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MATHEMATICAL ANALYSIS AND NUMERICAL METHODS FOR MULTISCALE KINETIC EQUATIONS WITH UNCERTAINTIES

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Abstract

Kinetic modeling and computation face the challenges of multiple scales and uncertainties. Developing efficient multiscale computational methods, and quantifying uncertainties arising in their collision kernels or scattering coefficients, initial or boundary data, forcing terms, geometry, etc. have important engineering and industrial applications. In this article we will report our recent progress in the study of multiscale kinetic equations with uncertainties modelled by random inputs. We first study the mathematical properties of uncertain kinetic equations, including their regularity and long-time behavior in the random space, and sensitivity of their solutions with respect to the input and scaling parameters. Using the hypocoercivity of kinetic operators, we provide a general framework to study these mathematical properties for general class of linear and nonlinear kinetic equations in various asymptotic regimes. We then approximate these equations in random space by the stochastic Galerkin methods, study the numerical accuracy and long-time behavior of the methods, and furthermore, make the methods "stochastically asymptotic preserving", in order to handle the multiple scales efficiently.

1 Introduction

Kinetic equations describe the probability density function of a gas or system comprised of a large number of particles. In multiscale modeling hierarchy, they serve as the bridge between atomistic and continuum models. On one hand, since they model the collective dynamics of particles, thus are more efficient than molecular dynamics; on the other

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hand, they provide more accurate solutions when the macroscopic fluid mechanics laws of Navier-Stokes and Fourier become inadequate. The most fundamental kinetic equation is the Boltzmann equation, an integro-differential equation describing particle transport with binary collisions Chapman and Cowling [1991] and Cercignani [1988]. Now kinetic theory has seen expanding applications from rarefied gas dynamics Cercignani [2000], radiative transfer Chandrasekhar [1960], medical imaging Arridge [1999], plasma physics Degond and Deluzet [2017a], to microfabrication technology Markowich, Ringhofer, and Schmeiser [1990] and Jüngel [2009], biological and even social sciences Naldi, Pareschi, and Toscani [2010].

There are three main computational challenges in kinetic modeling and simulation: *Dimension curse, multiple scales,* and *uncertainty*.

A kinetic equation solves the particle density distribution $f(t, x, v)$, which depends on time $t \in \mathbb{R}^+$, space $x \in \mathbb{R}^d$, and particle velocity $v \in \mathbb{R}^d$. Typically, $d = 3$, therefore one has to solve a six dimensional differential-integral equation plus time.

Kinetic equations often have multiple scales, characterized by the Knudsen number ε, the ratio of particle mean free path over a typical length scale, which can vary spatially dramatically. In these problems, multiscale and multi physics modelings are essential. For example, in the space shuttle reentry problem, along the vehicle trajectory, one encounters free streaming, rarefied gas (described by the Boltzmann equation), transition to the macroscopic hydrodynamic (described by the Euler or Navier-Stokes equations) regimes. In this process the mean free path changes from $O(1)$ meters to $O(10^{-8})$ meters Rivell [2006]. In plasma physics, one has to match the plasma and sheath where the quasineutral (which allows macroscopic modeling) and non-quasineutral (which needs kinetic modeling) models need to be coupled Franklin and Ockendon [1970]. These multiscale and multi-physics problems pose tremendous numerical challenges, with stiff collision terms, strong (electric or magnetic) fields, fast advection speed, and long-time behavior that require prohibitively small time step and mesh size in order to obtain reliable computational results.

Another challenge, which has been ignored in the community, is the issue of *uncertainties* in kinetic models. In reality, there are many sources of uncertainties that can arise in these equations, such as collision kernels, scattering coefficients, initial or boundary data, geometry, source or forcing terms Bird [1994], Berman, Haverkort, and Woerdman [1986], and Koura and Matsumoto [1991]. Understanding the impact of these uncertainties is crucial to the simulations of the complex kinetic systems in order to validate and improve these models.

To characterize the uncertainty, we assume that certain quantities depend on a random vector $\mathbf{z} \in \mathbb{R}^n$ in a properly defined probability space $(\Sigma, \mathfrak{A}, \mathbb{P})$, whose event space is Σ and is equipped with σ-algebra $\mathfrak{A}$ and probability measure $\mathbb{P}$. We also assume

the components of $\mathbf{z}$ are mutually independent random variables with known probability $\omega(\mathbf{z}) : I_{\mathbf{z}} \longrightarrow \mathbb{R}^+$, obtained already through some dimension reduction technique, e.g., Karhunen-Loève (KL) expansion Loève [1977], and do not pursue further the issue of random input parameterization.

Although uncertainty quantification (UQ) has been a popular field in scientific and engineering computing in the last two decades, UQ for kinetic equations has been largely an open area until recently. In this article we will present some of our recent results in UQ for multiscale kinetic equations. We will use *hypocoercivity* of kinetic operators to study the regularity and long-time behavior in the random space, as well as sensitivity of the solutions with respect to the random input parameters. Our results are fairly general, covering most important linear and nonlinear kinetic equations, including the Boltzmann, Landau, semi-classical relaxation models, and the Vlasov-Poisson-Fokker-Planck equations. We then introduce the stochastic Galerkin method for random kinetic equations, and study their numerical accuracy and long-time behavior, and formulate them as *stochastic asymptotic-preserving* methods for multiscale kinetic equations with uncertainties, which allows one to solve these problems with *all* numerical parameters–including the degree of orthogonal polynomials used in the polynomial chaos expansions–*independent of the Kundsen number*.

2 Basic mathematical theory for uncertain kinetic equations

2.1 The linear transport equation with isotropic scattering. We first introduce the linear transport equation in one dimensional slab geometry:

$$\text{(2-1)}\quad \varepsilon\partial_t f + v\partial_x f = \frac{\sigma}{\varepsilon}\mathcal{L}f - \varepsilon\sigma^a f + \varepsilon S, \quad t > 0,\ x \in [0,1],\ v \in [-1,1],\ z \in I_z,$$

$$\text{(2-2)}\quad \mathcal{L}f(t,x,v,z) = \frac{1}{2}\int_{-1}^{1} f(t,x,v',z)\,\mathrm{d}v' - f(t,x,v,z),$$

with the initial condition

$$\text{(2-3)}\qquad f(0,x,v,z) = f^0(x,v,z).$$

This equation arises in neutron transport, radiative transfer, etc. and describes particles (for example neutrons) transport in a background media (for example nuclei). $\mathcal{L}$ is the collision operator, $v = \Omega \cdot e_x = \cos\theta$ where θ is the angle between the moving direction and x-axis. $\sigma(x,z)$, $\sigma^a(x,z)$ are total and absorption cross-sections respectively. $S(x,z)$ is the source term. For $\sigma(x,z)$, we assume

$$\text{(2-4)}\qquad \sigma(x,z) \geq \sigma_{\min} > 0.$$

The equation is scaled in long time with strong scattering.

Denote

$$[\phi] = \frac{1}{2}\int_{-1}^{1} \phi(v)\,\mathrm{d}v \tag{2-5}$$

as the average of a velocity dependent function ϕ.

Define in the Hilbert space $L^2\big([-1,1];\ \phi^{-1}\,\mathrm{d}v\big)$ the inner product and norm

$$\langle f, g\rangle_\phi = \int_{-1}^{1} f(v)g(v)\phi^{-1}\,\mathrm{d}v, \qquad \|f\|_\phi^2 = \langle f, f\rangle_\phi. \tag{2-6}$$

The linear operator $\mathfrak{L}$ satisfies the following *coercivity* properties Bardos, Santos, and Sentis [1984]: $\mathfrak{L}$ is non-positive self-adjoint in $L^2([-1,1];\phi^{-1}\,\mathrm{d}v)$, i.e., there is a positive constant s_m such that

$$\langle f, \mathfrak{L}f\rangle_\phi \le -2s_m\|f\|_\phi^2, \qquad \forall\ f \in \mathfrak{N}(\mathfrak{L})^\perp; \tag{2-7}$$

with $\mathfrak{N}(\mathfrak{L}) = \mathrm{span}\,\{\phi \mid \phi = [\phi]\}$ the null space of $\mathfrak{L}$.

Let $\rho = [f]$. For each fixed z, the classical diffusion limit theory of linear transport equation Larsen and Keller [1974], Bensoussan, Lions, and Papanicolaou [1979], and Bardos, Santos, and Sentis [1984] gives that, as $\varepsilon \to 0$, ρ solves the following diffusion equation:

$$\partial_t\rho = \partial_x\left(\frac{1}{3}\sigma(x,z)^{-1}\partial_x\rho\right) - \sigma^a(x,z)\rho + S(x,z). \tag{2-8}$$

To study the regularity and long-time behavior in the random space of the linear transport Equation (2-1)-(2-3), we use the Hilbert space of the random variable

$$H(I_z;\ \omega\,\mathrm{d}z) = \left\{ f \mid I_z \to \mathbb{R}^+,\ \int_{I_z} f^2(z)\omega(z)\,\mathrm{d}z < +\infty \right\}, \tag{2-9}$$

equipped with the inner product and norm defined as

$$\langle f, g\rangle_\omega = \int_{I_z} fg\,\omega(z)\,\mathrm{d}z, \quad \|f\|_\omega^2 = \langle f, f\rangle_\omega\,. \tag{2-10}$$

We also define the kth order differential operator with respect to z as

$$D^k f(t,x,v,z) := \partial_z^k f(t,x,v,z), \tag{2-11}$$

and the Sobolev norm in z as

$$\|f(t,x,v,\cdot)\|_{H^k}^2 := \sum_{\alpha\le k}\|D^\alpha f(t,x,v,\cdot)\|_\omega^2. \tag{2-12}$$

Finally, we introduce norms in space and velocity as follows,

$$\|f(t,\cdot,\cdot,\cdot)\|_{\Gamma}^2 := \int_Q \|f(t,x,v,\cdot)\|_{\omega}^2 \,\mathrm{d}x\,\mathrm{d}v, \qquad t \geq 0, \tag{2-13}$$

$$\|f(t,\cdot,\cdot,\cdot)\|_{\Gamma^k}^2 := \int_Q \|f(t,x,v,\cdot)\|_{H^k}^2 \,\mathrm{d}x\,\mathrm{d}v, \qquad t \geq 0, \tag{2-14}$$

where $Q = [0,1] \times [-1,1]$ denotes the domain in the phase space. For simplicity of notations, we will suppress the dependence of t and just use $\|f\|_{\Gamma}$, $\|f\|_{\Gamma^k}$ in the following results, which were established in Jin, J.-G. Liu, and Ma [n.d.].

