11})\left(B + B_x\right) \\ &+ 2\left(\left|k_{11}(1 - k_{00})\right| + \left|k_{00}(1 - k_{11})\right|\right)\left(\sqrt{AB} + \sqrt{A_x B_x}\right) \\ &+ 4a\left(\left|1 - k_{11}\right|\sqrt{B} + \left|1 - k_{00}\right|\sqrt{A}\right)\left(\left|k_{00}\right|\sqrt{A_x} + \left|k_{11}\right|\sqrt{B_x}\right) \le 0,\end{aligned}$$

The above assumption is a technique assumption following from the energy estimate (3.3), so that one could has the exponential decay of $\|h(t)\|$ in time.

The next lemma shows that the conditions on $K(\epsilon)$ given in Theorem 2.4 and Theorem 2.5 fulfill the previous assumptions. Depending on the size of the electric field C_E we can have different conditions on $K(\epsilon)$.

Lemma 3.4. *Let $a \leq C_B(t)$ for all $t \geq 0$ and assume that the entries of $K(\epsilon)$ are given by equation (2.22). Then, the assumptions 3.2, 3.2 and 3.3 are fulfilled.*

Alternatively, assume the entries of $K(\epsilon)$ are given by equation (2.25). Then, the assumptions 3.1, 3.2 and 3.3 are fulfilled.

Proof. In fact, the only nontrivial part is to compute Assumption 3.3. In the case $h = 0$ (then $A = B = 0$), the inequality is trivially fulfilled. Since $k_{00} = k_{11}$, Assumption 3.3 can be written as

$$\begin{aligned}
&2k_{00}(1-k_{00})\left(-A - A_x - B - B_x + 2\sqrt{AB} + 2\sqrt{A_x B_x} + 2a\left(\sqrt{B}+\sqrt{A}\right)\left(\sqrt{A_x}+\sqrt{B_x}\right)\right)\\
=&2k_{00}(1-k_{00})\left(-(\sqrt{A}-\sqrt{B})^2 - (\sqrt{A_x}-\sqrt{B_x})^2 + 2a\left(\sqrt{B}+\sqrt{A}\right)\left(\sqrt{A_x}+\sqrt{B_x}\right)\right)\\
=&4k_{00}(1-k_{00})\left(\sqrt{B}+\sqrt{A}\right)\left(\sqrt{A_x}+\sqrt{B_x}\right)(a - C_B) \leq 0.
\end{aligned}$$

By the above equation and if we have $a \leq C_B$, then there is no additional assumption on K. The condition (2.22) is due to Assumptions 3.1 and 3.2. On the other hand, if $a > C_B$, then the above inequality holds true if $k_{00} = 1$ or $k_{00} = 0$. Since the case $k_{00} = 1$ is a contradiction to Assumption 3.1 for sufficiently small $\epsilon = 0$, hence, we obtain (2.22). □

Summarizing, provide that $K = K(\epsilon)$ satisfies the assumptions 3.1 - 3.3, and assuming (2.19), (2.20) and (2.21), we obtain the following **uniform** in ϵ estimate of the energy E_h:

$$\partial_t E_h + \xi \|h\|_V^2 \leq 0, \ \forall 0 \leq \epsilon \leq 1, \tag{3.4}$$

where ξ is defined by equation (2.24). The result will be proven below in Lemma 3.9. Assume for now that (3.4) holds. Then, integrating over t and using the equivalence (1.11), we obtain an estimate for $\|h(t)\|_V^2$ as

$$\frac{1-a\epsilon}{2}\|h(t)\|_V^2 \leq E_h(t) \leq E_h(0) - \xi\int_0^t \|h(s)\|_V^2\, ds, \tag{3.5}$$

$$\leq \frac{1+a\epsilon}{2}\|h(0)\|_V^2 - \xi\int_0^t \|h(s)\|_V^2\, ds. \tag{3.6}$$

This is a Gronwall inequality for the norm of h and therefore,

$$\|h(t)\|_V^2 \leq \frac{1+a\epsilon}{1-a\epsilon}\|h(0)\|_V^2 e^{-\frac{2\xi}{1-a\epsilon}t}. \tag{3.7}$$

In Lemma 3.9 we establish that $4a \leq (\lambda - C_E)$. Since $C_E \geq 0$ and $\lambda \leq 1$, see Section 1.1, we have $a \leq \frac{1}{4}$. Since we are interested in small Knudsen numbers, we may assume $\epsilon \leq 1$ and obtain

$$0 \leq a\epsilon \leq \frac{1}{4}. \tag{3.8}$$

Hence, we obtain a **uniform** bound on $\|h(t)\|^2$ as

$$\|h(t)\|_V^2 \leq \frac{5}{4}\|h(0)\|_V^2 e^{-2\xi t}. \tag{3.9}$$

Due to the definition of h in equation (1.6) the same decay rate is obtained for f.

3.2 Alternative conditions on $K(\epsilon)$. Clearly, the particular choice of $K(\epsilon)$ in Theorem 2.4 and Theorem 2.5 are not the only possibility to fulfill the assumptions 3.1 - 3.3. In the following section we give alternative conditions on $K(\epsilon)$ such that previously introduced assumptions 3.1 - 3.3 hold true. For readability reasons we drop now the dependence of the coefficients of $K(\epsilon)$ on ϵ whenever there is no confusion. We start by simplifying assumption 3.3 in the case that $k_{00}, k_{11} \leq 1$.

Provided that

$$0 \leq k_{00}, k_{11} \leq 1 \tag{3.10}$$

holds true, then assumption 3.3 holds true provided that

$$\begin{aligned} &-2\left(k_{00}\sqrt{A} - k_{11}\sqrt{B}\right)\left((1-k_{00})\sqrt{A} - (1-k_{11})\sqrt{B}\right) \\ &-2\left(k_{00}\sqrt{A_x} - k_{11}\sqrt{B_x}\right)\left((1-k_{00})\sqrt{A_x} - (1-k_{11})\sqrt{B_x}\right) \\ &+4a\left((1-k_{00})\sqrt{A} + (1-k_{11})\sqrt{B}\right)\left(k_{00}\sqrt{A_x} + k_{11}\sqrt{B_x}\right) \leq 0. \end{aligned} \tag{3.11}$$

Even though the equation (3.11) seems more complicated, it allows for a refined analysis in the following two cases summarized in the following corollary.

