

A Domain Decomposition Method for Semilinear Hyperbolic Systems with Two-scale Relaxations

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Abstract

We present a domain decomposition method on a semilinear hyperbolic system with multiple relaxation times. In the region where the relaxation time is small, an asymptotic equilibrium equation can be used for computational efficiency. An interface condition based on the sign of the characteristic speed at the interface is provided to couple the two systems in a domain decomposition setting. A rigorous analysis, based on the Laplace Transform, on the L^2 error estimate is presented for the linear case, which shows how the error of the domain decomposition method depends on the smaller relaxation time, and the boundary and interface layer effects. The given convergence rate is optimal. We present a numerical implementation of this domain decomposition method, and give some numerical results in order to study the performance of this method.

MSC subject classifications. 65M55 35L50

1 Introduction

Consider the hyperbolic system

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, & (1.1a) \\ v_t^\epsilon + u_x^\epsilon = -\frac{1}{\epsilon(x)}(v^\epsilon - f(u^\epsilon)), & (1.1b) \end{cases}$$

where $\epsilon(x)$ is the relaxation time and $f(x)$ satisfies the sub-characteristic condition:

$$|f'(x)| < 1. \quad (1.2)$$

The problem is posed for $x \in [-L, L]$ and $t > 0$ with initial data

$$u^\epsilon(x, 0) = u_0(x), \quad v^\epsilon(x, 0) = v_0(x) \quad (1.3)$$

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and the order of the relaxation time varies considerably over the domain $[-L, L]$. In this paper, we consider the case when $\epsilon(x)$ is given by:

$$\epsilon(x) = 1, \quad x \in [-L, 0); \quad \epsilon(x) = \epsilon, \quad x \in (0, L], \quad (1.4)$$

where $\epsilon \ll 1$ is a small parameter. For the boundary condition, we simply choose the Dirichlet condition for u , i.e:

$$u^\epsilon(x_L, t) = b_L(t), \quad u^\epsilon(x_R, t) = b_R(t). \quad (1.5)$$

More general boundary conditions can also be analyzed by the method of the present paper. The initial data and boundary data are required to be compatible, i.e., $b_1(0) = u_0(x_L)$, $b_2(0) = u_0(x_R)$.

Since the relaxation time is small in the region $(0, L]$, numerical computation of this system becomes very costly. On the other hand, in $(0, L]$, the solution is, to leading order in ϵ , governed by the equilibrium equation

$$u_t + f(u)_x = 0, \quad (1.6)$$

which can be more efficiently solved numerically. Thus a domain decomposition method, which couples the relaxation system (1.1) for $x \in [-L, 0)$, where $\epsilon(x) = O(1)$, with the equilibrium equation (1.6) for $x \in (0, L]$, is computationally competitive. Interface conditions at $x = 0$ must be provided for this coupling.

System (1.1) was first proposed by Jin-Xin [15] for numerical purpose, which supplies a new and powerful approximation to equilibrium conservation law (1.6). There have been many works concerning the asymptotic convergence of the relaxation systems (1.1) to the corresponding conservation laws (1.6) as the relaxation time tends to zero. Most of the results dealt with the Cauchy problem. In particular, Natalini [24] gave a rigorous proof that the solution to Cauchy problem (1.1) with initial condition (1.3) converges strongly in $C([0, \infty), L^1_{loc}(\mathbb{R}))$ to the unique entropy solution of (1.6) when $\epsilon \rightarrow 0$. See also [25] for a review in this direction, and results for larger systems [2] and on more general hyperbolic systems with relaxations [7].

In the presence of physical boundary conditions, Kriess and some others first gave the suitability of boundary conditions for linear hyperbolic systems when the source term is not stiff, see, for examples [16], [14], [23], [27]. Wang and Xin [31] later gave a similar result of the system (1.1) (1.3) with boundary condition (1.5). They proved that when the initial and boundary data satisfy a strict version of the subcharacteristic condition (1.2), the solution of the relaxation system converges as $\epsilon \rightarrow 0$ to a unique weak solution of the conservation law (1.6) which satisfies the boundary-entropy condition. [34] and [33] then gave an explicit necessary and sufficient condition (the so-called "Stiff Kriess Condition") on the boundary that guarantees the uniform well-posedness of the IBVP, and also revealed the boundary layer structures. [33] dealt with the linear cases while [34] considered the nonlinear one.

Domain decomposition methods connecting kinetic equation and its hydrodynamic or diffusion limit have received a lot of attention in the past 20 years. Our paper is strongly motivated by [13]. Others can refer to [1], [29], [3], [12], [36], [17], [18], [9], [11]. A thorough study on the problem of this paper provides a better understanding of the more general

coupling problem of kinetic and hydrodynamic equations, since indeed the Jin-Xin relaxation system (1.1) can be viewed as a discrete-velocity kinetic model, while (1.6) resembles some important features of hydrodynamic (compressible Euler) equations.

Relaxation systems themselves are important in many physical situations, such as kinetics theories [5], gases not in thermodynamic equilibrium [30], phase transitions with small transition time [20], river flows, traffic flows and more general waves [32].

In this paper, we give a domain decomposition method for system (1.1) - (1.4) by providing the interface condition at $x = 0$. The interface condition depends on the sign of $f'(u)$ at the interface. When $f'(u(0, t)) < 0$, there will be an interface layer in u around $x = 0^+$ when approximating the original system (1.1) by (1.6), then one can solve (1.6) in the right region first and then transfers the value of $v(0, t)$ to the left as one boundary condition for (1.1) in the left region, see (3.1) - (3.2). On the other hand, when $f'(u(0, t)) > 0$, one just uses $v(0, t) = f(u(0, t))$ as one boundary condition for (1.1) in the left region, and solves it first, then uses the value $u(0, t)$ as the boundary condition for (1.6) in the right region. The details are given in (3.3) - (3.4). For the linear case, i.e., $f(u) = \lambda u$, where $|\lambda| < 1$ a constant, we first prove the stiff well-posedness of the original system (1.1) in Theorem 3 in the sense that the L^2 norm of the solution is controlled by the L^2 norm of the initial and boundary data. Then we prove the asymptotic convergence in Theorem 4 to show that the difference between the solution to our domain decomposition system and the solution to the original system is asymptotically small. Sharp error estimates are also given.

This domain decomposition can be directly extended to more general cases, such as the coupling of multiple regions, $f'(u(0, t))$ changing sign in time, ϵ depending on both time and space [8], and more complicated cases such as when the equilibrium equation is a hyperbolic system instead of the scalar conservation law, and in higher space dimensions. Some details are given in section 6.

The paper is organized as follows. In Section 2 we show the formal expansion of the initial boundary value problem (1.1) in the upper half plane $\{x > 0, t > 0\}$ in which the boundary layer may exist. We also refer to the theorems in [34] which validate this expansion. Section 3 is devoted to present the domain decomposition method, and the corresponding interface condition is given. We then prove the stiff well-posedness and asymptotic convergence for the linear case. The theorems are proved in two parts: one for homogeneous initial data (Section 4) and the other the inhomogeneous one (Section 5). For the homogeneous one, we simply use the Laplace Transform to obtain the solution, while for the inhomogeneous case, we construct several auxiliary systems to decompose the solution into two parts, one generated by the initial data, and the other by the interface condition. With this decomposition, we are able to use some existing results for the Cauchy problem to avoid the difficulties raised by the Laplace Transform. Finally in Section 6, we present the corresponding numerical algorithms and some extensions of the domain decomposition method, and finally give some numerical examples to validate the theoretical analysis.