Theorem 2.1 (Uniform regularity). *If for some integer $m \geq 0$,*

$$\|D^k\sigma(z)\|_{L^\infty} \leq C_\sigma, \qquad \|D^k f_0\|_{\Gamma} \leq C_0, \qquad k = 0,\ldots,m, \tag{2-15}$$

then the solution f to the linear transport Equation (2-1)–(2-3), with $\sigma^a = S = 0$ and periodic boundary condition in x, satisfies,

$$\|D^k f\|_{\Gamma} \leq C, \qquad k = 0,\cdots,m, \qquad \forall t > 0, \tag{2-16}$$

where C_σ, C_0 and C are constants independent of ε.

The above theorem shows that, under some smoothness assumption on σ, the regularity of the initial data is preserved in time and the Sobolev norm of the solution is bounded uniformly in ε.

Theorem 2.2 (ε^2-estimate on $[f]-f$). *With all the assumptions in Theorem 2.1 and furthermore, $\sigma \in W^{k,\infty} = \{\sigma \in L^\infty([0,1]\times I_z) | D^j\sigma \in L^\infty([0,1]\times I_z) \quad \text{for all } j \leq k\}$. For a given time $T > 0$, the following regularity result of $[f]-f$ holds:*

$$\|D^k([f]-f)\|_{\Gamma}^2 \leq \mathrm{e}^{-\sigma_{\min} t/2\varepsilon^2}\|D^k([f_0]-f_0)\|_{\Gamma}^2 + C'\varepsilon^2 \tag{2-17}$$

for any $t \in (0,T]$ and $0 \leq k \leq m$,, where C' and C are constants independent of ε.

The first term on the right hand side of (2-17) is the behavior of the initial layer, which is damped exponentially in t/ε^2. After the initial layer, the high order derivatives in z of the difference between f and its local equilibrium $[f]$ is of $O(\varepsilon)$.

Such results have been generalized to linear anisotropic collision operators in L. Liu [n.d.]. For general linear collision operators conserving mass, the hypocoercivity framework of Dolbeault, Mouhot, and Schmeiser [2015] was first used by Li and Wang [2016] to prove regularity in the random space with sharp constants.

2.2 General collisional nonlinear kinetic equations with random uncertainties. Consider the initial value problem for kinetic equations of the form

$$\text{(2-18)} \qquad \begin{cases} \partial_t f + \dfrac{1}{\varepsilon^\alpha} v \cdot \nabla_x f = \dfrac{1}{\varepsilon^{1+\alpha}} \mathcal{Q}(f), \\ f(0,x,v,z) = f_{in}(x,v,z), \qquad x \in \Omega \subset \mathbb{T}^d,\ v \in \mathbb{R}^d, z \in I_z \subset \mathbb{R}. \end{cases}$$

The operator $\mathcal{Q}$ models the collisional interactions of particles, which is either binary or between particles and a surrounding medium. $\alpha = 1$ is referred to the *incompressible Navier-Stokes* scaling, while $\alpha = 0$ corresponds to the Euler (or acoustic) scaling. The periodic boundary conditions for the spatial domain $\Omega = \mathbb{T}^d$ is assumed here for theoretical purpose. In the sequel $\mathfrak{L}$ is used for both the linear collision operator and the linearized collision operator for nonlinear equations. Consider the linearized equation

$$\text{(2-19)} \qquad \partial_t g + \frac{1}{\varepsilon^\alpha} v \cdot \nabla_x g = \frac{1}{\varepsilon^{1+\alpha}} \mathfrak{L}(g),$$

Since $\mathfrak{L}$ is not fully dissipative, as summarized in S. Daus, Jüngel, Mouhot, and Zamponi [2016] and Dolbeault, Mouhot, and Schmeiser [2015], the idea is to use the hypocoercivity of the linearized kinetic operator

$$\mathfrak{S} = \frac{1}{\varepsilon^{1+\alpha}} \mathfrak{L} - \frac{1}{\varepsilon^\alpha} \mathfrak{T},$$

where $\mathfrak{T} = v \cdot \nabla_x$ is the streaming operator, using the dissipative properties of $\mathfrak{L}$ and the conservative properties of $\mathfrak{T}$. The aim is to find a Lyaponov type functional $\eta[h]$ which is equivalent to the square of the norm of a Banach space, for example

$$H^1_{x,v} = \left\{ f \,\middle|\, \int_{\Omega \times \mathbb{R}^d} \sum_{|i|+|j| \le 1} ||\partial_{x_i} \partial_{v_j} g||^2_{L^2_{x,v}} \, dx dv < \infty \right\},$$

such that

$$\kappa_1 \, ||g||_{H^1_{x,v}} \le \eta[g] \le \kappa_2 \, ||g||_{H^1_{x,v}}, \qquad \text{for } g \in H^1_{x,v},$$

which leads to

$$\frac{d}{dt} \eta[g(t)] \le -\kappa \, ||g(t)||_{H^1_{x,v}}, \qquad t > 0,$$

with constants $\kappa_1, \kappa_2, \kappa > 0$. Then one concludes the exponential convergence of g in $H^1_{x,v}$. The obvious choice of $\eta[g] = c_1 \, ||g||^2_{L^2_{x,v}} + c_2 \, ||\nabla_x g||^2_{L^2_{x,v}} + c_3 \, ||\nabla_v g||_{L^2_{x,v}}$ does not work, since the collision operator is not coercive. The key idea, first seen in Villani [2009] and implemented in Mouhot and Neumann [2006], is to add the "mixing term" $c \langle \nabla_x g, \nabla_v g \rangle_{L^2_{x,v}}$ to the definition of $\eta[g]$, that is

$$\frac{d}{dt} \langle \nabla_x g, \nabla_v g \rangle_{L^2_{x,v}} = -||\nabla_x g||^2_{L^2_{x,v}} + 2 \langle \nabla_x \mathfrak{L}(g), \nabla_v g \rangle_{L^2_{x,v}}.$$

Mouhot and Neumann [ibid.] discusses the linearized equation $\partial_t g + v \cdot \nabla_x g = \mathfrak{L}(g)$ and proves that if the linear operator $\mathfrak{L}$ satisfies some assumptions, then $\mathfrak{L} - v \cdot \nabla_x$ generates a strongly continuous evolution semi-group $e^{t\mathfrak{G}}$ on $H^s_{x,v}$, which satisfies

$$||e^{t\mathfrak{G}}(\mathbb{I} - \Pi_{\mathfrak{G}})||_{H^s_{x,v}} \leq C \exp[-\tau t], \tag{2-20}$$

for some explicit constants C, $\tau > 0$ depending only on the constants determined by the equation itself. Here $\Pi_{\mathfrak{G}}$ is the orthogonal projection in L^2_v onto the null space of $\mathfrak{L}$. This result shows that apart from 0, the spectrum of $\mathfrak{G}$ is included in

$$\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) \leq -\tau\}.$$

For nonliner kinetic equations, the main idea is to use *the perturbative setting* Guo [2006] and Strain and Guo [2008]. Equations defined in (2-18) admit a unique global equilibrium in the torus, denoted by $\mathfrak{M}$ which is independent of t, x. Now consider the linearization around this equilibrium and perturbation of the solution of the form

$$f = \mathfrak{M} + \varepsilon M h \tag{2-21}$$

with $\mathfrak{M}$ being the global equilibrium (or global) Maxwellian, and $M = \sqrt{\mathfrak{M}}$. Then h satisfies

$$\partial_t h + \frac{1}{\varepsilon^{\alpha}} v \cdot \nabla_x h = \frac{1}{\varepsilon^{1+\alpha}} \mathfrak{L}(h) + \frac{1}{\varepsilon^{\alpha}} \mathfrak{T}(h, h). \tag{2-22}$$

$\mathfrak{L}$ is the linearized (around $\mathfrak{M}$) collision operator acting on $L^2_v = \{f \mid \int_{\mathbb{R}^d} f^2\, dv < \infty\}$, with the kernel denoted by $N(\mathfrak{L}) = \mathrm{span}\{\psi_1, \cdots, \psi_d\}$. $\{\psi_i\}_{1 \leq i \leq d}$ is an orthonormal family of polynomials in v corresponding to the manifold of local equilibria for the linearized kinetic models. The orthogonal projection on $N(\mathfrak{L})$ in L^2_v is defined by

$$\Pi_{\mathfrak{L}}(h) = \sum_{i=1}^{n} \left(\int_{\mathbb{R}^d} h \psi_i \, dv \right) \psi_i, \tag{2-23}$$

where $\Pi_{\mathfrak{L}}$ is the projection on the 'fluid part' and $\mathbb{I} - \Pi_{\mathfrak{L}}$ is the projection on the kinetic part, with $\mathbb{I}$ the identity operator. The global equilibrium is then

$$\mathfrak{M} = \Pi_{\mathfrak{G}}(h) = \sum_{i=1}^{n} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} h \psi_i \, dx \, dv \right) \psi_i, \tag{2-24}$$

which is independent of x and t and is the orthogonal projection on $N(\mathfrak{G}) = N(\mathfrak{L})$ in $L^2_{x,v} = \{f \mid \int_{\Omega \times \mathbb{R}^d} f^2\, dx dv < \infty\}$.