Corollary 3.5. *Assume that the matrix $K(\epsilon) \in \mathbb{R}^{2\times 2}$ defined by equation (2.3) fulfills (3.10). Assume that for all $\epsilon \geq 0$,*

$$k_{00}(\epsilon) = k_{11}(\epsilon). \tag{3.12}$$

Then, assumption (3.11) is fulfilled provided that

$$a \leq C_B(t) = \frac{\left(\sqrt{A} - \sqrt{B}\right)^2 + \left(\sqrt{A_x} - \sqrt{B_x}\right)^2}{2(\sqrt{A} + \sqrt{B})(\sqrt{A_x} + \sqrt{B_x})}. \tag{3.13}$$

In fact, equation (3.11) reads under assumption (3.12) for $k_{00} = k_{00}(\epsilon)$

$$2k_{00}(1-k_{00})\left[-(\sqrt{A} - \sqrt{B})^2 - (\sqrt{A_x} - \sqrt{B_x})^2 + 2a(\sqrt{A} + \sqrt{B})(\sqrt{A_x} + \sqrt{B_x})\right] \leq 0. \tag{3.14}$$

Since $k_{00}(1-k_{00}) \geq 0$ according to (3.10), the above inequality holds true provided that either $k_{00}(1-k_{00}) = 0$ or

$$a \leq C_B(t), \quad C_B(t) = \frac{\left(\sqrt{A} - \sqrt{B}\right)^2 + \left(\sqrt{A_x} - \sqrt{B_x}\right)^2}{2(\sqrt{A} + \sqrt{B})(\sqrt{A_x} + \sqrt{B_x})}. \tag{3.15}$$

In the subsequent Lemma 3.9 we obtain the following bounds on the constants a and C_E :

$$\frac{4C_E}{3C_s - 2C_E} < a < \frac{\lambda - C_E}{4}, \quad C_E \leq \frac{\lambda C_s}{8}. \tag{3.16}$$

Furthermore, note that $0 \leq \lambda \leq 1$. If we combine the inequalities (3.16) and estimate (3.15) we obtain

$$C_B \geq \frac{4C_E}{3C_s - 2C_E} \implies C_E < \min\left\{\frac{\lambda C_s}{8}, \frac{3}{2}C_s - \frac{3C_s}{2 + C_B}\right\}. \tag{3.17}$$

In the case condition (3.17) does not hold we have the following result which follows directly by equation (3.14). In this case the boundary conditions are either periodic or reflective.

Corollary 3.6. *For the matrix* $K(\epsilon) \in \mathbb{R}^{2\times 2}$ *defined by equation* (2.3)*, assume further that*

$$k_{00}(\epsilon) = k_{11}(\epsilon) \text{ and } k_{00}(\epsilon) \in \{0,\ 1\}. \tag{3.18}$$

Then, assumption (3.11) *is fulfilled.*

Finally, we have the following remark on the case $h = 0$. In this case $A = B = 0$ and therefore (3.13) is not defined. However, the state $h = 0$ corresponds to zero energy $E_h(t) = 0$ and it is precisely the state that we stabilize. Therefore, as long as $h \neq 0$, we may assume $A, B > 0$.

3.3 Limiting equation and boundary layer. The particular case of Theorem 2.4 requires an analysis of the spatial derivative of $\partial_x h$. A formal differentiation of the boundary conditions (2.8) to the VFP equation (2.6) leads to the following result.

Lemma 3.7. *Let* h *be a sufficient smooth solution to systems* (2.6)*,* (2.7) *and* (2.8) *. Then,* $\partial_x h$ *fulfills the following boundary conditions.*

$$\begin{cases} \partial_x h(t,0,v) = -k_{00}\partial_x h(t,0,-v) + k_{10}\partial_x h(t,1,v), & v > 0; \\ \partial_x h(t,1,v) = k_{01}\partial_x h(t,0,v) - k_{11}\partial_x h(t,1,-v), & v < 0. \end{cases} \tag{3.19}$$

Proof. Differentiating the boundary condition (2.8) with respect to time t and with respect to velocity v we obtain the relations

$$\begin{aligned} \partial_t h(t,0,v) &= k_{00}\partial_t h(t,0,-v) + k_{10}\partial_t h(t,1,v), \quad v > 0, \\ \partial_v h(t,0,v) &= -k_{00}\partial_v h(t,0,-v) + k_{10}\partial_v h(t,1,v), \quad v > 0, \\ v h(t,0,v) &= -k_{00} v h(t,0,-v) + k_{10} v h(t,1,v), \quad v > 0, \\ \partial_v^2 h(t,0,v) &= k_{00}\partial_v^2 h(t,0,-v) + k_{10}\partial_v^2 h(t,1,v), \quad v > 0. \end{aligned} \tag{3.20}$$

The strong form of the linearized Fokker–Planck operator $\mathcal{L}$ is given by

$$\mathcal{L}h = \frac{1}{\sqrt{M}}\partial_v\left(M\partial_v \frac{h}{\sqrt{M}}\right). \tag{3.21}$$

If we differentiate h we obtain

$$\mathcal{L}h = \frac{1}{2}h - \frac{v^2}{4}h + \partial_v^2 h. \tag{3.22}$$

Evaluating the VFP equation (2.6) at $x = 0$ and assumption (2.20) that $E = 0$ at boundary yields

$$\epsilon\partial_t h(t,0,v) + v\partial_x h(t,0,v) - \frac{1}{\epsilon}\left(\frac{1}{2}h - \frac{v^2}{4}h + \partial_v^2 h\right)(t,0,v) = 0, \quad v > 0. \tag{3.23}$$

Reformulating the equation in terms of $v\partial_x h(t,0,v)$ we obtain

$$v\partial_x h(t,0,v) = \left[-\epsilon\partial_t h + \frac{1}{\epsilon}\left(\frac{1}{2}h - \frac{v^2}{4}h + \partial_v^2 h\right)\right](t,0,v) \tag{3.24}$$

$$= k_{00}\left[-\epsilon\partial_t h + \frac{1}{\epsilon}\left(\frac{1}{2}h - \frac{v^2}{4}h + \partial_v^2 h\right)\right](t,0,-v) + \tag{3.25}$$

$$+ k_{10}\left[-\epsilon\partial_t h + \frac{1}{\epsilon}\left(\frac{1}{2}h - \frac{v^2}{4}h + \partial_v^2 h\right)\right](t,1,v) \tag{3.26}$$

$$= -k_{00} v\partial_x h(t,0,-v) + k_{10} v\partial_x h(t,1,v), \tag{3.27}$$

where the last line follows by replacing v by $-v$ for $v<0$ in equation (3.23), i.e., for $v<0$

$$\epsilon\partial_t h(t,0,-v) - v\partial_x h(t,0,-v) - \frac{1}{\epsilon}\left(\frac{1}{2}h - \frac{v^2}{4}h + \partial_v^2 h\right)(t,0,-v) = 0 \tag{3.28}$$

$$\implies -v\partial_x h(t,0,-v) = \left(-\epsilon\partial_t h + \frac{1}{\epsilon}\left(\frac{1}{2}h - \frac{v^2}{4}h + \partial_v^2 h\right)\right)(t,0,-v) \tag{3.29}$$

A similar computation can be performed for the boundary condition at $x=1$. $\square$

In the following lemma we discuss the limiting equation for vanishing Knudsen numbers and solutions f to the original VFP equation (2.1).