2 The local equilibrium limit

In this section, we recall the asymptotic analysis proposed in [34]. Here we only consider the boundary layer effect, and let

$$v_0(x) = f(u_0(x))$$

in order to avoid the initial layer effect. When $x \in [0, L]$ where ϵ is small, one can use the hyperbolic conservation law (1.6) to approximate the relaxation system. Away from $x = 0$ and $t = 0$, use the expansion

$$\begin{aligned} u^\epsilon(x, t) &\sim u^0(x, t) + \epsilon u^1(x, t) + \epsilon^2 u^2(x, t) + \dots, \\ v^\epsilon(x, t) &\sim v^0(x, t) + \epsilon v^1(x, t) + \epsilon^2 v^2(x, t) + \dots, \end{aligned}$$

then matching the orders of ϵ , one obtains:

$$\begin{aligned} v^0 &= f(u^0), \\ \partial_t u^0 + \partial_x v^0 &= 0, \\ \partial_t v^0 + \partial_x u^0 &= -(v^1 - f'(u^0)u^1). \end{aligned} \tag{2.1}$$

...

Thus the leading order of the expansion gives

$$\partial_t u^0 + \partial_x f(u^0) = 0, \quad v^0 = f(u^0), \tag{2.2}$$

which is the equilibrium limit (the zero relaxation limit) (1.6).

Near $x = 0$, introduce the stretched variable $\zeta = x/\epsilon$, and write the asymptotic expansion of $u^\epsilon(x, t)$ as

$$\begin{aligned} u^\epsilon(x, t) &\sim u^0(x, t) + \epsilon u^1(x, t) + \dots + \Gamma_u^0(\zeta, t) + \epsilon \Gamma_u^1(\zeta, t) + \dots, \\ v^\epsilon(x, t) &\sim v^0(x, t) + \epsilon v^1(x, t) + \dots + \Gamma_v^0(\zeta, t) + \epsilon \Gamma_v^1(\zeta, t) + \dots, \end{aligned}$$

here $\Gamma_u^0, \Gamma_v^0, \Gamma_u^1, \Gamma_v^1, \dots$, depending on ζ and t , are the boundary layer correctors near $x = 0$. Apply this ansatz to (1.1), and expand the nonlinear term $f(u^\epsilon)$ near $x = 0$ as

$$\begin{aligned} f(u^\epsilon) &= f(u^0(x, t) + \Gamma_u^0(\zeta, t) + \epsilon u^1(x, t) + \epsilon \Gamma_u^1(\zeta, t) + \dots) \\ &= f(u^0(0, t) + \epsilon \zeta \partial_x u^0(0, t) + \dots + \Gamma_u^0(\zeta, t) + \epsilon u^1(x, t) + \epsilon \Gamma_u^1(\zeta, t) + \dots) \\ &= f(u^0(0, t) + \Gamma_u^0(\zeta, t)) + \epsilon f'(u^0(0, t) + \Gamma_u^0(\zeta, t))(\zeta \partial_x u^0(0, t) + u^1(0, t) + \Gamma_u^1(\zeta, t)) + \epsilon^2 \dots \end{aligned}$$

where the second equality comes from the relation $x = \epsilon \zeta$. By using (2.1) and (2.2) one has the equation to the leading order $O(\frac{1}{\epsilon})$

$$\partial_\zeta \Gamma_v^0 = 0, \tag{2.3}$$

$$\partial_\zeta \Gamma_u^0 = -(v^0(0, t) + \Gamma_v^0 - f(\Gamma_u^0 + u^0(0, t))). \tag{2.4}$$

(2.3) implies $\Gamma_v^0 \equiv 0$ because the boundary layer $\Gamma_v^0(\zeta, 0)$ should decay as $\zeta \rightarrow 0$. Also, (2.4) can be written as

$$(\Gamma_u^0)_\zeta = -(v^0(0, t) - f(u^0(0, t) + \Gamma_u^0)) \simeq f'(u^0(0, t))\Gamma_u^0(\zeta, t),$$

thus one gets the behavior of the boundary layer in u :

$$\Gamma_u^0(\zeta, t) = \exp(f'(u^0(0, t))\zeta)\Gamma_u^0(0, t). \quad (2.5)$$

Since the boundary layer has to decay exponentially fast, one needs $f'(u^0(0, t)) < 0$. In other words, if $f'(u^0(0, t)) < 0$, there will be a boundary layer, otherwise there will not be a boundary layer.

The above analysis was rigorously validated in [34].

3 A domain decomposition method

In section 2, one sees that when ϵ goes to 0, the hyperbolic system (1.1) can be approximated by the equilibrium equation (1.6) that does not have any stiff term. But the interface condition that connects the two regions should be provided. In this section, we will give the detailed algorithm that approximates the solution of the two-scale problem. We will consider the case with $f'(u(0, t)) < 0$ and $f'(u(0, t)) > 0$ separately.

3.1 $f'(u(0, t)) < 0$

In this case, there will be an interface layer in u near the interface $x = 0$, so one can not simply use u obtained from $(0, L]$ to solve (1.6) in domain $[-L, 0)$. Instead we can use the information of v at $x = 0$ directly from the equation in $(0, L]$ since there is no $O(1)$ interface layer in v . Here is the coupling algorithm.

- **Step 1.** For $x \in (0, L]$, solve

$$\begin{cases} u_t^r + f(u^r)_x = 0, & (3.1a) \end{cases}$$

$$\begin{cases} v^r(x, t) = f(u^r(x, t)), & (3.1b) \end{cases}$$

$$\begin{cases} u^r(x, 0) = u_0(x), & (3.1c) \end{cases}$$

$$\begin{cases} u^r(L, t) = b_R(t). & (3.1d) \end{cases}$$

Note in this case one can solve (3.1) first to get $v^r(0, t)$, and then solve (3.2).

- **Step 2.** For $x \in [-L, 0)$, solve

$$\begin{cases} u_t^l + v_x^l = 0, & (3.2a) \end{cases}$$

$$\begin{cases} v_t^l + u_x^l = -(v^l - f(u^l)), & (3.2b) \end{cases}$$

$$\begin{cases} u^l(x, 0) = u_0(x), \quad v^l(x, 0) = v_0(x), & (3.2c) \end{cases}$$

$$\begin{cases} u^l(-L, t) = b_L(t), & (3.2d) \end{cases}$$

$$\begin{cases} v^l(0, t) = v^r(0, t); & (3.2e) \end{cases}$$

where $v^r(0, t)$ is obtained from Step 1.

3.2 $f'(u(0, t)) > 0$

In this case, at the interface $x = 0$ there is no $O(1)$ interface layer in u and v . In other words, u and v are in local equilibrium $v = f(u)$, and we can just use this as the interface condition. We give the following algorithm.

- **Step 1.** For $x \in [-L, 0)$, solve

$$\begin{cases} u_t^l + v_x^l = 0, & (3.3a) \\ v_t^l + u_x^l = -(v^l - f(u^l)), & (3.3b) \\ u^l(x, 0) = u_0(x), \quad v^l(x, 0) = v_0(x), & (3.3c) \\ u^l(-L, t) = b_L(t), & (3.3d) \\ f(u^l(0, t)) = v^l(0, t); & (3.3e) \end{cases}$$

- **Step 2.** For $x \in (0, L]$, solve

$$\begin{cases} u_t^r + f(u^r)_x = 0, & (3.4a) \\ v^r(x, t) = f(u^r(x, t)), & (3.4b) \\ u^r(x, 0) = u_0(x), & (3.4c) \\ u^r(0, t) = u^l(0, t), & (3.4d) \end{cases}$$

where $u^l(0, t)$ is obtained from Step 1.