Since the linear part $\frac{1}{\varepsilon^{1+\alpha}}\mathfrak{L}$ part has one extra factor of $\frac{1}{\varepsilon}$ than the nonlinear part $\frac{1}{\varepsilon^{\alpha}}\mathfrak{T}$, one hopes to use the hypocoercivity from the linear part to control the nonlinear part in order to come up with the desired decay estimate. This is only possible for initial data close to $\mathfrak{M}$, as in (2-21). In addition, one needs some assumptions on these operators, which can be checked for a number of important collision kernels, such as the Boltzmann, Landau, and semi-classical relaxation models Briant [2015].

Assumption on the linear operator $\mathfrak{L}$. *$\mathfrak{L}$ has the local coercivity property: There exists $\lambda > 0$ such that $\forall\, h \in L^2_v$,*

$$\langle \mathfrak{L}(h),\, h\rangle_{L^2_v} \le -\lambda\, ||h^{\perp}||^2_{\Lambda_v}\,, \tag{2-25}$$

where

$$h^{\perp} = h - \Pi_{\mathfrak{L}}(h)$$

stands for the microscopic part of h, which satisfies $h^{\perp} \in N(\mathfrak{L})^{\perp}$ in L^2_v. Here Λ_v-norm is collision operator specific. For the Boltzmann collision operator, it is given in (2-33).

To extend to higher-order Sobolev spaces, let us first introduce some notations of multi-indices and Sobolev norms. For two multi-indices j and l in $\mathbb{N}^d$, define

$$\partial_l^j = \partial/\partial v_j\, \partial/\partial x_l.$$

For $i \in \{1, \cdots, d\}$, denote by $c_i(j)$ the value of the i-th coordinate of j and by $|j|$ the l^1 norm of the multi-index, that is, $|j| = \sum_{i=1}^d c_i(j)$. Define the multi-index δ_{i_0} by: $c_i(\delta_{i_0}) = 1$ if $i = i_0$ and 0 otherwise. We use the notation

$$\partial_z^{\alpha} h = \partial^{\alpha} h.$$

Denote $||\cdot||_{\Lambda} := ||\,||\cdot||_{\Lambda_v}\,||_{L^2_x}$. The Sobolev norms on $H^s_{x,v}$ and H^s_{Λ} are defined by

$$||h||^2_{H^s_{x,v}} = \sum_{|j|+|l|\le s} ||\partial_l^j h||^2_{L^2_{x,v}}\,, \qquad ||h||^2_{H^s_{\Lambda}} = \sum_{|j|+|l|\le s} ||\partial_l^j h||^2_{\Lambda}\,.$$

Define the sum of Sobolev norms of the z derivatives by

$$||h||^2_{H^{s,r}_{x,v}} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_{x,v}}\,, \qquad ||h||^2_{H^{s,r}_{\Lambda}} = \sum_{|m|\le r} ||\partial^m h||^2_{H^s_{\Lambda}}\,, \qquad ||h||^2_{H^{s,r}_x L^2_v} = \sum_{|m|\le r} ||\partial^m h||^2_{H}$$

Note that these norms are all functions of z. Define the norms in the (x, v, z) space

$$||h(x,v,\cdot)||^2_{H^s_z} = \int_{I_z} ||h||^2_{H^s_{x,v}}\, \pi(z)\, dz\,, \qquad ||h(x,v,\cdot)||^2_{H^s_{x,v} H^r_z} = \int_{I_z} ||h||_{H^{s,r}_{x,v}}\, \pi(z)\, dz\,,$$

in addition to the sup norm in z variable,

$$||h||_{H^s_{x,v}L^\infty_z} = \sup_{z\in I_z} ||h||_{H^s_{x,v}} .$$

Assumptions on the nonlinear term $\mathfrak{F}$: *$\mathfrak{F} : L^2_v \times L^2_v \to L^2_v$ is a bilinear symmetric operator such that for all multi-indexes j and l such that $|j| + |l| \le s$, $s \ge 0$, $m \ge 0$,*

$$\left|\langle \partial^m \partial_l^j \mathfrak{F}(h,h),\ f\rangle_{L^2_{x,v}}\right| \le \begin{cases} \mathfrak{G}^{s,m}_{x,v,z}(h,h)\,||f||_\Lambda , & \text{if } j \neq 0, \\ \mathfrak{G}^{s,m}_{x,z}(h,h)\,||f||_\Lambda , & \text{if } j = 0. \end{cases} \tag{2-26}$$

Sum up $m = 0, \cdots, r$, then $\exists\, s_0 \in \mathbb{N}$, $\forall s \ge s_0$, there exists a z-independent $C_{\mathfrak{F}} > 0$ such that for all z,

$$\sum_{|m|\le r} (\mathfrak{G}^{s,m}_{x,v,z}(h,h))^2 \le C_{\mathfrak{F}}\, ||h||^2_{H^{s,r}_{x,v}}\, ||h||^2_{H^{s,r}_\Lambda}, \qquad \sum_{|m|\le r} (\mathfrak{G}^{s,m}_{x,z}(h,h))^2 \le C_{\mathfrak{F}}\, ||h||^2_{H^{s,r}_x L^2_v}\, ||h||^2_{H^{s,r}_\Lambda} .$$

With uncertainty in the equation, following the deterministic framework in Briant [ibid.], we define a Lyapunov type functional

$$\begin{aligned} ||\cdot||^2_{\mathcal{H}^s_{\varepsilon\perp}} = &\sum_{|j|+|l|\le s,\,|j|\ge 1} b^{(s)}_{j,l}\, ||\partial_l^j(\mathbb{I}-\Pi_{\mathfrak{L}})\cdot||^2_{L^2_{x,v}} + \sum_{|l|\le s} \alpha^{(s)}_l\, ||\partial_l^0 \cdot||^2_{L^2_{x,v}} \\ &+ \sum_{|l|\le s,\, i,\, c_i(l)>0} \varepsilon\, a^{(s)}_{i,l}\, \langle \partial^{\delta_i}_{l-\delta_i}\cdot,\ \partial^0_l \cdot\rangle_{L^2_{x,v}} , \end{aligned} \tag{2-27}$$

and the corresponding Sobolev norms

$$|||h|||^2_{\mathcal{H}^{s,r}_{\varepsilon\perp}} = \sum_{|m|\le r} ||\partial^m h||^2_{\mathcal{H}^s_{\varepsilon\perp}}, \qquad ||h||_{\mathcal{H}^{s,r}_{\varepsilon\perp}L^\infty_z} = \sup_{z\in I_z} ||h||_{\mathcal{H}^{s,r}_{\varepsilon\perp}} .$$

The following theorem is from L. Liu and Jin [2017]:

Theorem 2.3. *For all $s \ge s_0$, $\exists\, (b^{(s)}_{j,l}), (\alpha^{(s)}_l), (a^{(s)}_{i,l}) > 0$ and $0 \le \varepsilon_d \le 1$, such that for all $0 \le \varepsilon \le \varepsilon_d$,*
(1) $||\cdot||_{\mathcal{H}^s_{\varepsilon\perp}} \sim ||\cdot||_{H^s_{x,v}}$;
(2) Assume $||h_{in}||_{H^s_{x,v}L^\infty_z} \le C_I$, then if h_ε is a solution of (2-22) in $H^{s,r}_{x,v}$ for all z, we have

$$||h_\varepsilon||_{H^{s,r}_{x,v}L^\infty_z} \le C_I\, e^{-\tau_s t}, \quad ||h_\varepsilon||_{H^s_{x,v}H^r_z} \le C_I\, e^{-\tau_s t}, \quad \text{for} \quad \alpha = 1 ; \tag{2-28}$$

$$||h_\varepsilon||_{H^{s,r}_{x,v}L^\infty_z} \le C_I\, e^{-\varepsilon\tau_s t}, \quad ||h_\varepsilon||_{H^s_{x,v}H^r_z} \le C_I\, e^{-\varepsilon\tau_s t}, \quad \text{for} \quad \alpha = 0 , \tag{2-29}$$

where C_I, τ_s are positive constants independent of ε.