Lemma 3.8. *For fixed initial data $f_0(x,v) = \sigma_0(x)M(v) \in V$ for some given function $\sigma_0(x)$ and fixed $\epsilon > 0$ we assume there exist a sufficiently smooth solution to equations (2.1) and (2.2). The solution is denoted by $f^\epsilon = f^\epsilon(t,x,v)$. We assume that a sufficiently smooth function $f^0(t,x,v)$ exists as limit of the sequence of solutions f^ϵ for $\epsilon \to 0$. The following system of differential equation is then formally fulfilled by $f^0(t,x,v)$. The solution $f^0(t,x,v)$ is given by*

$$f^0(t,x,v) = \sigma^0(t,x)M(v) \tag{3.30}$$

where σ^0 satisfies for all $x \in [0,1], t \geq 0$,

$$\partial_t\sigma^0 - \partial_x\left(\partial_x\sigma^0 + E\sigma^0\right) = 0. \tag{3.31}$$

The initial condition is given by $\sigma(0,x) = \sigma_0(x)$. Only if

$$\left(1-k_{00}^0\right)\left(1-k_{11}^0\right) = k_{10}^0 k_{01}^0, \quad \left(1+k_{00}^0\right)\left(1+k_{11}^0\right) = k_{10}^0 k_{01}^0, \tag{3.32}$$

then the function σ^0 fulfills the boundary condition

$$\sigma^0(t,0) = k_{10}\sigma^0(t,1), \quad \partial_x\sigma^0(t,0) = k_{10}\partial_x\sigma^0(t,1).$$

The condition (3.32) shows that in the zero Knudsen number limit the limiting system (3.31) and (3.32) determine the solution $f^0(t,x,v)$. Hence, f^0 is defined up to the boundary by the macroscopic quantity $\sigma(t,x)$ and the global Maxwellian. We therefore do **not** observe a boundary layer as transition phase between the kinetic distribution at the boundary and the small Knudsen limit in the interior.

Proof. The proof is given by a formal expansion of f in a Hilbert series in terms of the parameter ϵ. Assume $f = h\sqrt{M}$ can be expanded in terms of the Knudsen number $f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + O(\epsilon^3)$. Recalling equation (2.1) we have

$$\epsilon\partial_t f + v\partial_x f - E\partial_v f = \frac{1}{\epsilon}\mathcal{F}f, \quad (x,v)\in\Omega, \tag{3.33}$$

we obtain for the expansion terms of the following order in ϵ

$$\begin{aligned} O\left(\frac{1}{\epsilon}\right): &\quad \mathcal{F}f_0 = 0,\\ O(1): &\quad v\partial_x f_0 - E\partial_v f_0 = \mathcal{F}(f_1),\\ O(\epsilon): &\quad \partial_t f_0 + v\partial_x f_1 - E\partial_v f_1 = \mathcal{F}(f_2).\end{aligned} \tag{3.34}$$

If f_0 has the following form

$$f_0 = \sigma^0(t,x)M(v), \ (x,v)\in\Omega \tag{3.35}$$

we obtain $\mathcal{F}f_0 = 0$. We define $\sigma^0(t,x) := \int_{\mathbb{R}} f_0(t,x,v)dv$ for all $x\in[0,1]$ and $t\geq 0$.

In the following computations we derive an equation for σ^0. For example, the specific form of f^0 given by equation (3.35) leads in equation (3.34) to

$$v\partial_x\sigma_0 M + vE\sigma_0 M = \partial_v\left(M\partial_v\left(\frac{f_1}{M}\right)\right). \tag{3.36}$$

Multiplying v and integrating over v yields

$$\int vf_1 dv = -\left(\partial_x\sigma^0 + E\sigma^0\right). \tag{3.37}$$

Integration of the $O(\epsilon)$ term in equation (3.34) and using the previous relation (3.37) the closed formulation (3.31) in terms of s^0 is obtained:

$$\partial_t\sigma^0 - \partial_x\left(\partial_x\sigma^0 + E\sigma^0\right) = 0. \tag{3.38}$$

We define boundary condition for both σ^0 and $\partial_x\sigma^0$. Inserting $\lim_{\epsilon\to 0} f^\epsilon = \sigma^0 M$ into (2.2) and using that the global Maxwellian is symmetric, $M(v) = M(-v)$, we obtain

$$\begin{cases}\sigma^0(t,0) = k_{00}\sigma^0(t,0) + k_{10}\sigma^0(t,1),\\ \sigma^0(t,1) = k_{01}\sigma^0(t,0) + k_{11}\sigma^0(t,1),\end{cases}$$

which is equivalent to

$$\begin{cases}\left(1-k^0_{00}\right)\sigma^0(t,0) = k^0_{10}\sigma^0(t,1),\\ \left(1-k^0_{11}\right)\sigma^0(t,1) = k^0_{01}\sigma^0(t,0).\end{cases}$$

The above equation holds for any $\sigma^0(t,0), \sigma^0(t,1)$ if and only if

$$\left(1-k^0_{00}\right)\left(1-k^0_{11}\right) = k^0_{10}k^0_{01}.$$

For boundary conditions $\partial_x\sigma^0$ we repeat the previous computation for the $\lim_{\epsilon\to 0}\partial_x f^\epsilon = \partial_x\sigma^0 M$ and obtain

$$\begin{cases}\partial_x\sigma^0(t,0) = -k_{00}\partial_x\sigma^0(t,0) + k_{10}\partial_x\sigma^0(t,1),\\ \partial_x\sigma^0(t,1) = k_{01}\partial_x\sigma^0(t,0) - k_{11}\partial_x\sigma^0(t,1).\end{cases}$$