Remark 1. In this case there will be a boundary layer in u near $x = L^-$, which is why in Theorem 4 that the convergence rate is $O(\epsilon)$.

In both cases, we define the solution to the domain decomposition system as follows:

$$\begin{cases} u(x, t) = u^l(x, t), \quad v(x, t) = v^l(x, t), & (x, t) \in [-L, 0) \times [0, T], & (3.5a) \\ u(x, t) = u^r(x, t), \quad v(x, t) = v^r(x, t), & (x, t) \in (0, L] \times [0, T]. & (3.5b) \end{cases}$$

Remark 2. If $f'(u(0, t))$ changes sign at the interface, one can check the sign of $f'(u(0, t))$ at the current time step, and then use either (3.1) - (3.2) or (3.3) - (3.4) to continue to the next step. More general cases, such as time-dependent ϵ or higher space dimensions, are discussed in section 6.3.

The detailed numerical implementation of this domain decomposition method is given in section 6.

Now we state the main theorems in the paper about the stiff well-posedness of the original relaxation system and asymptotic convergence of our domain decomposition system.

Theorem 3. Let $U^\epsilon = (u^\epsilon, v^\epsilon)^T$ be the solution of the original system (1.1). If $u_0(x), v_0(x), b_L(t), b_R(t) \in L^2$, and $U_0(\pm L) = 0$, $b_L(0) = b_R(0) = 0$, then the solution to the original system (1.1), with variable $\epsilon(x)$ given in (1.4), is stiffly well-posed in the sense:

$$\begin{aligned} & \int_0^T \int_{-L}^L |U^\epsilon(x, t)|^2 dx dt + \int_0^T |U^\epsilon(-L, t)|^2 dt + \int_0^T |U^\epsilon(L, t)|^2 dt \\ & \leq K_T \left[\int_0^T |b_L(t)|^2 dt + \int_0^T |b_R(t)|^2 dt + \int_{-L}^L |U_0(x)|^2 dx \right], \end{aligned}$$

where K_T is a positive constant independent of ϵ . Moreover, if $u_0(x), v_0(x), b_L(t)$ and $b_R(t)$ are continuous, then the solution U^ϵ is continuous in x .

Theorem 4. Assume $b_L(t), b_R(t) \in L^2(\mathbb{R}^+)$, $U_0(\pm L) = 0$, $U_0(x) \in H^3([-L, L])$ and $U_0(0) = U_0'(0) = U_0''(0) = 0$, then there exists a unique solution $U = (u, v)^T$ of the domain decomposition system (3.1) - (3.2) or (3.3) - (3.4) such that

$$\int_{-L}^L \int_0^\infty |U^\epsilon - U|^2 e^{-2\alpha t} dt dx \rightarrow 0$$

as $\epsilon \rightarrow 0$ for any $\alpha > 0$. Moreover, if we assume $b_L(t), b_R(t) \in H^2(\mathbb{R}^+)$, $b_L(0) = b_L'(0) = b_R(0) = b_R'(0)$, and $U_0'(\pm L) = 0$, then

$$\begin{aligned} & \int_{-L}^L \int_0^\infty |U^\epsilon - U|^2 e^{-2\alpha t} dt dx \\ & \leq O(1)\epsilon \|b_L\|_{L^2}^2 + O(1)\epsilon \|b_R\|_{L^2}^2 + O(1)\epsilon^2 \|b_L\|_{H^2}^2 \\ & \quad + O(1)\epsilon^2 \|b_R\|_{H^2}^2 + O(1)\epsilon \|v_0 - \lambda u_0\|_{L^2[0, L]}^2 \\ & \quad + \begin{cases} O(1)\epsilon^2 \|U_0\|_{H^3}^2, & \text{for } \lambda > 0, \\ O(1)\epsilon \|U_0\|_{L^2}^2 + O(1)\epsilon^2 \|U_0\|_{H^3}^2, & \text{for } \lambda < 0. \end{cases} \end{aligned}$$

Remark 5. (1) In the $\lambda < 0$ case, there is an interface layer near $x = 0^+$, while in the $\lambda > 0$ case, there is a boundary layer near $x = L^-$, so in both cases, the optimal convergence rate due to the boundary data is $O(1)\epsilon$, which is where the terms $O(1)\epsilon \|b_L(t)\|_{L^2}^2 + O(1)\epsilon \|b_R(t)\|_{L^2}^2$ come from.

(2) The lower convergence rate in the case of $\lambda < 0$ is due to the presence of an interface layer near $x = 0^+$ generated by the initial data.

(3) $O(1)\epsilon \|v_0 - \lambda u_0\|_{L^2[0, L]}^2$ comes from the initial layer in v .

4 Error estimate for the domain decomposition method for the linear case: the Homogeneous initial data

In this and the next sections, we will give a rigorous justification of the domain decomposition method for linear problems, where $f(u) = \lambda u$, for $|\lambda| < 1$ a constant. The main results are given in Theorems 3 and 4. We first represent the exact solution to the original system (1.1) - (1.4) by the Laplace Transform, we then study the stiff wellposedness and the asymptotic convergence followed by direct calculations.

Denote

$$U^\epsilon = \begin{pmatrix} u^\epsilon \\ v^\epsilon \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ \lambda & -1 \end{pmatrix}.$$

Here we consider system (1.1) with zero initial data (1.3), i.e., $u_0(x) = 0$, $v_0(x) = 0$ and nonzero boundary data (1.5). In this case one can focus on the boundary layer effects and avoid the interactions between the initial and boundary layers.

4.1 Solution by the Laplace Transform

When (1.1) is linear, i.e., $f(u) = \lambda u$, one can find the exact solution of (1.1) - (1.4) by the Laplace Transform. Let

$$\hat{U}^\epsilon(x, \xi) = \mathcal{L}(U^\epsilon) = \int_0^\infty e^{-\xi t} U^\epsilon(x, t) dt, \quad \text{Re}(\xi) > 0.$$

Here $\xi = \alpha + i\beta$, then $\mathcal{L}(\partial_t U^\epsilon) = \xi \hat{U}^\epsilon - U^\epsilon(x, 0) = \xi \hat{U}^\epsilon(x, \xi)$. With the homogeneous initial condition, system (1.1) - (1.5) becomes

$$\partial_x \hat{U}^\epsilon = \frac{1}{\epsilon(x)} A^{-1} (S - \epsilon(x) \xi I) \hat{U}^\epsilon = \frac{1}{\epsilon(x)} M(\epsilon(x) \xi) \hat{U}^\epsilon, \quad (4.1)$$

$$\hat{u}^\epsilon(-L, \xi) = \hat{b}_L(\xi), \quad \hat{u}^\epsilon(L, \xi) = \hat{b}_R(\xi), \quad (4.2)$$

where matrix

$$M(\xi) = A^{-1} (S - \epsilon(x) \xi I) \quad (4.3)$$

has two eigenvalues

$$\mu_\pm(\xi) = \frac{\lambda \pm \sqrt{\lambda^2 + 4\xi(1 + \xi)}}{2}, \quad (4.4)$$

and two corresponding eigenvectors

$$\begin{pmatrix} 1 \\ \frac{\mu_\mp(\xi)}{1 + \xi} \end{pmatrix} = \begin{pmatrix} 1 \\ g_\mp(\xi) \end{pmatrix}. \quad (4.5)$$