Remark 2.4. *Theorem 2.3 provides the regularity of h (thus f) in the random space, which preserves the regularity of the initial data in time. Furthermore, it shows that the uncertainty from the initial datum will eventually diminish and the solution will exponentially decay to the deterministic global equilibrium in the long time, with a decay rate of $\mathcal{O}(e^{-t})$ under the incompressible Navier-Stokes scaling and $\mathcal{O}(e^{-\varepsilon t})$ under the acoustic scaling.*

2.3 The Boltzmann equation with uncertainties. As an example of the general theory in subSection 2.2, we consider the Boltzmann equation with uncertain initial data and uncertain collision kernel:

$$\text{(2-30)} \qquad \begin{cases} \partial_t f + \dfrac{1}{\varepsilon^\alpha} v\cdot\nabla_x f = \dfrac{1}{\varepsilon^{1+\alpha}}\mathcal{Q}(f,f), \\ f(0,x,v,z) = f^0(x,v,z), \qquad x\in\Omega\subset\mathbb{T}^d,\ v\in\mathbb{R}^d,\ z\in I_z\,. \end{cases}$$

The collision operator is

$$\mathcal{Q}(f,f) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B(|v-v_*|,\cos\theta,z)\,(f'f'_* - ff_*)\,dv_*\,d\sigma.$$

We adopt notations $f' = f(v')$, $f_* = f(v_*)$ and $f'_* = f(v'_*)$, where

$$v' = (v+v_*)/2 + (|v-v_*|/2)\sigma, \qquad v'_* = (v+v_*)/2 - (|v-v_*|/2)\sigma$$

are the post-collisional velocities of particles with pre-collisional velocities v and v_*. $\theta\in[0,\pi]$ is the deviation angle between $v'-v'_*$ and $v-v_*$. The global equilibrium distribution is given by the Maxwellian distribution

$$\text{(2-31)} \qquad \mathfrak{M}(\rho_\infty,u_\infty,T_\infty) = \frac{\rho_\infty}{(2\pi T_\infty)^{N/2}}\,\exp\left(-\frac{|u_\infty - v|^2}{2T_\infty}\right),$$

where ρ_∞, u_∞, T_∞ are the density, mean velocity and temperature of the gas

$$\rho_\infty = \int_{\Omega\times\mathbb{R}^d} f(v)\,dxdv, \qquad u_\infty = \frac{1}{\rho_\infty}\int_{\Omega\times\mathbb{R}^d} vf(v)\,dxdv,$$
$$T_\infty = \frac{1}{N\rho_\infty}\int_{\Omega\times\mathbb{R}^d} |u_\infty - v|^2 f(v)\,dxdv,$$

which are all determined by the initial datum due to the conservation properties. We will consider hard potentials with B satisfying Grad's angular cutoff, that is,

$$\text{(2-32)} \qquad B(|v-v_*|,\cos\theta,z) = \phi(|v-v_*|)\,b(\cos\theta,z), \qquad \phi(\xi) = C_\phi\,\xi^\gamma, \text{ with } \gamma\in[0,1],$$
$$\forall\eta\in[-1,1],\ |b(\eta,z)|\le C_b,\ |\partial_\eta b(\eta,z)|\le C_b,\ |\partial_z^k b(\eta,z)|\le C_b^*,\ \forall\,0\le k\le r\,.$$

where b is non-negative and not identically equal to 0. Recall that h solves (2-22), with the linearized collision operator given by

$$\mathfrak{L}(h) = M^{-1}\left[\mathfrak{Q}(Mh, \mathfrak{M}) + \mathfrak{Q}(\mathfrak{M}, Mh)\right],$$

while the bilinear part is given by

$$\mathfrak{T}(h,h) = 2M^{-1}\mathfrak{Q}(Mh, Mh) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \phi(|v-v_*|)\, b(\cos\theta, z) M_* (h'_* h' - h_* h)\, dv_* d\sigma .$$

The the coercivity norm used in (2-25) is

$$||h||_\Lambda = ||h(1+|v|)^{\gamma/2}||_{L^2}. \tag{2-33}$$

The coercivity argument of $\mathfrak{L}$ is proved in Mouhot [2006]:

$$-\langle h,\ \mathfrak{L}(h)\rangle_{L^2_v} \geq \lambda\, ||h^\perp||^2_{\Lambda_v}. \tag{2-34}$$

Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials have been obtained in Mouhot and Baranger [2005] and extended to estimates given in Mouhot [2006]. Proofs of $\mathfrak{L}$ satisfying Equation (2-25) and $\mathfrak{T}$ satisfying (2-26), even for random collision kernel satisfying conditions given in (2-32), were given in L. Liu and Jin [2017]. Thus Theorem 2.3 holds for the Boltzmann equation with random initial data and collision kernel. Similar results can be extended to Landau equation and semi-classical relaxation model, see L. Liu and Jin [ibid.].

2.4 The Vlasov-Poisson-Fokker-Planck system. One kinetic equation which does not fit the collisional framework presented in subSection 2.2 is the Vlasov-Poisson-Fokker-Planck (VPFP) system that arises in the kinetic modeling of the Brownian motion of a large system of particles in a surrounding bath Chandrasekhar [1943]. One application of such system is the electrostatic plasma, in which one considers the interactions between the electrons and a surrounding bath via the Coulomb force. With the electrical potential $\phi(t, \mathbf{x}, \mathbf{z})$, the equations read

$$\begin{cases} \partial_t f + \frac{1}{\delta}\partial_x f - \frac{1}{\varepsilon}\partial_x\phi\partial_v f = \frac{1}{\delta\varepsilon}\mathfrak{T} f, \\ -\partial_{xx}\phi = \rho - 1, \quad t > 0, \quad x \in \Omega \subset \mathbb{R},\ v \in \mathbb{R},\ z \in I_z, \end{cases} \tag{2-35}$$

with initial condition

$$f(0, x, v, z) = f^0(x, v, z). \tag{2-36}$$

Here, $\mathcal{F}$ is the Fokker-Planck operator describing the Brownian motion of the particles,

$$\mathcal{F} f = \partial_v \left(\mathfrak{M} \nabla_v \left(\frac{f}{\mathfrak{M}} \right) \right), \tag{2-37}$$

where $\mathfrak{M}$ is the *global equilibrium* or *global Maxwellian*,

$$\mathfrak{M} = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}}. \tag{2-38}$$

δ is the reciprocal of the scaled thermal velocity, ε represents the scaled thermal mean free path. There are two different regimes for this system. One is the *high field regime*, where $\delta = 1$. As $\varepsilon \to 0$, f goes to the local Maxwellian $\mathfrak{M}_l = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v-\nabla_x \phi|^2}{2}}$, and the VPFP system converges to a hyperbolic limit Arnold, Carrillo, Gamba, and C.-W. Shu [2001], Goudon, Nieto, Poupaud, and Soler [2005], and Nieto, Poupaud, and Soler [2001]:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho \nabla_x \phi) = 0, \\ -\Delta_{\mathbf{x}} \phi = \rho - 1. \end{cases} \tag{2-39}$$

Another regime is the *parabolic regime*, where $\delta = \varepsilon$. When $\varepsilon \to 0$, f goes to the global Maxwellian $\mathfrak{M}$, and the VPFP system converges to a parabolic limit Poupaud and Soler [2000]:

$$\begin{cases} \partial_t \rho - \nabla_x \cdot (\nabla_x \rho - \rho \nabla_x \phi) = 0, \\ -\Delta_{\mathbf{x}} \phi = \rho - 1. \end{cases} \tag{2-40}$$

Define the L^2 space in the measure of

$$\mathrm{d}\mu = \mathrm{d}\mu(x, v, z) = \omega(z)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}z. \tag{2-41}$$

With this measure, one has the corresponding Hilbert space with the following inner product and norms:

(2-42)

$$< f, g >= \int_\Omega \int_{\mathbb{R}} \int_{I_z} fg \, \mathrm{d}\mu(x, v, z), \quad \text{or,} \quad < \rho, j >= \int_\Omega \int_{I_z} \rho j \, \mathrm{d}\mu(x, z),$$

with norm

$$\| f \|^2 =< f, f > .$$

In order to get the convergence rate of the solution to the global equilibrium, define

$$h = \frac{f - \mathfrak{M}}{\sqrt{\mathfrak{M}}}, \quad \sigma = \int_{\mathbb{R}} h \sqrt{M}\, dv, \quad u = \int_{\mathbb{R}} h\, v \sqrt{M}\, dv, \tag{2-43}$$

where h is the (microscopic) *fluctuation* around the equilibrium, σ is the (macroscopic) density fluctuation, and u is the (macroscopic) velocity fluctuation. Then the microscopic quantity h satisfies,

$$\varepsilon\delta\partial_t h + \beta v\partial_x h - \delta\partial_x\phi\partial_v h + \delta\frac{v}{2}\partial_x\phi h + \delta v\sqrt{M}\partial_x\phi = \mathfrak{L}^F h, \tag{2-44}$$

$$\partial_x^2\phi = -\sigma, \tag{2-45}$$

while the macroscopic quantities σ and u satisfy

$$\delta\partial_t\sigma + \partial_x u = 0, \tag{2-46}$$

$$\varepsilon\delta\partial_t u + \varepsilon\partial_x\sigma + \varepsilon\int v^2\sqrt{M}(1-\Pi)\partial_x h\,dv + \delta\partial_x\phi\sigma + u + \delta\partial_x\phi = 0\,, \tag{2-47}$$

where $\mathfrak{L}^F$ is the so-called linearized Fokker-Planck operator,

$$\mathfrak{L}^F h = \frac{1}{\sqrt{\mathfrak{m}}}\mathfrak{F}\left(\mathfrak{m}+\sqrt{\mathfrak{m}}h\right) = \frac{1}{\sqrt{\mathfrak{m}}}\partial_v\left(\mathfrak{m}\partial_v\left(\frac{h}{\sqrt{\mathfrak{m}}}\right)\right). \tag{2-48}$$

Introduce projection operator

$$\Pi h = \sigma\sqrt{\mathfrak{m}} + vu\sqrt{\mathfrak{m}}. \tag{2-49}$$

Furthermore, we also define the following norms and energies,

$$\|h\|^2_{L^2(v)} = \int_{\mathbb{R}} h^2\,dv, \quad \|f\|^2_{H^m} = \sum_{l=0}^{m}\|\partial_z^l f\|^2,$$
$$E_h^m = \|h\|^2_{H^m} + \|\partial_x h\|^2_{H^{m-1}}, \quad E_\phi^m = \|\partial_x\phi\|^2_{H^m} + \|\partial_x^2\phi\|^2_{H^{m-1}}.$$

We need the following hypocoercivity properties proved in Duan, Fornasier, and Toscani [2010]:

Proposition 2.5. *For $\mathfrak{L}^F$ defined in (2-48),*

(a) $-\langle\mathfrak{L}^F h, h\rangle = -\langle L(1-\Pi)h, (1-\Pi)h\rangle + \|u\|^2$;

(b) $-\langle\mathfrak{L}^F(1-\Pi)h, (1-\Pi)h\rangle = \|\partial_v(1-\Pi)h\|^2 + \frac{1}{4}\|v(1-\Pi)h\|^2 - \frac{1}{2}\|(1-\Pi)h\|^2$;

(c) $-\langle\mathfrak{L}^F(1-\Pi)h, (1-\Pi)h\rangle \geq \|(1-\Pi)h\|^2$;

(d) There exists a constant $\lambda_0 > 0$, such that the following hypocoercivity holds,

$$-\langle\mathfrak{L}^F h, h\rangle \geq {}_0\|(1-\Pi)h\|_v^2 + \|u\|^2, \tag{2-50}$$

and the largest $\lambda_0 = \frac{1}{7}$ in one dimension.