This holds true only if

$$\left(1+k^0_{00}\right)\left(1+k^0_{11}\right) = k^0_{10}k^0_{01},$$

which finishes the formal proof. □

3.4 Estimates on the Lyapunov Function. In this section we establish estimates on the Lyapunov function E_h defined by equation (1.9) and recalled here for convenience:

$$E_h(t) := \frac{1}{2}\left\|h(t,\cdot,\cdot)\right\|^2_V + \epsilon a\left\langle u(t,\cdot), \partial_x\sigma(t,\cdot)\right\rangle,$$

where h fulfills equation (2.6), σ and u defined by equation (2.10), i.e.,

$$\sigma(t,x) = \int_{\mathbb{R}} h(t,x,v)\sqrt{M}(v)dv,\ u(t,x) = \int_{\mathbb{R}} vh(t,x,v)\sqrt{M}(v)dv.$$

Lemma 3.9. *Assume that Assumption 2.2 holds true and assume $1 \geq \epsilon > 0$ and let any matrix $K(\epsilon) \in \mathbb{R}^{2\times 2}$ be given. Then, E_h defined by equation (1.9) fulfills* *the following estimates for any solution h to (2.6) with initial data $h_0(\cdot,\cdot) \in V$ as in equation (2.7) and boundary conditions (2.8),*

$$\begin{aligned}\partial_t E_h + \xi \|h\|_V^2 \leq &- \int_{\mathbb{R}} \frac{v}{2} \left(h^2(t,1,v) - h^2(t,0,v)\right) dv \\ &- \int_{\mathbb{R}} \frac{v}{2} \left((\partial_x h)^2(t,1,v) - (\partial_x h)^2(t,0,v)\right) dv \\ &- a\left(u\partial_x u(t,1) - u\partial_x u(t,0)\right) + \frac{aC_s}{2\epsilon^2}\left(\int_0^t (u(s,1)-u(s,0))\,ds\right)^2 .\end{aligned} \tag{3.39}$$

The constant ξ is given by

$$\xi = \min\left\{\frac{\lambda - C_E - 4a}{\epsilon^2}, \frac{a(3C_s - 2C_E) - 4C_E}{8}\right\} > 0, \tag{3.40}$$

and a and C_E are chosen such that

$$\frac{4C_E}{3C_s - 2C_E} < a < \frac{\lambda - C_E}{4}, \quad C_E \leq \frac{\lambda C_s}{8}, \tag{3.41}$$

hold. The constants C_s, λ are defined in (1.5) and (1.2), respectively.

Proof. The proof is divided into two parts.

Consider equation (2.6), multiply by $\frac{h}{2}$, integrate on Ω and use the estimate (1.2) to obtain

$$\frac{\epsilon}{2}\partial_t \|h\|^2 + \int \frac{v}{2}\partial_x(h^2)dxdv + \frac{\lambda}{\epsilon}\|(1-\Pi)h\|_\omega^2 \leq \left\langle E\left(\partial_v - \frac{v}{2}\right)h, h\right\rangle. \tag{3.42}$$

In the following computation we use $< g, h >= \int_\Omega ghdxdv$ for simplified notations. We use the decomposition of h into the macroscopic kernel $\Pi h(t,x,v) = \sigma(t,x)M(v)$ and its complement to simplify the right-hand side,

$$\begin{aligned}\left\langle E\left(\partial_v - \frac{v}{2}\right)h, h\right\rangle &= \left\langle E\left(\partial_v - \frac{v}{2}\right)h, \sigma\sqrt{M}\right\rangle + \left\langle E\left(\partial_v - \frac{v}{2}\right)h, (1-\Pi)h\right\rangle \\ &= \left\langle Eh, \left(-\partial_v - \frac{v}{2}\right)\sigma\sqrt{M}\right\rangle + \left\langle Eh, \left(-\partial_v - \frac{v}{2}\right)(1-\Pi)h\right\rangle \\ &\leq \left\langle Eh, \left(\frac{v}{2} - \frac{v}{2}\right)\sigma\sqrt{M}\right\rangle + \|E\|_{L_x^\infty}\left(\frac{\epsilon}{2}\|h\|^2 + \frac{1}{2\epsilon}\left\|(\partial_v + \frac{v}{2})(1-\Pi)h\right\|^2\right) \\ &\leq \|E\|_{L_x^\infty}\left(\frac{\epsilon}{2}\|h\|^2 + \frac{1}{2\epsilon}\|(1-\Pi)h\|_\omega^2\right).\end{aligned} \tag{3.43}$$

Note that we only integrate by parts in v which is defined in the whole space, therefore no boundary conditions appear. Further, note that we used in the last line Young's inequality with the same fixed value of ϵ as in the equation. The estimate for terms involving the electric field E yields in equation (3.42) the following preliminary result

$$\begin{aligned}\frac{\epsilon}{2}\partial_t \|h\|^2 + \frac{\lambda}{\epsilon}\|(1-\Pi)h\|_\omega^2 \leq &\frac{1}{2}\|E\|_{L_x^\infty}\left(\epsilon\|h\|^2 + \frac{1}{\epsilon}\|(1-\Pi)h\|_\omega^2\right) \\ &- \int \frac{v}{2}\left(h^2(t,1,v) - h^2(t,0,v)\right)dv.\end{aligned} \tag{3.44}$$

The previous computation is repeated for the spatial derivative of (2.6). Integration on Ω yields

$$\begin{aligned} &\frac{\epsilon}{2}\partial_t \|\partial_x h\|^2 + \int \frac{v}{2}\partial_x((\partial_x h)^2)dxdv + \frac{\lambda}{\epsilon}\|(1-\Pi)\partial_x h\|_\omega^2 \\ &\le \left\langle (\partial_x E)\left(\partial_v - \frac{v}{2}\right)h, \partial_x h\right\rangle + \left\langle E\left(\partial_v - \frac{v}{2}\right)\partial_x h, \partial_x h\right\rangle. \end{aligned} \tag{3.45}$$

The second part on the right–hand side can be bounded in a similar fashion as in equation (3.43). For first part of the right–hand side we apply Young's inequality and use the fact that $\partial_x E$ is also bounded. Hence, we obtain

$$\begin{aligned} &\left\langle \partial_x E\left(\partial_v - \frac{v}{2}\right)h, \partial_x h\right\rangle + \left\langle E\left(\partial_v - \frac{v}{2}\right)\partial_x h, \partial_x h\right\rangle \\ \le&\frac{1}{2}\|\partial_x E\|_{L^\infty_x}\left(\epsilon\|h\|^2 + \frac{1}{\epsilon}\|(1-\Pi)\partial_x h\|_\omega^2\right) + \frac{1}{2}\|E\|_{L^\infty_x}\left(\epsilon\|\partial_x h\|^2 + \frac{1}{\epsilon}\|(1-\Pi)\partial_x h\|_\omega^2\right). \end{aligned} \tag{3.46}$$