Thus the solution of (4.1) (4.2) can be written as:

$$\begin{cases} \hat{U}^\epsilon(x, \xi) = c_1 e^{\mu_-(\xi)x} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} + c_2 e^{\mu_+(\xi)x} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix} & \text{for } x < 0, \epsilon(x) = 1; \\ \hat{U}^\epsilon(x, \xi) = c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} \begin{pmatrix} 1 \\ g_+(\epsilon\xi) \end{pmatrix} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}} \begin{pmatrix} 1 \\ g_-(\epsilon\xi) \end{pmatrix} & \text{for } x > 0, \epsilon(x) = \epsilon, \end{cases} \quad (4.6)$$

where the coefficient c_1, c_2, c_3, c_4 are determined by the boundary conditions:

$$c_1 e^{-\mu_-(\xi)L} + c_2 e^{-\mu_+(\xi)L} = \hat{b}_L(\xi), \quad (4.7)$$

$$c_3 e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + c_4 e^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}} = \hat{b}_R(\xi). \quad (4.8)$$

By continuity at the interface, one has

$$c_1 + c_2 = c_3 + c_4, \quad (4.9)$$

$$c_1 g_+(\xi) + c_2 g_-(\xi) = c_3 g_+(\epsilon\xi) + c_4 g_-(\epsilon\xi). \quad (4.10)$$

From (4.7) - (4.10), one sees that $c_1 - c_4$ are uniquely determined. Denote

$$c_3 = E c_1 + F c_2, \quad (4.11)$$

$$c_4 = G c_1 + H c_2, \quad (4.12)$$

where

$$E = \frac{g_+(\xi) - g_-(\epsilon\xi)}{g_+(\epsilon\xi) - g_-(\epsilon\xi)}, \quad F = \frac{g_-(\xi) - g_-(\epsilon\xi)}{g_+(\epsilon\xi) - g_-(\epsilon\xi)}, \quad G = \frac{g_+(\xi) - g_+(\epsilon\xi)}{g_-(\epsilon\xi) - g_+(\epsilon\xi)}, \quad H = \frac{g_-(\xi) - g_+(\epsilon\xi)}{g_-(\epsilon\xi) - g_+(\epsilon\xi)}.$$

Plugging (4.11) (4.12) into (4.7) (4.8), gives

$$c_1 = \frac{\hat{b}_R(\xi)e^{-\mu_+(\xi)L} - \hat{b}_L(\xi)(Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})}{(Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_+(\xi)L} - (Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_-(\xi)L}}, \quad (4.13)$$

$$c_2 = \frac{\hat{b}_R(\xi)e^{-\mu_-(\xi)L} - \hat{b}_L(\xi)(Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})}{(Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_-(\xi)L} - (Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_+(\xi)L}}. \quad (4.14)$$

4.2 Stiff well-posedness

We first summarize some properties of the eigenvalues $\mu_{\pm}(\xi)$ in (4.4) and $g_{\pm}(\xi)$ appeared in the eigenvector in (4.5), which will be used many times later. We then prove the stiff well-posedness stated in Theorem 3.

First, we give some bounds on $\mu_{\pm}(\xi)$.

Lemma 6. *Under the subcharacteristic condition $|\lambda| < 1$, one has*

$$(1) \quad |\lambda|(1 + 2\alpha) \leq \operatorname{Re} \sqrt{\lambda^2 + 4\xi(1 + \xi)} \leq 1 + 2\alpha, \quad \text{for } \operatorname{Re}(\xi) = \alpha \geq 0; \quad (4.15)$$

$$(2) \quad \operatorname{Re} \mu_+(\xi) > 0, \quad \operatorname{Re} \mu_-(\xi) < 0; \quad (4.16)$$

$$(3) \quad \text{when } \lambda < 0, \quad 2\operatorname{Re} \mu_-(\epsilon\xi) \leq -2|\lambda|, \quad 2\operatorname{Re} \mu_+(\epsilon\xi) \geq -2\epsilon\lambda\alpha; \quad (4.17)$$

$$\text{when } \lambda > 0, \quad 2\operatorname{Re} \mu_-(\epsilon\xi) \leq -2\epsilon\lambda\alpha, \quad 2\operatorname{Re} \mu_+(\epsilon\xi) \geq 2\lambda. \quad (4.18)$$

For the proof of the lemma, please refer to [33].

Now we give bounds and asymptotic behavior of $g_{\pm}(\epsilon\xi)$.

Lemma 7. *Under the subcharacteristic condition $|\lambda| < 1$, one has*

(1) For $\lambda > 0$, $g_-(\epsilon\xi) = O(1)\epsilon\xi$, and $0 < C_1 \leq |g_+(\epsilon\xi)| \leq C_2$, here C_1 and C_2 are two positive constants, and $g_+(\epsilon\xi) - \lambda = O(1)\epsilon\xi$;

(2) For $\lambda < 0$, $g_+(\epsilon\xi) = O(1)\epsilon\xi$, and $0 < C_3 \leq |g_-(\epsilon\xi)| \leq C_4$, here C_3 and C_4 are two positive constants, and $g_-(\epsilon\xi) - \lambda = O(1)\epsilon\xi$;

(3) $g_{\pm}(\xi) - g_{\pm}(\epsilon\xi)$, $g_+(\xi) - g_-(\epsilon\xi)$, $g_+(\epsilon\xi) - g_-(\xi)$ are uniformly bounded with respect to both ϵ and ξ , and they are bounded away from zero for $\operatorname{Re}\xi = \alpha > 0$.

Proof. (1). When $\lambda > 0$, from definition (4.5), one sees

$$g_-(\epsilon\xi) = \frac{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}}{2(1 + \epsilon\xi)} = \frac{-2\epsilon\xi}{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}} = O(1)\epsilon\xi,$$

and

$$g_+(\epsilon\xi) = \frac{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}}{2(1 + \epsilon\xi)}.$$

In order to prove that $g_+(\epsilon\xi)$ is uniformly bounded with respect to $\epsilon\xi$, and the denominator is nonzero, one just needs to check what happens when $|\epsilon\xi|$ goes to 0 or ∞ . Let $\epsilon\xi = re^{i\theta}$, one sees that when $|\epsilon\xi| \rightarrow 0$, i.e., when $r \rightarrow 0$, $|g_+(\epsilon\xi)| \rightarrow \lambda$; when $|\epsilon\xi| \rightarrow \infty$, i.e., when $|r| \rightarrow \infty$, one has

$$|g_+(\epsilon\xi)| \rightarrow \left| \frac{\sqrt{\lambda^2 + 4re^{i\theta}(1 + re^{i\theta})}}{2(1 + re^{i\theta})} \right| \rightarrow (\cos^2 2\theta + \sin^4 \theta)^{\frac{1}{4}},$$

which is bounded and nonzero. Moreover,

$$g_+(\epsilon\xi) - \lambda = \frac{2(1 - \lambda^2)\epsilon\xi}{\sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)} + \lambda(1 + 2\epsilon\xi)} = O(1)\epsilon\xi.$$

(2). When $\lambda < 0$, similarly one has

$$g_+(\epsilon\xi) = \frac{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}}{2(1 + \epsilon\xi)} = \frac{-2\epsilon\xi}{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}} = O(1)\epsilon\xi,$$

and

$$g_-(\epsilon\xi) = \frac{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}}{2(1 + \epsilon\xi)}.$$

In the same way as in (1), one can prove that $g_-(\epsilon\xi)$ is uniformly bounded in $\epsilon\xi$.