The following results were obtained in Jin and Y. Zhu [n.d.].

Theorem 2.6. *For the high field regime ($\delta = 1$), if*

$$E_h^m(0) + \frac{1}{\varepsilon^2} E_\phi^m(0) \le \frac{C_0}{\varepsilon}, \tag{2-51}$$

then,

(2-52)

$$E_h^m(t) \le \frac{3}{\lambda_0} e^{-\frac{t}{\varepsilon^2}} \left(E_h^m(0) + \frac{1}{\varepsilon^2} E_\phi^m(0) \right), \qquad E_\phi^m(t) \le \frac{3}{\lambda_0} e^{-t} \left(\varepsilon^2 E_h^m(0) + E_\phi^m(0) \right);$$

For the parabolic regime ($\delta = \varepsilon$), if

$$E_h^m(0) + \frac{1}{\varepsilon^2} E_\phi^m(0) \le \frac{C_0}{\varepsilon^2}, \tag{2-53}$$

then,

(2-54)

$$E_h^m(t) \le \frac{3}{\lambda_0} e^{-\frac{t}{\varepsilon}} \left(E_h^m(0) + \frac{1}{\varepsilon^2} E_\phi^m(0) \right). \qquad E_\phi^m(t) \le \frac{3}{\lambda_0} e^{-t} \left(\varepsilon^2 E_h^m(0) + E_\phi^m(0) \right).$$

Here $C_0 = 2\lambda_0/(32BC_1^2\sqrt{\varepsilon})^2$, $B = 48\sqrt{m}\binom{m}{[m/2]}$ is a constant only depending on m, $[m/2]$ is the smallest integer larger or equal to $\frac{m}{2}$, and C_1 is the Sobolev constant in one dimension, and $m \ge 1$.

These results show that the solution will converge to the global Maxwellian $\mathfrak{M}$. Since $\mathfrak{M}$ is independent of z, one sees that the impact of the randomness dies out exponentially in time, in both asymptotic regimes. One should also note the small initial data requirement: $E_\phi^m = O(\varepsilon)$ for $\delta = 1$.

The above theorem also leads to the following regularity result for the solution to VPFP system:

Theorem 2.7. *Under the same condition given in Theorem 2.6, for $x \in [0, l]$, one has*

$$\|f(t)\|_{H_z^m}^2 \le \frac{3}{\lambda_0} E^m(0) + 2l^2, \tag{2-55}$$

where $E^m(0) = E_h^m(0) + \frac{1}{\varepsilon^2} E_\phi^m$.

This Theorem shows that the regularity of the initial data in the random space is preserved in time. Furthermore, the bound of the Sobolev norm of the solution is independent of the small parameter ε.

3 Stochastic Galerkin methods for random kinetic equations

In order to quantify the uncertainty of kinetic equation we will use polynomial chaos expansion based stochastic Galerkin (SG) method Ghanem and Spanos [1991] and Xiu and Karniadakis [2002]. As is well known, the SG methods can achieve spectral accuracy if the solution has the regularity. This makes it very efficient if the dimension of the random space is not too high, compared with the classical Monte-Carlo method.

Due to its Galerkin formulation, mathematical analysis of the SG methods can be conducted more conveniently. Indeed many of the analytical methods well-established in kinetic theory can be easily adopted or extended to study the SG system of the random kinetic equations. For example, the study of regularity, and hypocoercivity based sensitivity analysis, as presented in Section 2, can been used to analyze the SG methods. Furthermore, for multiscale kinetic equations, the SG methods allow one to extend the deterministic *Asymptotic-preserving* framework–a popular computational paradigm for multiscale kinetic and hyperbolic problems–to the random problem naturally. Finally, kinetic equations often contain small parameters such as the mean free path/time which asymptotically lead to macroscopic hyperbolic/diffusion equations. We are interested in developing the stochastic analogue of the asymptotic-preserving (AP) scheme, a scheme designed to capture the asymptotic limit at the discrete level. The SG method yields systems of deterministic equations that *resemble the deterministic kinetic equations*, although in vector forms. Thus it allows one to easily use the deterministic AP framework for the random problems, and allowing minimum "intrusion" to the legacy deterministic codes.

3.1 The generalized polynomial chaos expansion based SG methods. In the generalized polynomial chaos (gPC) expansion, one approximates the solution of a stochastic problem via an orthogonal polynomial series by seeking an expansion in the following form:

$$f(t,\mathbf{x},v,z) \approx \sum_{|\mathbf{k}|=0}^{KM} f_{\mathbf{k}}(t,\mathbf{x},v)\Phi_{\mathbf{k}}(z) := f^K(t,\mathbf{x},v,z), \tag{3-1}$$

where $\mathbf{k} = (k_1,\ldots,k_n)$ is a multi-index with $|\mathbf{k}| = k_1 + \cdots + k_n$. $\{\Phi_{\mathbf{k}}(z)\}$ are from $\mathbb{P}_K^n$, the set of all n-variate polynomials of degree up to M and satisfy

$$< \Phi_{\mathbf{k}}, \Phi_{\mathbf{j}} >_\omega = \int_{I_z} \Phi_{\mathbf{k}}(z)\Phi_{\mathbf{j}}(z)\omega(z)\,\mathrm{d}z = \delta_{\mathbf{kj}}, \quad 0 \le |\mathbf{k}|, |\mathbf{j}| \le K.$$

Here $\delta_{\mathbf{kj}}$ is the Kronecker delta function. The orthogonality with respect to $\omega(z)$, the probability density function of z, then defines the orthogonal polynomials. For example, the

Gaussian distribution defines the Hermite polynomials; the uniform distribution defines the Legendre polynomials, etc.

Now inserting (3-1) into a general kinetic equation

$$\text{(3-2)}\qquad \begin{cases} \partial_t f + v\cdot\nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}}\phi\cdot\nabla_v f = \mathcal{Q}(f), \quad t>0,\ \mathbf{x}\in\Omega,\ v\in\mathbb{R}^d,\ z\in I_z,\\ f(0,\mathbf{x},v) = f^0(\mathbf{x},v), \quad \mathbf{x}\in\Omega,\ v\in\mathbb{R}^d,\ z\in I_z. \end{cases}$$

Upon a standard Galerkin projection, one obtains for each $0\le|\mathbf{k}|\le M$,

$$\text{(3-3)}\qquad \begin{cases} \partial_t f_{\mathbf{k}} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_{\mathbf{k}} - \displaystyle\sum_{|\mathbf{j}|=0}^{K} \nabla_{\mathbf{x}}\phi_{\mathbf{kj}}\cdot\nabla_{\mathbf{v}} f_{\mathbf{j}} = \mathcal{Q}_{\mathbf{k}}(f^K), \quad t>0,\ \mathbf{x}\in\Omega,\ \mathbf{v}\in\mathbb{R}^d,\\ f_{\mathbf{k}}(0,\mathbf{x},\mathbf{v}) = f^0_{\mathbf{k}}(\mathbf{x},\mathbf{v}), \quad \mathbf{x}\in\Omega,\ \mathbf{v}\in\mathbb{R}^d, \end{cases}$$

with

$$\mathcal{Q}_{\mathbf{k}}(f^K) := \int_{I_z} \mathcal{Q}(f^K)(t,\mathbf{x},\mathbf{v},z)\Phi_{\mathbf{k}}(z)\omega(z)\,\mathrm{d}z, \quad \phi_{\mathbf{kj}} := \int_{I_z} \phi(t,\mathbf{x},z)\Phi_{\mathbf{k}}(z)\Phi_{\mathbf{j}}(z)\omega(z)\,\mathrm{d}z,$$

$$f^0_{\mathbf{k}} := \int_{I_z} f^0(\mathbf{x},\mathbf{v},z)\Phi_{\mathbf{k}}(z)\omega(z)\,\mathrm{d}z.$$

We also assume that the potential $\phi(t,\mathbf{x},z)$ is given a priori for simplicity (the case that it is coupled to a Poisson equation can be treated similarly).