Combining the estimate on the terms of electric field with equation (3.45) we obtain a bound on the norm of $\partial_x h$ similar to (3.44),

$$\begin{aligned} &\frac{\epsilon}{2}\partial_t \|\partial_x h\|^2 + \frac{\lambda}{\epsilon}\|(1-\Pi)\partial_x h\|_\omega^2 \\ \le&\frac{1}{2}\|\partial_x E\|_{L^\infty_x}\left(\epsilon\|h\|^2 + \frac{1}{\epsilon}\|(1-\Pi)\partial_x h\|_\omega^2\right) + \frac{1}{2}\|E\|_{L^\infty_x}\left(\epsilon\|\partial_x h\|^2 + \frac{1}{\epsilon}\|(1-\Pi)\partial_x h\|_\omega^2\right) \\ &-\int \frac{v}{2}\left((\partial_x h)^2(t,1,v) - (\partial_x h)^2(t,0,v)\right)dv. \end{aligned} \tag{3.47}$$

Both bounds (3.44) and (3.47) together give a bound in the $V-$ and V,ω-norm defined in Section 1.1,

$$\begin{aligned} &\frac{\epsilon}{2}\partial_t \|h\|_V^2 + \frac{\lambda}{\epsilon}\|(1-\Pi)h\|_{V,\omega}^2 \\ \le&\frac{C_E}{2}\left(\epsilon\|h\|_V^2 + \frac{1}{\epsilon}\|(1-\Pi)h\|_{V,\omega}^2\right) - \int \frac{v}{2}\left(h^2(t,1,v) - h^2(t,0,v)\right)dv \\ &-\int \frac{v}{2}\left((\partial_x h)^2(t,1,v) - (\partial_x h)^2(t,0,v)\right)dv. \end{aligned} \tag{3.48}$$

In estimate (3.48) we estimate by C_E the maximum of the $L^\infty-$ norm of the electric field and its spatial derivative as stated in Assumption 2.2. Finally, note that $\|h\|_V^2 = \|\sigma\|_V^2 + \|(1-\Pi)h\|_V^2$, see Section 1.1. Hence, we obtain after multiplication by $\frac{1}{\epsilon}$

$$\begin{aligned} \frac{1}{2}\partial_t \|h\|_V^2 + \frac{(\lambda - C_E)}{\epsilon^2}\|(1-\Pi)h\|_{V,\omega}^2 \le& \frac{C_E}{2}\|\sigma\|_V^2 - \frac{1}{\epsilon}\int \frac{v}{2}\left(h^2(t,1,v) - h^2(t,0,v)\right)dv \\ &-\frac{1}{\epsilon}\int \frac{v}{2}\left((\partial_x h)^2(t,1,v) - (\partial_x h)^2(t,0,v)\right)dv. \end{aligned} \tag{3.49}$$

This completes the first part of the proof.

In the second step we derive an estimate on the macroscopic quantities σ and u. Recall that, provided h is a solution to equation (2.6), the corresponding macroscopic quantities fulfill (2.11) and equation (2.12), i.e.,

$$\epsilon\partial_t\sigma + \partial_x u = 0, \tag{3.50}$$

$$\epsilon\partial_t u + \partial_x\sigma + \int v^2\sqrt{M}(1-\Pi)\partial_x h dv + \frac{1}{\epsilon}u = -E\sigma. \tag{3.51}$$

Upon multiplication of $\partial_x\sigma$ to equation (3.51), and integration on x, we obtain

$$\epsilon\left\langle \partial_t u, \partial_x\sigma\right\rangle + \|\partial_x\sigma\|^2 + \left\langle v^2\sqrt{M}(1-\Pi)\partial_x h, \partial_x\sigma\right\rangle + \frac{1}{\epsilon}\left\langle u, \partial_x\sigma\right\rangle_x = -\left\langle E\sigma, \partial_x\sigma\right\rangle_x \tag{3.52}$$

Note that here $< \cdot, \cdot >_x$ denotes the integration on x only. By Young's inequality we bound the third and fourth terms in equation (3.52) as follows:

$$\begin{aligned}
\left\langle v^2\sqrt{M}(1-\Pi)\partial_x h, \partial_x\sigma\right\rangle &\leq \int \|v(1-\Pi)\partial_x h\|_{L^2_v}\left\|v\sqrt{M}\right\|_{L^2_v}\partial_x\sigma dx \\
&\leq \frac{1}{2}\left\|v\sqrt{M}\right\|_{L^2_v}\left(4\|v(1-\Pi)\partial_x h\|^2 + \frac{1}{4}\|\partial_x\sigma\|^2\right) \\
&\leq 2\|(1-\Pi)\partial_x h\|^2_\omega + \frac{1}{8}\|\partial_x\sigma\|^2; \\
\frac{1}{\epsilon}\left\langle u, \partial_x\sigma\right\rangle_x &\leq \frac{1}{2\epsilon}\left(\frac{4}{\epsilon}\|u\|^2 + \frac{\epsilon}{4}\|\partial_x\sigma\|^2\right) \leq \frac{2}{\epsilon^2}\|u\|^2 + \frac{1}{8}\|\partial_x\sigma\|^2; \\
\left\langle E\sigma, \partial_x\sigma\right\rangle_x &\leq \frac{1}{2}\|E\|_{L^\infty_x}\left(\|\sigma\|^2 + \|\partial_x\sigma\|^2\right) \leq \frac{C_E}{4}\|\sigma\|^2_V,
\end{aligned} \tag{3.53}$$

where we apply assumption (2.19) in the last inequality. Using the continuity equation (2.11) we simplify the term $< \partial_t u, \partial_x\sigma >$,

$$\begin{aligned}
&\epsilon\left\langle \partial_t u, \partial_x\sigma\right\rangle = \epsilon\partial_t\left\langle u, \partial_x\sigma\right\rangle + \left\langle u, \partial_x(\partial_x u)\right\rangle = \epsilon\partial_t\left\langle u, \partial_x\sigma\right\rangle + \left\langle u, \partial_x^2 u\right\rangle \\
=&\epsilon\partial_t\int u\partial_x\sigma dx + (u\partial_x u)(t,1) - (u\partial_x u)(t,0) - \|\partial_x u\|^2.
\end{aligned} \tag{3.54}$$

Combining the previous estimates we can simplify (3.52) as

$$\begin{aligned}
&\epsilon\partial_t\left\langle u, \partial_x\sigma\right\rangle + \|\partial_x\sigma\|^2 \leq -(u\partial_x u)(t,1) - (u\partial_x u)(t,0) + \|\partial_x u\|^2 + \\
&+\frac{C_E}{4}\|\sigma\|^2_V + \frac{2}{\epsilon^2}\|u\|^2 + \frac{2}{8}\|\partial_x\sigma\|^2 + 2\|(1-\Pi)\partial_x h\|^2_\omega, \\
&\epsilon\partial_t\left\langle u, \partial_x\sigma\right\rangle + \frac{3}{4}\|\partial_x\sigma\|^2 \leq \frac{C_E}{4}\|\sigma\|^2_V + \frac{4}{\epsilon^2}\|(1-\Pi)h\|^2_{V,,\omega} - \left(u\partial_x u(t,1,z) - u\partial_x u(t,0,z)\right),
\end{aligned} \tag{3.55}$$

where $\|u\|^2 \leq \|u\|^2_V \leq \|(1-\Pi)h\|^2_{V,\omega}$ is used.