(3). Note

$$g_+(\xi) - g_-(\epsilon\xi) = \frac{\lambda\xi(\epsilon - 1) + (1 + \epsilon\xi)\sqrt{\lambda^2 + 4\xi(1 + \xi)} + (1 + \xi)\sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}}{(1 + \xi)(1 + \epsilon\xi)}.$$

Let $\xi = re^\theta$, then when $\epsilon \rightarrow 0$, and $|r| \rightarrow 0$, one has $|g_+(\xi) - g_-(\epsilon\xi)| \rightarrow 2|\lambda|$. When $\epsilon \rightarrow 0$, $|r| \rightarrow \infty$, and $\epsilon|r| \rightarrow 0$, one has

$$|g_+(\xi) - g_-(\epsilon\xi)| \rightarrow \left| \frac{\lambda + \sqrt{\lambda^2 + 4\xi(1 + \xi)}}{1 + \xi} \right| \rightarrow \frac{1}{2}(\cos^2 2\theta + \sin^4 \theta)^{\frac{1}{4}},$$

which is bounded and nonzero. When $\epsilon \rightarrow 0$, $|r| \rightarrow \infty$, and $\epsilon|r| \rightarrow \infty$, one can still prove that $|g_+(\xi) - g_-(\epsilon\xi)|$ is uniformly bounded away from 0, but the detailed calculation will be omitted. Similarly, one can prove the same result for $g_+(\epsilon\xi) - g_-(\xi)$. As for $g_+(\xi) - g_+(\epsilon\xi)$, notice

$$g_+(\xi) - g_+(\epsilon\xi) = \frac{\lambda\xi(\epsilon - 1) + (1 + \epsilon\xi)\sqrt{\lambda^2 + 4\xi(1 + \xi)} - (1 + \xi)\sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}}{(1 + \xi)(1 + \epsilon\xi)},$$

then following the same procedure as above, it is not hard to check that it is uniformly bounded as $\epsilon \rightarrow 0$, and $|\xi| \rightarrow \infty$. Moreover, when $\epsilon \rightarrow 0$, $|\xi| \rightarrow \alpha$,

$$|g_+(\xi) - g_+(\epsilon\xi)| \rightarrow \left| \frac{-\lambda\alpha + \sqrt{\lambda^2 + 4\alpha(1+\alpha)} - (1+\alpha)\lambda}{1+\alpha} \right|,$$

which is nonzero, one can arrive at the same conclusion for $g_-(\xi) - g_-(\epsilon\xi)$. \square

Remark 8. (1) We will fix $\text{Re}\xi = \alpha > 0$ from now on.

(2) From definition (4.6), one sees that, when $\lambda > 0$, by (4.18), there is a boundary layer near $x = L$, and on the other hand, when $\lambda < 0$, by (4.17), there is an interface layer near $x = 0$. This observation will play an important role in subsequent proofs.

Now we prove theorem about the stiff well-posedness.

Proof. We use solution (4.6) of the original system (1.1) given by the Laplace Transform. Consider the integral:

$$\begin{aligned} \int_{-L}^L dx \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x, \xi)|^2 d\beta &= \int_{-L}^0 e^{2\text{Re}\mu_-(\xi)x} dx \int_{-\infty}^{\infty} |c_1|^2 (1 + |g_+(\xi)|^2) d\beta \\ &+ \int_{-L}^0 e^{2\text{Re}\mu_+(\xi)x} dx \int_{-\infty}^{\infty} |c_2|^2 (1 + |g_-(\xi)|^2) d\beta \\ &+ \int_0^L e^{2\text{Re}\mu_-(\epsilon\xi)\frac{x}{\epsilon}} dx \int_{-\infty}^{\infty} |c_3|^2 (1 + |g_+(\epsilon\xi)|^2) d\beta \\ &+ \int_0^L dx \int_{-\infty}^{\infty} |c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}}|^2 (1 + |g_-(\epsilon\xi)|^2) d\beta. \end{aligned}$$

By Lemma 4 one sees E, F, G , and H in (4.11) (4.12) are uniformly bounded away from 0. And from (4.13), (4.14), (4.11) and (4.12) one gets

$$c_1, c_2, c_3, c_4 = O(1)(\hat{b}_L(\xi) + \hat{b}_R(\xi)),$$

and moreover, from (4.8),

$$e^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}} c_4 = (\hat{b}_R(\xi) - c_3 e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}}), \quad (4.19)$$

so $e^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}} c_4 = O(1)e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} \hat{b}_L(\xi) + O(1)\hat{b}_R(\xi)$. Therefore

$$\int_{-L}^L dx \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x, \xi)|^2 d\beta \leq O(1) \int_{-\infty}^{\infty} (|\hat{b}_L(\xi)|^2 + |\hat{b}_R(\xi)|^2) d\beta. \quad (4.20)$$

Then by Parseval's identity:

$$\int_0^\infty e^{-2\alpha t} |U^\epsilon(x, t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x, \alpha + i\beta)|^2 d\beta, \quad (4.21)$$

the stiff well-posedness, as stated in Theorem 3, now follows. \square

4.3 Asymptotic convergence and error estimates

Next we turn to the question of the asymptotic convergence and error estimate stated in Theorem 4

To prove the theorem, still we compare the analytical solution of the domain decomposition problem (3.1) - (3.4) with the original problem given in section 4.1 with the help of the Laplace Transform.

Proof. Consider the case $\lambda < 0$ first. The solution of (3.1) is

$$u^r(x, t) = \begin{cases} 0, & x - L \leq \lambda t, \\ b_R(t + \frac{1}{\lambda}(L - x)), & x - L \geq \lambda t, \quad 0 \leq x \leq L. \end{cases}$$

Using the Laplace Transform, it becomes

$$\hat{u}^r(x, \xi) = \hat{b}_R(\xi) e^{\frac{\xi}{\lambda}(L-x)}, \quad \text{for } x > 0. \quad (4.22)$$

The solution of (3.2) is

$$\hat{U}^l(x, \xi) = d_1 e^{\mu_-(\xi)x} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} + d_2 e^{\mu_+(\xi)x} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix}, \quad (4.23)$$

where d_1 and d_2 are determined by

$$d_1 e^{-\mu_-(\xi)L} + d_2 e^{-\mu_+(\xi)L} = \hat{b}_L(\xi), \quad (4.24)$$

$$d_1 g_+(\xi) + d_2 g_-(\xi) = \lambda \hat{b}_R(\xi) e^{\frac{\xi}{\lambda}L}. \quad (4.25)$$

Now compare the first expression of (4.6) with (4.23), and the second with (4.22) respectively. For $x \in [0, L]$, using (4.6) and (4.22), one gets for the first component u

$$\begin{aligned} & \int_0^L dx \int_{-\infty}^{\infty} |\hat{u}^r - \hat{u}^\epsilon|^2 d\beta = \int_0^L dx \int_{-\infty}^{\infty} |c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}} - \hat{b}_R e^{\frac{\xi}{\lambda}(L-x)}|^2 d\beta \\ & \leq \int_0^L dx \int_{-\infty}^{\infty} |c_3 (e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}})|^2 d\beta + \int_0^L dx \int_{-\infty}^{\infty} |\hat{b}_R(\xi)|^2 |e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} - e^{\frac{\xi}{\lambda}(L-x)}|^2 d\beta \\ & = I_1 + I_2. \end{aligned}$$

Here the first inequality was derived by substituting c_4 in (4.19). For I_1 , it is easy to see:

$$I_1 \leq O(1) \int_{-\infty}^{\infty} |c_3(\xi)|^2 d\beta \left(\int_0^L e^{2\text{Re}\mu_-(\epsilon\xi)\frac{x}{\epsilon}} dx + e^{2\text{Re}\mu_-(\epsilon\xi)\frac{L}{\epsilon}} \int_0^L e^{2\text{Re}\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} dx \right).$$

Then by (4.17) one gets the estimate for I_1 as:

$$\begin{aligned} I_1 & \leq O(1)\epsilon \int_{-\infty}^{\infty} |c_3(\xi)|^2 d\beta \\ & = O(1)\epsilon \int_{-\infty}^{\infty} (|\hat{b}_L|^2 + |\hat{b}_R|^2) d\beta \leq O(1)\epsilon (\|b_L\|_{L^2}^2 + \|b_R\|_{L^2}^2). \end{aligned} \quad (4.26)$$

Note here in I_1 , the term that contains $e^{\mu - (\epsilon\xi)\frac{x}{\epsilon}}$ is the result of interface layer, which drops the L^2 convergence rate down to $\epsilon^{\frac{1}{2}}$.