Therefore, one has a system of *deterministic* equations to solve and the unknowns are gPC coefficients $f_{\mathbf{k}}$, which are *independent of* z. Mostly importantly, the resulting SG system is just a vector analogue of its deterministic counterpart, thus allowing straightforward extension of the existing deterministic kinetic solvers. Once the coefficients $f_{\mathbf{k}}$ are obtained through some numerical procedure, the statistical information such as the mean, covariance, standard deviation of the true solution f can be approximated as

$$\mathbb{E}[f]\approx f_{\mathbf{0}}, \quad \mathrm{Var}[f]\approx \sum_{|\mathbf{k}|=1}^{K} f_{\mathbf{k}}^2, \quad \mathrm{Cov}[f]\approx \sum_{|\mathbf{i}|,|\mathbf{j}|=1}^{K} f_{\mathbf{i}} f_{\mathbf{j}}.$$

3.2 Hypocoercivity estimate of the SG system. The hypocoercivity theoy presented in Section 2.2 can be used to study the properties of the SG methods. Here we take $\phi=0$. Assume the random collision kernel has the assumptions given by (2-32). Consider the perturbative form

$$\text{(3-4)}\qquad f_{\mathbf{k}} = \mathfrak{M} + \varepsilon M h_{\mathbf{k}},$$

where $h_{\mathbf{k}}$ is the coefficient of the following gPC expansion

$$h(t,\mathbf{x},v,z) \approx \sum_{|\mathbf{k}|=0}^{M} h_{\mathbf{k}}(t,\mathbf{x},v)\Phi_{\mathbf{k}}(z) := h^K(t,\mathbf{x},v,z)\,.$$

Inserting ansatz (3-4) into (3-3) and conducting a standard Galerkin projection, one obtains the gPC-SG system for $h_{\mathbf{k}}$:

$$\text{(3-5)} \qquad \begin{cases} \partial_t h_{\mathbf{k}} + \dfrac{1}{\varepsilon} v\cdot\nabla_x h_{\mathbf{k}} = \dfrac{1}{\varepsilon^2}\mathfrak{L}_{\mathbf{k}}(h^K) + \dfrac{1}{\varepsilon}\mathfrak{T}_{\mathbf{k}}(h^K,h^K), \\ h_{\mathbf{k}}(0,x,v) = h^0_{\mathbf{k}}(x,v), \qquad x\in\Omega\subset\mathbb{T}^d,\ v\in\mathbb{R}^d, \end{cases}$$

for each $1\le|\mathbf{k}|\le K$, with a periodic boundary condition and the initial data given by

$$h^0_{\mathbf{k}} := \int_{I_z} h^0(x,v,z)\psi_{\mathbf{k}}(z)\,\pi(z)dz.$$

For the Boltzmann equation, the collision parts are given by

$$\mathfrak{L}_{\mathbf{k}}(h^K) = \mathfrak{L}^+_{\mathbf{k}}(h^K) = \sum_{|\mathbf{i}|=1}^{K}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widetilde{S}_{\mathbf{ki}}\,\phi(|v-v_*|)\,(h_{\mathbf{i}}(v')M(v'_*)+h_{\mathbf{i}}(v'_*)M(v'))\,M(v_*)\,dv$$

$$- M(v)\sum_{|\mathbf{i}|=1}^{K}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widetilde{S}_{\mathbf{ki}}\,\phi(|v-v_*|)\,h_{\mathbf{i}}(v_*)M(v_*)\,dv_*d\sigma - \sum_{|\mathbf{i}|=1}^{K}\nu_{\mathbf{ki}}h_{\mathbf{i}}\,,$$

$$\mathfrak{T}_{\mathbf{k}}(h^K,h^K)(t,x,v) = \sum_{|\mathbf{i}|,|\mathbf{j}|=1}^{K}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} S_{\mathbf{kij}}\,\phi(|v-v_*|)\,M(v_*)\,(h_{\mathbf{i}}(v')h_{\mathbf{j}}(v'_*)-h_{\mathbf{i}}(v)h_{\mathbf{j}}(v_*))\,dv$$

with

$$\widetilde{S}_{\mathbf{ki}} := \int_{I_z} b(\cos\theta,z)\,\psi_{\mathbf{k}}(z)\psi_{\mathbf{i}}(z)\,\pi(z)dz, \qquad \nu_{\mathbf{ki}} := \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widetilde{S}_{\mathbf{ki}}\,\phi(|v-v_*|)\,\mathfrak{M}(v_*)\,dv_*d\sigma$$

and $$S_{\mathbf{kij}} := \int_{I_z} b(\cos\theta,z)\,\psi_{\mathbf{k}}(z)\psi_{\mathbf{i}}(z)\psi_{\mathbf{j}}(z)\,\pi(z)dz.$$

For technical reasons, we assume $z\in I_z$ is one dimensional and I_z has finite support $|z|\le C_z$ (which is the case, for example, for the uniform and Beta distribution). In L. Liu and Jin [2017] the following results are given:

Theorem 3.1. *Assume the collision kernel B satisfies (2-32) and is linear in z, with the form of*

$$\text{(3-6)} \qquad b(\cos\theta,z) = b_0(\cos\theta) + b_1(\cos\theta)z\,,$$

with $|\partial_z b| \leq= O(\varepsilon)$. *We also assume the technical condition*

$$||\psi_k||_{L^\infty} \leq C k^p, \qquad \forall\, k, \tag{3-7}$$

with a parameter $p > 0$. *Let* $q > p + 2$, *define the energy* E^K *by*

$$E^K(t) = E^K_{s,q}(t) = \sum_{k=1}^{K} ||k^q h_k||^2_{H^s_{x,v}}, \tag{3-8}$$

with the initial data satisfying $E^K(0) \leq \eta$. *Then for all* $s \geq s_0$, $0 \leq \varepsilon_d \leq 1$, *such that for* $0 \leq \varepsilon \leq \varepsilon_d$, *if* h^K *is a gPC solution of (3-5) in* $H^s_{x,v}$, *we have the following:*
(i) Under the incompressible Navier-Stokes scaling ($\alpha = 1$),

$$E^K(t) \leq \eta\, e^{-\tau t}.$$

(ii) Under the acoustic scaling ($\alpha = 0$),,

$$E^K(t) \leq \eta\, e^{-\varepsilon \tau t},$$

where η, τ *are all positive constants that only depend on* s *and* q, *independent of* K *and* z.

Remark 3.2. *The choice of energy* E^K *in (3-8) enables one to obtain the desired energy estimates with initial data independent of* K *R. Shu and Jin [2017].*

From here, one also concludes that, $||h^K||_{H^s_{x,v} L^\infty_z}$ also decays exponentially in time, with the same rate as $E^K(t)$, namely

$$||h^K||_{H^s_{x,v} L^\infty_z} \leq \eta\, e^{-\tau t} \tag{3-9}$$

in the incompressible Navier-Stokes scaling, and

$$||h^K||_{H^s_{x,v} L^\infty_z} \leq \eta\, e^{-\varepsilon \tau t}$$

in the acoustic scaling.

For other kinetic models like the Landau equation, the proof is similar and we omit it here.

L. Liu and Jin [2017] also gives the following error estimates on the SG method for the uncertain Boltzmann equations.

Theorem 3.3. *Suppose the assumptions on the collision kernel and basis functions in Theorem 3.1 are satisfied, and the initial data are the same in those in Theorem 2.3, then*
(i) Under the incompressible Navier-Stokes scaling,

$$||h - h^K||_{H^s_z} \leq C_e \frac{e^{-\lambda t}}{K^r}, \tag{3-10}$$

(ii) Under the acoustic scaling,

$$||h - h^K||_{H_z^s} \le C_e \frac{e^{-\varepsilon\lambda t}}{K^r}, \tag{3-11}$$

with the constants C_e, $\lambda > 0$ *independent of* K *and* ε.

The above results not only give the regularity of the SG solutions, which are the same as the initial data, but also show that the numerical fluctuation h^K converges with spectral accuracy to h, and the numerical error will also decay exponentially in time in the random space.

For more general solution (not the perturbative one given by (2-21) to the uncertain Boltzmann equation, one cannot obtain similar estimates. Specifically, for $\alpha = 1$, as $\varepsilon \to 0$, the moments of f is governed by the compressible Euler equations whose solution may develop shocks, thus the Sobolev norms used in this paper are not adequate. For $\varepsilon = O(1)$, Hu and Jin [2016] proved that, in the space homogeneous case, the regularity of the initial data in the random space is preserved in time. They also introduced a fast algorithm to compute the collision operator Q_k. When the random variable is in higher dimension, sparse grids can be used, see R. Shu, Hu, and Jin [2017].

4 Stochastic asymptotic-preserving (sAP) schemes for multiscale random kinetic equations

When ε is small, numerically solving the kinetic equations is challenging since time and spatial discretizations need to resolve ε. Asymptotic-preserving (AP) schemes are those that mimic the asymptotic transitions from kinetic equations to their hydrodynamic/diffusion limits in the discrete setting Jin [1999, 2012]. The AP strategy has been proved to be a powerful and robust technique to address multiscale problems in many kinetic problems. The main advantage of AP schemes is that they are very efficient even when ε is small, since they do not need to resolve the small scales numerically, and yet can still capture the macroscopic behavior governed by the limiting macroscopic equations. Indeed, it was proved, in the case of linear transport with a diffusive scaling, an AP scheme converges uniformly with respect to the scaling parameter Golse, Jin, and Levermore [1999]. This is expected to be true for all AP schemes Jin [2012], although specific proofs are needed for specific problems. AP schemes avoid the difficulty of coupling a microscopic solver with a macroscopic one, as the micro solver *automatically* becomes a macro solver as $\varepsilon \to 0$. Interested readers may also consult earlier reviews in this subject Jin [2012], Degond and Deluzet [2017b], and Hu, Jin, and Li [2017].

Here we are interested in the scenario when the uncertainty (random inputs) and small scaling both present in a kinetic equation. Since the SG method makes the random kinetic

equations into deterministic systems which are vector analogue of the original scalar deterministic kinetic equations, one can naturally utilize the deterministic AP machinery to solve the SG system to achieve the desired AP goals. To this aim, the notion of *stochastic asymptotic preserving (sAP)* was introduced in Jin, Xiu, and X. Zhu [2015]. A scheme is sAP if an SG method for the random kinetic equation becomes an SG approximation for the limiting macroscopic, random (hydrodynamic or diffusion) equation as $\varepsilon \to 0$, with highest gPC degree, mesh size and time step all held fixed. Such schemes guarantee that even for $\varepsilon \to 0$, *all* numerical parameters, including the number of gPC modes, can be chosen only for accuracy requirement and *independent* of ε.

Next we use the linear transport Equation (2-1) as an example to derive an sAP scheme. It has the merit that rigorous convergence and sAP theory can be established, see Jin, J.-G. Liu, and Ma [n.d.].