Next we turn to estimates of σ and $\partial_x\sigma$. By Poincare's inequality (1.5), there exists a constant $C_s \leq 1$, such that

$$\|\partial_x\sigma\|^2 \geq \frac{C_s}{2}\left\|\sigma - \int_0^1\sigma dx\right\|^2 \geq \frac{C_s}{2}\left(\|\sigma\|^2_V - \left(\int_0^1\sigma\, dx\right)^2\right). \tag{3.56}$$

In order to estimate also σ we integrate the continuity equation (2.11) with respect to x,

$$\begin{aligned} &\epsilon\partial_t \int_0^1 \sigma dx = -\left(u(t,1)-u(t,0)\right) \\ \Longrightarrow &\int_0^1 \sigma(t,x)dx - \int_0^1 \sigma_0(x)dx = -\frac{1}{\epsilon}\int_0^t \left(u(s,1)-u(s,0)\right)ds. \end{aligned} \tag{3.57}$$

Due to Assumption 2.3 we have that $\int_\Omega h_0(x,v)dxdv = 0$. Therefore, $\int_0^1 \sigma_0(x)dx = 0$ and inequality (3.56) simplifies to

$$\|\partial_x\sigma\|^2 \geq \frac{C_s}{2}\left(\|\sigma\|_V^2 - \frac{1}{\epsilon^2}\left(\int_0^t \left(u(s,1)-u(s,0)\right)ds\right)^2\right). \tag{3.58}$$

Inserting the above inequality to (3.55) gives:

$$\begin{aligned} &\epsilon\partial_t \langle u, \partial_x\sigma\rangle + \frac{3C_s}{8}\left(\|\sigma\|_V^2 - \frac{1}{\epsilon^2}\left(\int_0^t \left(u(s,1)-u(s,0)\right)ds\right)^2\right) \\ \leq &\frac{C_E}{4}\|\sigma\|_V^2 + \frac{4}{\epsilon^2}\|(1-\Pi)h\|_{V,\omega}^2 - \left(u\partial_x u(t,1,z) - u\partial_x u(t,0,z)\right). \end{aligned}$$

This in turn implies the final estimate for σ.

$$\begin{aligned} \epsilon\partial_t \langle u, \partial_x\sigma\rangle + \frac{3C_s - 2C_E}{8}\|\sigma\|_V^2 \leq \frac{C_s}{2\epsilon^2}\left(\int_0^t \left(u(s,1)-u(s,0)\right)ds\right)^2 + \\ + \frac{4}{\epsilon^2}\|(1-\Pi)h\|_{V,\omega}^2 - \left(u\partial_x u(t,1,z) - u\partial_x u(t,0,z)\right). \end{aligned} \tag{3.59}$$

In the last step of the proof we add the estimate for h obtained in equation (3.49) and the previous estimate (3.59) (multiplied by $a \geq 0$).

$$\begin{aligned} &\frac{1}{2}\partial_t \|h\|_V^2 + a\epsilon\partial_t\langle u, \partial_x\sigma\rangle + a\frac{3C_s - 2C_E}{8}\|\sigma\|_V^2 + \frac{(\lambda - C_E)}{\epsilon^2}\|(1-\Pi)h\|_{V,\omega}^2 \\ \leq &\frac{C_E}{2}\|\sigma\|_V^2 + \frac{4a}{\epsilon^2}\|(1-\Pi)h\|_{V,\omega}^2 \\ &+ \frac{C_s a}{2\epsilon^2}\left(\int_0^t \left(u(s,1)-u(s,0)\right)ds\right)^2 + -a\left(u\partial_x u(t,1,z) - u\partial_x u(t,0,z)\right) \\ &- \frac{1}{\epsilon}\int \frac{v}{2}\left((\partial_x h)^2(t,1,v) - (\partial_x h)^2(t,0,v)\right)dv. \end{aligned} \tag{3.60}$$

This implies that for ξ sufficiently small

$$\begin{aligned} &\partial_t E_h(t) + \xi\|h\|_V^2 \leq \partial_t E_h(t) + \xi\|h\|_{V,\omega}^2 \\ \leq &\partial_t E_h(t) + \left(a\frac{3C_s - 2C_E}{8} - \frac{C_E}{2}\right)\|\sigma\|_V^2 + \frac{(\lambda - C_E - 4a)}{\epsilon^2}\|(1-\Pi)h\|_{V,\omega}^2 \\ \leq &-\frac{1}{\epsilon}\int \frac{v}{2}\left(h^2(t,1,v) - h^2(t,0,v)\right)dv - \frac{1}{\epsilon}\int \frac{v}{2}\left((\partial_x h)^2(t,1,v) - (\partial_x h)^2(t,0,v)\right)dv \\ &+ \frac{C_s a}{2\epsilon^2}\left(\int_0^t \left(u(s,1)-u(s,0)\right)ds\right)^2 - a\left(u\partial_x u(t,1,z) - u\partial_x u(t,0,z)\right). \end{aligned} \tag{3.61}$$

More precisely, ξ can be chosen as

$$\xi = \min\left\{\frac{\lambda - C_E - 4a}{\epsilon^2}, \frac{a(3C_s - 2C_E) - 4C_E}{8}\right\}.$$

In order to obtain exponential decay of $\|h\|_V^2$, we require $\xi > 0$, which implies the following condition on a :

$$\frac{4C_E}{3C_s - 2C_E} < a < \frac{\lambda - C_E}{4}. \tag{3.62}$$

In order for a to exist we require that the previous bounds are ordered. A sufficient condition for $\frac{4C_E}{3C_s-2C_E} < \frac{\lambda-C_E}{4}$ is for example given by

$$C_E \leq \frac{\lambda C_s}{8} \tag{3.63}$$

since $\lambda \leq 1, C_s \leq \frac{1}{2}$. ☐

3.5 Constraints on the feedback control. In the previous section we derived an estimate for the decay of E_h. The final equation still includes boundary terms. In the following two Lemmas, we derive conditions on $K(\epsilon)$ such that in the estimate (3.39) terms involving the boundary values of h at $x \in \{0, 1\}$ are non-positive. This fact allows to conclude the proofs of Theorem 2.4 and Theorem 2.5, respectively. Note that although the following lemmas are stated for function f, they also remain valid for h. In fact, at the boundary f and h are the same up to a scaling by $\sqrt{M(v)}$.