For I_2 , notice

$$\begin{aligned} & \int_0^L |e^{\mu + (\epsilon\xi)\frac{x-L}{\epsilon}} - e^{\frac{\xi}{\lambda}(L-x)}|^2 dx \\ &= \int_0^L |e^{\mu + (\epsilon\xi)\frac{-x}{\epsilon}} - e^{\frac{\xi}{\lambda}x}|^2 dx \leq \int_0^\infty |e^{\mu + (\epsilon\xi)\frac{-x}{\epsilon}} - e^{\frac{\xi}{\lambda}x}|^2 dx = O(1) \left| \frac{\mu + (\epsilon\xi)}{\epsilon} + \frac{\xi}{\lambda} \right|^2 \end{aligned} \quad (4.27)$$

one has

$$\begin{aligned} I_2 &\leq \int_{-\infty}^\infty O(1) \left| \frac{\mu + (\epsilon\xi)}{\epsilon} + \frac{\xi}{\lambda} \right|^2 |\hat{b}_R(\xi)|^2 d\xi = O(1)\epsilon^2 \int_{-\infty}^\infty |\xi|^4 |\hat{b}_R(\xi)|^2 d\xi \\ &\leq O(1)\epsilon^2 \|b_R\|_{H^2}^2. \end{aligned} \quad (4.28)$$

Here we use the fact

$$\frac{\mu + (\epsilon\xi)}{\epsilon} + \frac{\xi}{\lambda} = \frac{2\epsilon\xi^2(1 - \lambda^2)}{\lambda(\lambda^2 + 2\epsilon\xi - \lambda\sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)})} = O(1)\epsilon\xi^2, \quad (4.29)$$

and also we assume that $b_R(t) \in H^2(\mathbb{R}^+)$ and $b_R(t)$ satisfies the compatibility condition $b_R(0) = b'_R(0) = 0$. Adding I_1 and I_2 yields

$$\int_0^L dx \int_{-\infty}^\infty |\hat{u}^l - \hat{u}^\epsilon|^2 d\xi \leq O(1)\epsilon(\|b_L\|_{L^2}^2 + \|b_R\|_{L^2}^2) + O(1)\epsilon^2 \|b_R\|_{H^2}^2. \quad (4.30)$$

When $x \in [-L, 0]$ the difference between (4.6) and (4.23) is the difference between the coefficients, i.e.

$$\int_{-L}^0 dx \int_{-\infty}^\infty |\hat{U}^l - \hat{U}^\epsilon|^2 d\xi = O(1) \int_{-\infty}^\infty (|d_1 - c_1|^2 + |d_2 - c_2|^2) d\xi.$$

Compare (4.7) - (4.10) with (4.24)(4.25), one finds

$$|c_1 - d_1| = O(1)\epsilon\xi(\hat{b}_L + \hat{b}_R), \quad |c_2 - d_2| = O(1)\epsilon\xi(\hat{b}_L + \hat{b}_R),$$

after using Lemma 7 and some basic calculations. The details are omitted.

Therefore,

$$\begin{aligned} \int_{-L}^0 dx \int_{-\infty}^\infty |\hat{U}^l - \hat{U}^\epsilon|^2 d\xi &\leq O(1)\epsilon^2 \int_{-\infty}^\infty (|\xi\hat{b}_L(\xi)|^2 + |\xi\hat{b}_R(\xi)|^2) d\xi \\ &\leq O(1)\epsilon^2 (\|b_L\|_{H^1}^2 + \|b_R\|_{H^1}^2). \end{aligned} \quad (4.31)$$

Here we used the assumption that $b_L(t) \in H^1(\mathbb{R}^+)$, and $b_L(t)$ satisfies $b_L(0) = 0$. Now we are done with the $\lambda < 0$ case.

For $\lambda > 0$, the proof is similar. First the solution to (3.3) is

$$\hat{U}^l(x, \xi) = k_1 e^{\mu - (\epsilon\xi)x} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} + k_2 e^{\mu + (\epsilon\xi)x} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix}, \quad -L \leq x \leq 0, \quad (4.32)$$

where k_1 and k_2 are determined by

$$k_1 e^{-\mu_-(\xi)L} + k_2 e^{-\mu_+(\xi)L} = \hat{b}_L(\xi), \quad (4.33)$$

$$k_1(\lambda - g_+(\xi)) + k_2(\lambda - g_-(\xi)) = 0. \quad (4.34)$$

When $0 \leq x \leq L$, the solution to (3.4) is

$$u^r(x, t) = \begin{cases} 0, & \lambda t \leq x \leq L, \\ u^l(0, t - \frac{x}{\lambda}), & 0 \leq x \leq \lambda t, 0 \leq x \leq L. \end{cases}$$

So after using the Laplace Transform, one gets:

$$\hat{u}^r(x, \xi) = e^{-\xi \frac{x}{\lambda}} \hat{u}^l(0^-, \xi) = e^{-\xi \frac{x}{\lambda}} (k_1 + k_2). \quad (4.35)$$

Now compare (4.32) and (4.35) with (4.6). The difference between (4.32) and the first expression of (4.6) is again the difference between the coefficients. Thus

$$\int_{-L}^0 dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta = O(1) \int_{-\infty}^{\infty} (|k_1 - c_1|^2 + |k_2 - c_2|^2) d\beta.$$

Comparing (4.7)–(4.10) with (4.33) (4.34), one finds

$$|c_1 - k_1| = O(1)\epsilon\xi\hat{b}_L, \quad |c_2 - k_2| = O(1)\epsilon\xi\hat{b}_L.$$

Therefore,

$$\begin{aligned} \int_{-L}^0 dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta &\leq O(1)\epsilon^2 \int_{-\infty}^{\infty} |\xi\hat{b}_L(\xi)|^2 d\xi \\ &\leq O(1)\epsilon^2 \|b_L\|_{H^1}^2. \end{aligned} \quad (4.36)$$

The difference between (4.35) and the second expression of (4.6) is estimated as follows:

$$\begin{aligned} &\int_0^L dx \int_{-\infty}^{\infty} |\hat{u}^r - \hat{u}^\epsilon|^2 d\beta \\ &= \int_0^L dx \int_{-\infty}^{\infty} |c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}} - (k_1 + k_2) e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ &= \int_0^L dx \int_{-\infty}^{\infty} |c_3 (e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}}) + \hat{b}_R e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} - (k_1 + k_2) e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ &\leq J_1 + J_2 + J_3. \end{aligned}$$