4.1 An sAP-SG method for the linear transport equation. We assume the complete orthogonal polynomial basis in the Hilbert space $H(I_z;\omega(z)\,\mathrm{d}z)$ corresponding to the weight $\omega(z)$ is $\{\phi_i(z), i = 0, 1, \cdots, \}$, where $\phi_i(z)$ is a polynomial of degree i and satisfies the orthonormal condition:

$$\langle \phi_i, \phi_j \rangle_\omega = \int \phi_i(z)\phi_j(z)\omega(z)\,\mathrm{d}z = \delta_{ij}.$$

Here $\phi_0(z) = 1$, and δ_{ij} is the Kronecker delta function. Since the solution $f(t,\cdot,\cdot,\cdot)$ is defined in $L^2([0,1]\times[-1,1]\times\mathbb{I}_z; d\mu)$, one has the gPC expansion

$$f(t,x,v,z) = \sum_{i=0}^{\infty} f_i(t,x,v)\,\phi_i(z), \quad \hat{f} = (f_i)_{i=0}^{\infty} := (\bar{f}, \hat{f}_1).$$

The mean and variance of f can be obtained from the expansion coefficients as

$$\bar{f} = E(f) = \int_{I_z} f\omega(z)\,\mathrm{d}z = f_0, \quad \mathrm{var}(f) = |\hat{f}_1|^2.$$

Denote the SG solution by

$$f^K = \sum_{i=0}^{K} f_i\,\phi_i, \quad \hat{f}^K = (f_i)_{i=0}^{M} := (\bar{f}, \hat{f}_1^K), \tag{4-1}$$

from which one can extract the mean and variance of f^K from the expansion coefficients as

$$E(f^K) = \bar{f}, \quad \mathrm{var}(f^K) = |\hat{f}_1^K|^2 \le \mathrm{var}(f).$$

Furthermore, we define

$$\sigma_{ij} = \langle \phi_i, \sigma\phi_j \rangle_\omega, \quad \Sigma = \left(\sigma_{ij}\right)_{M+1,M+1}; \quad \sigma^a_{ij} = \langle \phi_i, \sigma^a\phi_j \rangle_\omega, \quad \Sigma^a = \left(\sigma^a_{ij}\right)_{M+1,M+1},$$

for $0 \le i, j \le M$. Let Id be the $(M+1)\times(M+1)$ identity matrix. Σ, Σ^a are symmetric positive-definite matrices satisfying (Xiu [2010])

$$\Sigma \ge \sigma_{\min} \text{ Id}\,.$$

If one applies the gPC ansatz (4-1) into the transport Equation (2-1), and conduct the Galerkin projection, one obtains

$$\varepsilon\partial_t \hat{f} + v\partial_x \hat{f} = -\frac{1}{\varepsilon}(I - [\cdot])\Sigma\hat{f} - \varepsilon\Sigma^a\hat{f} - \hat{S}, \tag{4-2}$$

where $\hat{S}$ is defined similarly as (4-1).

We now use the micro-macro decomposition (Lemou and Mieussens [2008]):

$$\hat{f}(t,x,v,z) = \hat{\rho}(t,x,z) + \varepsilon\hat{g}(t,x,v,z), \tag{4-3}$$

where $\hat{\rho} = [\hat{f}]$ and $[\hat{g}] = 0$, in (4-2) to get

$$\partial_t\hat{\rho} + \partial_x[v\hat{g}] = -\Sigma^a\hat{\rho} + \hat{S}, \tag{4-4a}$$

$$\partial_t\hat{g} + \frac{1}{\varepsilon}(I - [\cdot])(v\partial_x\hat{g}) = -\frac{1}{\varepsilon^2}\Sigma\hat{g} - \Sigma^a\hat{g} - \frac{1}{\varepsilon^2}v\partial_x\hat{\rho}, \tag{4-4b}$$

with initial data

$$\hat{\rho}(0,x,z) = \hat{\rho}_0(x,z), \quad \hat{g}(0,x,v,z) = \hat{g}_0(x,v,z)\,.$$

It is easy to see that system (4-4) formally has the diffusion limit as $\varepsilon \to 0$:

$$\partial_t\hat{\rho} = \partial_x(K\partial_x\hat{\rho}) - \Sigma^a\hat{\rho} + \hat{S}\,, \tag{4-5}$$

where

$$K = \frac{1}{3}\Sigma^{-1}\,. \tag{4-6}$$

This is the sG approximation to the random diffusion Equation (2-8). Thus the gPC approximation is sAP in the sense of Jin, Xiu, and X. Zhu [2015].

Let f be the solution to the linear transport Equation (2-1)–(2-2). Use the K-th order projection operator $P_M : P_K f = \sum_{i=0}^{K} f_i\phi_i(z)$, the error arisen from the gPC-sG can be split into two parts r_K and e_K,

$$f - f^K = f - P_K f + P_K f - f^K := r_K + e_K, \tag{4-7}$$

where $r_K = f - P_K f$ is the projection error, and $e_K = P_K M f - f^K$ is the SG error.

Here we summarize the results of Jin, J.-G. Liu, and Ma [n.d.].

Lemma 4.1 (Projection error). *Under all the assumption in Theorem 2.1 and Theorem 2.2, we have for $t \in (0, T]$ and any integer $k = 0, \dots, m$,*

$$\|r_K\|_\Gamma \le \frac{C_1}{K^k}. \tag{4-8}$$

Moreover,

$$\left\| [r_K] - r_K \right\|_\Gamma \le \frac{C_2}{K^k}\varepsilon, \tag{4-9}$$

where C_1 and C_2 are independent of ε.

Lemma 4.2 (SG error). *Under all the assumptions in Theorem 2.1 and Theorem 2.2, we have for $t \in (0, T]$ and any integer $k = 0, \dots, m$,*

$$\|e_M\|_\Gamma \le \frac{C(T)}{M^k}, \tag{4-10}$$

where $C(T)$ is a constant independent of ε.

Combining the above lemmas gives the uniform (in ε) convergence theorem:

Theorem 4.3. *If for some integer $m \ge 0$,*

$$\|\sigma(z)\|_{H^k} \le C_\sigma, \quad \|D^k f_0\|_\Gamma \le C_0, \quad \|D^k(\partial_x f_0)\|_\Gamma \le C_x, \quad k = 0, \dots, m, \tag{4-11}$$

then the error of the sG method is

$$\|f - f^K\|_\Gamma \le \frac{C(T)}{K^k}, \tag{4-12}$$

where $C(T)$ is a constant independent of ε.

Theorem 4.3 gives a uniformly in ε spectral convergence rate, thus one can choose K independent of ε, a very strong sAP property. Such a result is also obtained with the anisotropic scattering case, for the linear semiconductor Boltzmann equation (Jin and L. Liu [2017] and L. Liu [n.d.]).

4.2 A full discretization. By using the SG formulation, one obtains a vector version of the original deterministic transport equation. This enables one to use the deterministic AP methodology. Here, we adopt the micro-macro decomposition based AP scheme developed in Lemou and Mieussens [2008] for the gPC-sG system (4-4).

We take a uniform grid $x_i = ih, i = 0, 1, \cdots N$, where $h = 1/N$ is the grid size, and time steps $t^n = n\Delta t$. ρ_i^n is the approximation of ρ at the grid point (x_i, t^n) while $g_{i+\frac{1}{2}}^{n+1}$ is defined at a staggered grid $x_{i+1/2} = (i + 1/2)h, i = 0, \cdots N - 1$.

The fully discrete scheme for the gPC system (4-4) is

$$\frac{\hat{\rho}_i^{n+1} - \hat{\rho}_i^n}{\Delta t} + \left[v \frac{\hat{g}_{i+\frac{1}{2}}^{n+1} - \hat{g}_{i-\frac{1}{2}}^{n+1}}{\Delta x} \right] = -\Sigma_i^a \hat{\rho}_i^{n+1} + \hat{S}_i, \tag{4-13a}$$

$$\frac{\hat{g}_{i+\frac{1}{2}}^{n+1} - \hat{g}_{i+\frac{1}{2}}^{n}}{\Delta t} + \frac{1}{\varepsilon \Delta x}(I - [.]) \left(v^+ (\hat{g}_{i+\frac{1}{2}}^n - \hat{g}_{i-\frac{1}{2}}^n) + v^- (\hat{g}_{i+\frac{3}{2}}^n - \hat{g}_{i+\frac{1}{2}}^n) \right) = -\frac{1}{\varepsilon^2} \Sigma_i \hat{g}_{i+\frac{1}{2}}^{n+1} - \Sigma^a \hat{g}_{i+\frac{1}{2}}^{n+1} - \frac{1}{\varepsilon^2} v \frac{\hat{\rho}_{i+1}^n - \hat{\rho}_i^n}{\Delta x}. \tag{4-13b}$$

It has the formal diffusion limit when $\varepsilon \to 0$ given by

$$\frac{\hat{\rho}_i^{n+1} - \hat{\rho}_i^n}{\Delta t} - K \frac{\hat{\rho}_{i+1}^n - 2\hat{\rho}_i^n + \hat{\rho}_{i-1}^n}{\Delta x^2} = -\Sigma_i^a \hat{\rho}_i^{n+1} + \hat{S}_i, \tag{4-14}$$

where $K = \frac{1}{3}\Sigma^{-1}$. This is the fully discrete sG scheme for (4-5). Thus the fully discrete scheme is sAP.

One important property for an AP scheme is to have a stability condition independent of ε, so one can take $\Delta t \gg O(\varepsilon)$. The next theorem from Jin, J.-G. Liu, and Ma [n.d.] answers this question.