Lemma 3.10. *Let Assumption 2.3 hold true and let $\epsilon \geq 0$. Assume that f solves the VFP (2.1) with initial condition $f(0, x, v) = f_0(x, v)$ for some $f_0(\cdot, \cdot) \in V$ and boundary conditions (2.2).*

If the entries of $K(\epsilon)$ defined by (2.3) fulfill

$$k_{00}(\epsilon) + k_{01}(\epsilon) = 1, \quad k_{10}(\epsilon) + k_{11}(\epsilon) = 1, \tag{3.64}$$

then the boundary flux $u(t, x) = \int_{\mathbb{R}} vf(t, x, v)dv$ for $x \in \{0, 1\}$ fulfills

$$u(t, 1) = u(t, 0). \tag{3.65}$$

Proof. A direct computation shows that

$$\begin{aligned} u(t,0) &= \int_0^\infty vf(t,0,v)dv + \int_{-\infty}^0 vf(t,0,v)dv \\ &= \int_0^\infty vk_{00}f(t,0,-v)dv + \int_0^\infty vk_{10}f(t,1,v)dv + \int_{-\infty}^0 vf(t,0,v)dv \\ &= (1-k_{00})\int_{-\infty}^0 vf(t,0,v)dv + k_{10}\int_0^\infty vf(t,1,v)dv. \end{aligned} \tag{3.66}$$

Similarly, we obtain

$$u(t,1) = (1-k_{11})\int_0^\infty vf(t,1,v)dv + k_{01}\int_{-\infty}^0 vf(t,0,v)dv. \tag{3.67}$$

Subtracting (3.67) from (3.68) and inserting the assumption (3.68) gives

$$u(t,0) - u(t,1) = (1-k_{00}-k_{01})\int_{-\infty}^0 vf(t,0,v)dv + (k_{10}-1+k_{11})\int_0^\infty vf(t,1,v)dv = 0. \tag{3.68}$$

This finishes the proof. ☐

Finally, we derive an estimate on the boundary contributions in the estimate (3.39).

Lemma 3.11. *Let Assumption 2.3 hold true and let $\epsilon \geq 0$. Assume that h solves the VFP (3.33) with initial condition $h(0,x,v) = \frac{f_0(x,v)}{\sqrt{M}}$ for some $f_0(\cdot,\cdot) \in V$ and boundary conditions (2.8).*

Let assumption (3.2) hold true, i.e.,

$$k_{00}(\epsilon) + k_{01}(\epsilon) = 1, \quad k_{11}(\epsilon) + k_{10}(\epsilon) = 1. \tag{3.69}$$

Then, h fulfills the estimate (3.39) and we have

$$-\int_{\mathbb{R}} \frac{v}{2} \left[h^2\right]_0^1 dv - \int_{\mathbb{R}} \frac{v}{2} \left[(\partial_x h)^2\right]_0^1 dv - a\left[u\partial_x u\right]_0^1 \leq I(t,\epsilon),$$

where by $[g]_0^1 = g(t,1,v) - g(t,0,v)$. The term $I(t,\epsilon)$ is given by

$$\begin{aligned} I(t,\epsilon) =& -2k_{00}(1-k_{00})\left(A + A_x\right) - 2k_{11}(1-k_{11})\left(B + B_x\right) \\ &+ 2\left(|k_{11}(1-k_{00})| + |k_{00}(1-k_{11})|\right)\left(\sqrt{AB} + \sqrt{A_x B_x}\right) \\ &+ 4a\left(|1-k_{11}|\sqrt{B} + |1-k_{00}|\sqrt{A}\right)\left(|k_{00}|\sqrt{A_x} + |k_{11}|\sqrt{B_x}\right) \end{aligned} \tag{3.70}$$

where $A(t), B(t), A_x(t), B_x(t)$ are positive functions of time t defined by equation (1.7).

Proof. In the proof we drop the dependence of $K(\epsilon)$ on ϵ for notation clarity. By the boundary condition (2.8) we obtain

$$\begin{aligned} &-\int_{\mathbb{R}} \frac{v}{2} h^2(t,1,v)dv = -\int_0^\infty \frac{v}{2} h^2(t,1,v)dv - \int_{-\infty}^0 \frac{v}{2}\left(k_{11}h(t,1,-v) + k_{01}h(t,0,v)\right)^2 dv \\ &= -(1-k_{11}^2)B + k_{01}^2 A - 2k_{11}k_{01}\int_{-\infty}^0 \frac{v}{2} h(t,1,-v)h(t,0,v)dv, \end{aligned}$$

where we expanded the quadratic term and changed the variable $-v$ to v within the term $h(t,1,-v)$. Similarly,

$$\begin{aligned} &\int_{\mathbb{R}} \frac{v}{2} h^2(t,0,v)dv = \int_{-\infty}^0 \frac{v}{2} h^2(t,0,v)dv + \int_0^\infty \frac{v}{2}\left(k_{00}h(t,0,-v) + k_{10}h(t,1,v)\right)^2 dv \\ &= -(1-k_{00}^2)A + k_{10}^2 B - 2k_{00}k_{10}\int_{-\infty}^0 \frac{v}{2} h(t,1,-v)h(t,0,v)dv. \end{aligned}$$

Writing $v = -\sqrt{v}\sqrt{v}$ we obtain

$$\left|\int_{-\infty}^0 \frac{v}{2} h(t,1,-v)h(t,0,v)dv\right| \leq \sqrt{\int_{-\infty}^0 \frac{-v}{2} h^2(t,1,-v)dv}\sqrt{\int_{-\infty}^0 \frac{-v}{2} h^2(t,0,v)dv} = \sqrt{AB}.$$