To get the second equality, we again use (4.19). First,

$$\begin{aligned} J_1 &= \int_{-\infty}^{\infty} |\hat{b}_R(\xi)|^2 d\beta \int_0^L e^{2\text{Re}\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} dx \\ &\leq O(1)\epsilon \|b_R(t)\|_{L^2}^2, \end{aligned} \quad (4.37)$$

where the inequalities (4.16), (4.17) and (4.18) were used. For J_2 , one has:

$$\begin{aligned} J_2 &= \int_0^L dx \int_{-\infty}^{\infty} \left| [c_3 - (k_1 + k_2)] e^{\mu - (\epsilon\xi)\frac{x}{\epsilon}} - c_3 e^{\mu + (\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu - (\epsilon\xi)\frac{L}{\epsilon}} \right|^2 \\ &\leq \int_0^L dx \int_{-\infty}^{\infty} |c_3 - k_1 - k_2|^2 e^{2Re\mu - (\epsilon\xi)\frac{x}{\epsilon}} d\beta + O(1)\epsilon \int_{-\infty}^{\infty} |c_3|^2 d\beta. \end{aligned}$$

Since $c_3 + c_4 = c_1 + c_2 = k_1 + k_2 + O(1)\epsilon\xi\hat{b}_L(\xi)$, $c_4 = e^{-\mu + (\epsilon\xi)\frac{L}{\epsilon}}(\hat{b}_R(\xi) - c_3 e^{\mu - (\epsilon\xi)\frac{L}{\epsilon}})$, one has $|c_3 - k_1 - k_2|^2 = O(1)\epsilon^2 |\xi\hat{b}_L(\xi)|^2$. Therefore,

$$J_2 \leq O(1)\epsilon^2 \|b_L\|_{H^1}^2 + O(1)\epsilon \|b_L\|_{L^2}^2. \quad (4.38)$$

Note here the convergence rate is ϵ , which is caused by the boundary layer effect of $e^{\mu + (\epsilon\xi)\frac{x-L}{\epsilon}}$ in J_1 and J_2 . The remaining part J_3 is

$$\begin{aligned} J_3 &= \int_0^L dx \int_{-\infty}^{\infty} |(k_1 + k_2) e^{\mu - (\epsilon\xi)\frac{x}{\epsilon}} - (k_1 + k_2) e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ &\leq O(1) \int_0^L dx \int_{-\infty}^{\infty} |e^{\mu - (\epsilon\xi)\frac{x}{\epsilon}} - e^{-\xi\frac{x}{\lambda}}|^2 dx \int_{-\infty}^{\infty} |k_1 + k_2|^2 d\beta \\ &\leq O(1)\epsilon^2 \|b_L\|_{H^2}^2. \end{aligned} \quad (4.39)$$

The calculation here is similar to (4.28). In total, one gets

$$\int_0^L dx \int_{-\infty}^{\infty} |\hat{u}^r - \hat{u}^\epsilon|^2 d\beta \leq O(1)\epsilon (\|b_L\|_{L^2}^2 + \|b_R\|_{L^2}^2) + O(1)\epsilon^2 \|b_L\|_{H^2}^2. \quad (4.40)$$

To this end, we have proved Theorem 4 with zero initial data. \square

Remark 9. Here we jump to the estimation of the convergence rate, and omit the steps to prove the uniform convergence stated in Theorem 4, which is easily obtained by dominated convergence theorem.

5 Error estimate for the domain decomposition method for the linear case: the inhomogeneous initial data

The case with inhomogeneous initial data is much more complicated. For clarity, we consider instead the Cauchy problem here, that is, $x \in (-\infty, \infty)$ instead of $[-L, L]$. A new idea here is to construct some related initial value problem and make use of the existing results about the Cauchy problem [33] to overcome the difficulties arisen in the Laplace Transform. With these two results, the problem with both boundary and initial data is straightforward, and details will be omitted.

5.1 Solution by the Laplace Transform

Again, we solve system (1.1) with $L = \infty$ by the Laplace Transform. Then (1.1) (1.3) becomes:

$$\partial_x \hat{U}^\epsilon = \frac{1}{\epsilon(x)} M(\epsilon(x)\xi) \hat{U}^\epsilon + A^{-1} U_0(x), \quad (5.1)$$

where M is defined in (4.3). Then the general solution is:

$$\begin{cases} \hat{U}^\epsilon(x, \xi) = e^{M(\xi)x} (\hat{U}_L + \int_0^x e^{-M(\xi)y} A^{-1} U_0(y) dy) & \text{for } x < 0, \epsilon(x) = 1; \\ \hat{U}^\epsilon(x, \xi) = e^{M(\epsilon\xi)\frac{x}{\epsilon}} (\hat{U}_R + \int_0^x e^{-M(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y) dy) & \text{for } x > 0, \epsilon(x) = \epsilon, \end{cases} \quad (5.2)$$

where one can denote $e^{M(\xi)x}$ by

$$e^{M(\xi)x} = e^{\mu+(\xi)x} \Phi_+(\xi) + e^{\mu-(\xi)x} \Phi_-(\xi), \quad (5.3)$$

and Φ_\pm are defined by:

$$\Phi_+(\xi) = \frac{1}{g_+(\xi) - g_-(\xi)} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix} (g_+(\xi), -1), \quad (5.4)$$

$$\Phi_-(\xi) = \frac{1}{g_+(\xi) - g_-(\xi)} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} (-g_-(\xi), 1). \quad (5.5)$$

Then (5.2) can be rewritten as:

$$\begin{cases} \hat{U}^\epsilon(x, \xi) = e^{\mu+(\xi)x} \Phi_+(\xi) (\hat{U}_L(\xi) + \int_0^x e^{-\mu+(\xi)y} A^{-1} U_0(y) dy) \\ \quad + e^{\mu-(\xi)x} \Phi_-(\xi) (\hat{U}_L(\xi) + \int_0^x e^{-\mu-(\xi)y} A^{-1} U_0(y) dy) & \text{for } x < 0, \epsilon(x) = 1; \\ \hat{U}^\epsilon(x, \xi) = e^{\mu+(\epsilon\xi)\frac{x}{\epsilon}} \Phi_+(\epsilon\xi) (\hat{U}_R(\xi) + \int_0^x e^{-\mu+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y) dy) \\ \quad + e^{\mu-(\epsilon\xi)\frac{x}{\epsilon}} \Phi_-(\epsilon\xi) (\hat{U}_R(\xi) + \int_0^x e^{-\mu-(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y) dy) & \text{for } x > 0, \epsilon(x) = \epsilon. \end{cases} \quad (5.6)$$

Here $\hat{U}_L(\xi) = \begin{pmatrix} \hat{u}_L(\xi) \\ \hat{v}_L(\xi) \end{pmatrix}$ and $\hat{U}_R(\xi) = \begin{pmatrix} \hat{u}_R(\xi) \\ \hat{v}_R(\xi) \end{pmatrix}$ are two vectors independent of x , and defined by the boundary condition and interface conditions as follows.