Theorem 4.4. *Assume $\sigma^a = S = 0$. If Δt satisfies the following CFL condition*

$$\Delta t \le \frac{\sigma_{\min}}{3} \Delta x^2 + \frac{2\varepsilon}{3} \Delta x, \tag{4-15}$$

then the sequences $\hat{\rho}^n$ and $\hat{g}^n$ defined by scheme (4-13) satisfy the energy estimate

$$\Delta x \sum_{i=0}^{N-1} \left((\hat{\rho}_i^n)^2 + \frac{\varepsilon^2}{2} \int_{-1}^{1} \left(\hat{g}_{i+\frac{1}{2}}^n \right)^2 \mathrm{d}v \right) \le \Delta x \sum_{i=0}^{N-1} \left((\hat{\rho}_i^0)^2 + \frac{\varepsilon^2}{2} \int_{-1}^{1} \left(\hat{g}_{i+\frac{1}{2}}^0 \right)^2 \mathrm{d}v \right)$$

for every n, and hence the scheme (4-13) is stable.

Since the right hand side of (4-15) has a lower bound when $\varepsilon \to 0$ (and the lower bound being that of a stability condition of the discrete diffusion Equation (4-14)), the scheme is asymptotically stable and Δt remains finite even if $\varepsilon \to 0$.

A discontinuous Galerkin method based sAP scheme for the same problem was developed in Chen, L. Liu, and Mu [2017], where uniform stability and rigirous sAP property were also proven.

sAP schemes were also developed recently for other multiscale kinetic equations, for example the radiative heat transfer equations Jin and Lu [2017], and the disperse two-phase kinetic-fluid model Jin and R. Shu [2017].

4.3 Numerical examples. We now show one example from Jin, J.-G. Liu, and Ma [n.d.] to illustrate the sAP properties of the scheme. The random variable z is one-dimensional and obeys uniform distribution.

Consider the linear transport Equation (2-1) with $\sigma^a = S = 0$ and random coefficient $\sigma(z) = 2 + z$, subject to zero initial condition $f(0, x, v, z) = 0$ and boundary condition

$$f(t, 0, v, z) = 1, \quad v \geq 0; \qquad f(t, 1, v, z) = 0, \quad v \leq 0.$$

When $\varepsilon \to 0$, the limiting random diffusion equation is

$$\partial_t \rho = \frac{1}{3\sigma(z)} \partial_{xx} \rho, \tag{4-16}$$

with initial and boundary conditions:

$$\rho(0, x, z) = 0, \quad \rho(t, 0, z) = 1, \quad \rho(t, 1, z) = 0.$$

The analytical solution for (4-16) with the given initial and boundary conditions is

$$\rho(t, x, z) = 1 - \operatorname{erf}\left(x / \sqrt{\frac{4}{3\sigma(z)} t} \right). \tag{4-17}$$

When ε is small, we use this as the reference solution, as it is accurate with an error of $O(\varepsilon^2)$. For other implementation details, see Jin, J.-G. Liu, and Ma [ibid.].

In Figure 1, we plot the errors in mean and standard deviation of the SG numerical solutions at $t = 0.01$ with different gPC orders M. Three sets of results are included: solutions with $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles), $\Delta x = 0.01$ (stars). We always use $\Delta t = 0.0002/3$. One observes that the errors become smaller with finer mesh. One can see that the solutions decay rapidly in M and then saturate where spatial discretization error dominates. It is then obvious that the errors due to gPC expansion can be neglected at order $M = 4$ even for $\varepsilon = 10^{-8}$. From this simple example, we can see that using the properly designed sAP scheme, the time, spatial, and random domain discretizations can be chosen independently of the small parameter ε.

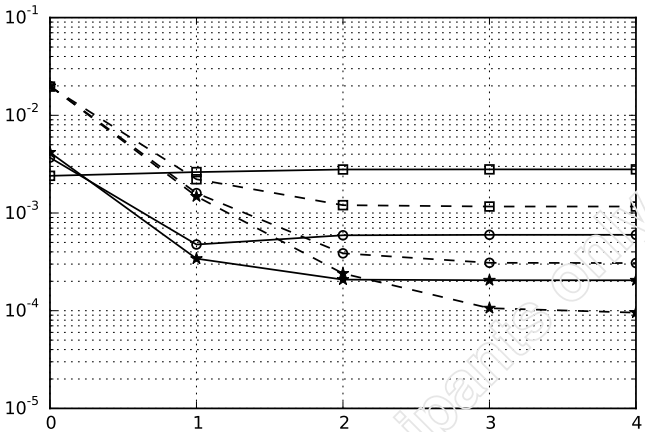

Figure 1: Errors of the mean (solid line) and standard deviation (dash line) of ρ with respect to the gPC order M at $\varepsilon = 10^{-8}$: $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles), $\Delta x = 0.01$ (stars). $\Delta t = 0.0002/3$.

In Figure 2, we examine the difference between the solution at $t = 0.01$ obtained by the 4th-order gPC method with $\Delta x = 0.01$, $\Delta t = \Delta x^2/12$ and the limiting analytical solution (4-17). As expected, we observe the differences become smaller as ε is smaller in a quadratic fashion, before the numerical errors become dominant. This shows the sAP scheme works uniformly for different ε.

5 Conclusion and open problems

In this article we have presented some of our recent development of uncertainty quantification (UQ) for multiscale kinetic equations. The uncertainties for such equations typically come from collision/scattering kernels, boundary data, initial data, forcing terms, among others. Using hypocoercivity theory of kinetic operators, we proved the regularity, sensitivity, and long-time behavior in the random space in a general framework, and then adopted the generalized polynomial chaos based stochastic Galerkin (gPC-SG) method to handle the random inputs which can be proved spectrally accurate, under some regularity assumption on the initial data and ramdom coefficients. When one needs to compute multiple scales, the SG method is constructed to possess the stochastic Asymptotic-Preserving (sAP) property, which allows all numerical parameters, including the gPC order, to be

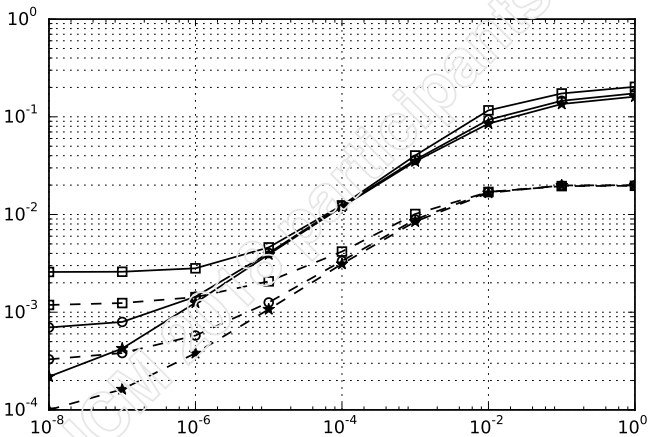

Figure 2: Differences in the mean (solid line) and standard deviation (dash line) of ρ with respect to ε^2, between the limiting analytical solution (4-17) and the 4th-order gPC solution with $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles) and $\Delta x = 0.01$ (stars).

chosen independently of the small parameter, hence is highly efficient when the scaling parameter, the Knudsen number, becomes small.

UQ for kinetic equations is a fairly recent research field, and many interesting problems remain open. We list a few such problems here:

- Whole space problem. Our hypocoercivity theory is developed for periodic spatial domain, which gives exponential decay towards the deterministic global Maxwellian. For the whole space problem, one cannot use the same abstract framework presented in subSection 2.2. For deterministic problems one can obtain only algebraic decay Duan and Strain [2011], Guo [2004], and Strain [2012]. It will be interesting to establish a corresponding theory for the uncertain Boltzmann equation.

- Boundary value problems. The uncertainty could also arise from boundary data. For the Maxwellian boundary condition, one can use the SG framework Hu and Jin [2016]. However, for small ε, the sensitivity analysis for random boundary input remain unexplored, even for the linear transport equation in the diffusive regime.

- Landau damping. While one can use hypocoercivity for collisonal operator or Fokker-Planck operator, for Vlasov type equation (such as the Vlasov-Poisson equations) from collisionless plasma, the system does not have any dissipation, yet one still observes the asymptotica decay of a perturbation around a stationary homogeneous solution and the vanishing of electric field, a phenomenon called the Landau damping Landau [1946]. It will be interesting to invest the impact of uncertainty on Landau damping, although a rigorous nonlinear mathematical theory is very challenging Mouhot and Villani [2011].

- High dimensional random space. When the dimension of the random parameter z is moderate, sparse grids have been introduced R. Shu, Hu, and Jin [2017] and Hu, Jin, and R. Shu [n.d.] using wavelet approximations. Since wavelet basis does not have high order accuracy, it remains to construct sparse grids with high (or spectral) order of accuracy in the random space. When the random dimension is much higher, new methods need to be introduced to reduce the dimension.

- Study of sampling based methods such as collocation and multi-level Monte-Carlo methods. In practice, sampling based non-intrusive methods are attractive since they are based on the deterministic, or legacy codes. So far there has been no analysis done for the stochastic collocation methods for random kinetic equations. Moreover, multi-level Monte-Carlo method could significantly reduce the cost of sampling based methods Giles [2015]. Its application to kinetic equations with uncertainty remains to be investigated.

Despite at its infancy, due to the good regularity and asymptotic behavior in the random space for kinetic equations with uncertain random inputs, the UQ for kinetic equations is a promising research direction that calls for more development in their mathematical theory, efficient numerical methods, and applications. Moreover, since the random parameters in uncertain kinetic equations share some properties of the velocity variable for a kinetic equation, the ideas from kinetic theory can be very useful for UQ Cho, Venturi, and Karniadakis [2016], and vice versa, thus the marrige of the two fields can be very fruitful.

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