Combining the previous estimates yields

$$
\begin{aligned}
&-\int_{\mathbb{R}} \frac{v}{2}\left(h^2(t,1,v)-h^2(t,0,v)\right)dv \\
=&-(1-k_{11}^2-k_{10}^2)B-(1-k_{00}^2-k_{01}^2)A \\
&-2k_{11}k_{01}\int_{-\infty}^{0}\frac{v}{2}h(t,1,-v)h(t,0,v)dv+2k_{00}k_{10}\int_{0}^{\infty}\frac{v}{2}h(t,1,v)h(t,0,-v)dv, \\
\leq&-(1-k_{00}^2-k_{01}^2)A-(1-k_{11}^2-k_{10}^2)B+2\left(|k_{11}k_{01}+k_{00}k_{10}|\right)\sqrt{AB} \\
=&-2k_{00}(1-k_{00})A-2k_{11}(1-k_{11})B+2\left(|k_{11}(1-k_{00})+k_{00}(1-k_{11})|\right)\sqrt{AB},
\end{aligned}
\tag{3.71}
$$

where we have used relation (3.69). In order to estimate the term $-\int \frac{v}{2}((\partial_x h)^2(t,1,v,z)-(\partial_x h)^2(t,0,v,z))dv$ we recall the boundary condition of $\partial_x h$ in Lemma 3.7:

$$
\begin{aligned}
&-\int_{\mathbb{R}}\frac{v}{2}(\partial_x h)^2(t,1,v)dv \\
=&-\int_0^{\infty}\frac{v}{2}(\partial_x h)^2(t,1,v)dv-\int_{-\infty}^{0}\frac{v}{2}\left(-k_{11}\partial_x h(t,1,-v)+k_{01}\partial_x h(t,0,v)\right)^2dv \\
=&-(1-k_{11}^2)B_x+k_{01}^2A_x+2k_{11}k_{01}\int_{-\infty}^{0}\frac{v}{2}\partial_x h(t,1,-v)\partial_x h(t,0,v)dv,
\end{aligned}
$$

and similarly

$$
\int_{\mathbb{R}}\frac{v}{2}(\partial_x h)^2(t,0,v)dv=-(1-k_{00}^2)A_x+k_{10}^2B_x+2k_{00}k_{10}\int_{-\infty}^{0}\frac{v}{2}\partial_x h(t,1,-v)\partial_x h(t,0,v)dv.
$$

The above inequalities imply that

$$
\begin{aligned}
&-\int_{\mathbb{R}}\frac{v}{2}\left((\partial_x h)^2(t,1,v)-(\partial_x h)^2(t,0,v)\right)dv \\
\leq&-(1-k_{00}^2-k_{01}^2)A_x-(1-k_{11}^2-k_{10}^2)B_x+2\left(|k_{11}k_{01}+k_{00}k_{10}|\right)\sqrt{A_xB_x} \\
=&-2k_{00}(1-k_{00})A_x-2k_{11}(1-k_{11})B_x+2\left(|k_{11}(1-k_{00})+k_{00}(1-k_{11})|\right)\sqrt{A_xB_x}.
\end{aligned}
\tag{3.72}
$$

Finally, we bound $-u\partial_x u(t,1)+u\partial_x(t,0)$. Use the definition of u in terms of h as

$$
\begin{aligned}
&-u\partial_x u(t,1)+u\partial_x u(t,0) \\
=&-\left((1-k_{11})\int_0^{\infty}v\sqrt{M}h(t,1,v)dv+k_{01}\int_{-\infty}^{0}v\sqrt{M}h(t,0,v)dv\right) \\
&\left((1+k_{11})\int_0^{\infty}v\sqrt{M}\partial_x h(t,1,v)dv+k_{01}\int_{-\infty}^{0}v\sqrt{M}\partial_x h(t,0,v)dv\right) \\
&+\left((1-k_{00})\int_{-\infty}^{0}v\sqrt{M}h(t,0,v)dv+k_{10}\int_0^{\infty}v\sqrt{M}h(t,1,v)dv\right) \\
&\left((1+k_{00})\int_{-\infty}^{0}v\sqrt{M}\partial_x h(t,0,v)dv+k_{10}\int_0^{\infty}v\sqrt{M}\partial_x h(t,1,v)dv\right).
\end{aligned}
$$

Due to the assumption $(1-k_{11}) = k_{10} = (1-k_{00})$, therefore,

$$
\begin{aligned}
&-u\partial_x u(t,1) + u\partial_x u(t,0) \\
&= \left((1-k_{11}) \int_0^\infty v\sqrt{M} h(t,1,v) dv + (1-k_{00}) \int_{-\infty}^0 v\sqrt{M} h(t,0,v) dv \right) \\
&\left(-2k_{11} \int_0^\infty v\sqrt{M} \partial_x h(t,1,v) dv + 2k_{00} \int_{-\infty}^0 v\sqrt{M} \partial_x h(t,0,v) dv \right).
\end{aligned}
\tag{3.73}
$$

The integral on $v\sqrt{M}h$ is bounded

$$
\int_0^\infty v\sqrt{M} h(t,1,v) dv \leq \sqrt{\int_0^\infty vM\, dv} \sqrt{\int_0^\infty vh^2(t,1,v) dv} = \sqrt{2B}.
$$

The $x=0$ case is handled similarly. Therefore (3.73) has an upper bound given by

$$
-u\partial_x u(t,1) + u\partial_x u(t,0) \leq 4\left(|1-k_{11}| \sqrt{B} + |1-k_{00}| \sqrt{A} \right) \left(|k_{00}| \sqrt{A_x} + |k_{11}| \sqrt{B_x} \right).
\tag{3.74}
$$

Combining equations (3.71), (3.72) and (3.74) we arrive at

$$
\begin{aligned}
&-\int_{\mathbb{R}} \frac{v}{2} \left[h^2\right]_0^1 dv - \int_{\mathbb{R}} \frac{v}{2} \left[(\partial_x h)^2\right]_0^1 dv - a\left[u\partial_x u\right]_0^1 \\
\leq &-2k_{00}(1-k_{00})(A+A_x) - 2k_{11}(1-k_{11})(B+B_x) \\
&+ 2\left(|k_{11}(1-k_{00})| + |k_{00}(1-k_{11})|\right) \left(\sqrt{AB} + \sqrt{A_x B_x} \right) \\
&+ 4a \left(|1-k_{11}| \sqrt{B} + |1-k_{00}| \sqrt{A} \right) \left(|k_{00}| \sqrt{A_x} + |k_{11}| \sqrt{B_x} \right).
\end{aligned}
$$

□

4 Conclusion. In this paper, we study feedback boundary conditions to guarantee stabilization of solutions to the Vlasov–Fokker–Planck equation. Using a novel Lyapunov function we are able to show that different linear feedback boundary conditions damp out perturbations of steady states exponentially fast in time. We discuss boundary conditions for the case where the electric field is suitably small as well as boundary conditions for the potentially large electric fields. Also, we study the interplay of the hydrodynamic limit with the derived feedback laws. In the hydrodynamic limit only periodic or reflective boundary conditions guarantee stabilization of steady states.

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