First, when $x \rightarrow \infty$, $\hat{U}^\epsilon(x, \xi) \rightarrow 0$, one gets

$$(g_+(\epsilon\xi), -1) \begin{pmatrix} \hat{u}_R(\xi) \\ \hat{v}_R(\xi) \end{pmatrix} + \int_0^\infty e^{-\mu+(\epsilon\xi)\frac{y}{\epsilon}} (g_+(\epsilon\xi), -1) \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} (y) dy = 0,$$

that is,

$$g_+(\epsilon\xi) \hat{u}_R(\xi) - \hat{v}_R(\xi) + \int_0^\infty e^{-\mu+(\epsilon\xi)\frac{y}{\epsilon}} (v_0(y) g_+(\epsilon\xi) - u_0(y)) dy = 0. \quad (5.7)$$

When $x \rightarrow -\infty$, $\hat{U}^\epsilon(x, \xi) \rightarrow 0$, thus

$$(-g_-(\xi), 1) \begin{pmatrix} \hat{u}_L(\xi) \\ \hat{v}_L(\xi) \end{pmatrix} + \int_0^{-\infty} e^{-\mu_-(\xi)y} (-g_-(\xi), 1) \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} (y) dy = 0,$$

that is,

$$-g_-(\xi)\hat{u}_L(\xi) + \hat{v}_L(\xi) + \int_0^{-\infty} e^{-\mu_-(\xi)y} (-v_0(y)g_-(\xi) + u_0(y)) dy = 0. \quad (5.8)$$

Then by continuity, $\Phi_+(\xi)\hat{U}_L + \Phi_-(\xi)\hat{U}_L = \Phi_+(\epsilon\xi)\hat{U}_R + \Phi_-(\epsilon\xi)\hat{U}_R$, it is easy to get:

$$\hat{u}_L = \hat{u}_R, \quad \hat{v}_L = \hat{v}_R. \quad (5.9)$$

Plugging (5.7) - (5.9) into (5.6), one ends up with a simplified version of (5.6):

$$\begin{aligned} \hat{U}^\epsilon(x, \xi) &= \frac{1}{g_+(\epsilon\xi) - g_-(\epsilon\xi)} \left\{ \begin{pmatrix} 1 \\ g_-(\epsilon\xi) \end{pmatrix} \int_x^\infty e^{\mu_+(\epsilon\xi)\frac{x-y}{\epsilon}} (u_0(y) - v_0(y)g_+(\epsilon\xi)) dy \right. \\ &\quad + \begin{pmatrix} 1 \\ g_+(\epsilon\xi) \end{pmatrix} \int_0^x e^{\mu_-(\epsilon\xi)\frac{x-y}{\epsilon}} (u_0(y) - v_0(y)g_-(\epsilon\xi)) dy \\ &\quad \left. + \begin{pmatrix} 1 \\ g_+(\epsilon\xi) \end{pmatrix} e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} (\hat{v}_R(\xi) - \hat{u}_R(\xi)g_-(\epsilon\xi)) \right\}, \quad \text{for } x > 0; \end{aligned} \quad (5.10)$$

and

$$\begin{aligned} \hat{U}^\epsilon(x, \xi) &= \frac{1}{g_+(\xi) - g_-(\xi)} \left\{ \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} \int_{-\infty}^x e^{\mu_-(\xi)(x-y)} (u_0(y) - v_0(y)g_-(\xi)) dy \right. \\ &\quad + \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix} \int_x^0 e^{\mu_+(\xi)(x-y)} (u_0(y) - v_0(y)g_+(\xi)) dy \\ &\quad \left. + \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix} e^{\mu_+(\xi)x} (-\hat{v}_L(\xi) + \hat{u}_L(\xi)g_+(\xi)) \right\}, \quad \text{for } x < 0. \end{aligned} \quad (5.11)$$

5.2 The stiff well-posedness

Due to the nonzero initial data, it is hard to estimate the L^2 norm of the solution from the expression (5.10) (5.11). So we take a detour to look at the initial value problem with initial data supported in the right (or left) half plane. For this initial value problem, one can solve it by the Fourier Transform, thus avoid the difficulties caused by the Laplace Transform. Without loss of generality, we consider $x > 0$ here. The $x < 0$ case is the same. First we have the following lemma.

Lemma 10. Assume $U_{IVP}^\epsilon = \begin{pmatrix} u_{IVP}^\epsilon \\ v_{IVP}^\epsilon \end{pmatrix}$ is the solution to

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, & (5.12a) \end{cases}$$

$$\begin{cases} v_t^\epsilon + u_x^\epsilon = -\frac{1}{\epsilon}(v^\epsilon - \lambda u^\epsilon), & (5.12b) \end{cases}$$

$$\begin{cases} u^\epsilon(x, 0) = u_0(x), v^\epsilon(x, 0) = v_0(x), & (5.12c) \end{cases}$$

here u_0 and v_0 are supported in $[0, \infty)$. Then the solution, after the Laplace Transform, is

$$\begin{aligned} \hat{U}_{IVP}^\epsilon(x, \xi) &= \frac{1}{g_+(\epsilon\xi) - g_-(\epsilon\xi)} \left\{ \left(\frac{1}{g_-(\epsilon\xi)} \right) \int_x^\infty e^{\mu_+(\epsilon\xi)\frac{x-y}{\epsilon}} (u_0(y) - v_0(y)g_+(\epsilon\xi)) dy \right. \\ &\quad \left. + \left(\frac{1}{g_+(\epsilon\xi)} \right) \int_0^x e^{\mu_-(\epsilon\xi)\frac{x-y}{\epsilon}} (u_0(y) - v_0(y)g_-(\epsilon\xi)) dy \right\}, \end{aligned} \quad (5.13)$$

and the following inequality holds:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty |\hat{U}_{IVP}^\epsilon(x, \xi)|^2 dx d\beta \leq O(1) \int_0^\infty |U_0(x)|^2 dx. \quad (5.14)$$

Proof. First solution (5.13) is obtained in the same way as (5.10), so we will omit the details. Then if the Fourier Transform w.r.t x is used instead of the Laplace Transform w.r.t t in this case, one gets [33]

$$\int_{-\infty}^\infty |U_{IVP}^\epsilon(x, t)|^2 dx \leq O(1) \int_0^\infty |U_0(x)|^2 dx, \quad \forall t > 0. \quad (5.15)$$

Integrating with respect to t gives

$$\int_0^\infty dt \int_{-\infty}^\infty e^{-2\alpha t} |U_{IVP}^\epsilon(x, t)|^2 dx \leq O(1) \int_0^\infty |U_0(x)|^2 dx.$$

Then by Parseval's identity (4.21), one can prove the inequality. For more details, see [33]. \square

One also needs to estimate $\int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} (u_0(y) - v_0(y)g_+(\epsilon\xi)) dy$ and $\int_0^{-\infty} e^{-\mu_-(\epsilon\xi)y} (u_0(y) - v_0(y)g_-(\epsilon\xi)) dy$ which appear in (5.7) and (5.8) respectively. The estimates of these two integrals are similar by using the energy estimate. So we only estimate the first integral here.

Lemma 11. *Let*

$$\hat{w}_{IBVP}(\epsilon\xi) = \int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} (u_0(y) - v_0(y)g_+(\epsilon\xi)) dy, \quad (5.16)$$

then

$$\int_{-\infty}^\infty |\hat{w}_{IBVP}(\epsilon\xi)|^2 d\beta \leq O(1) \int_0^\infty |U_0(x)|^2 dx. \quad (5.17)$$

Proof. The idea of the proof follows that in [33]. We construct the following initial boundary value problem on the right half plane $x > 0$. Later one can see that $\hat{w}_{IBVP}(\epsilon\xi)$ can be expressed by the Laplace Transform of the boundary value of the following problem, thus can be bounded by the initial data. This is the key motivation of constructing the following system:

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, & (5.18a) \\ v_t^\epsilon + u_x^\epsilon = -\frac{1}{\epsilon}(v^\epsilon - \lambda u^\epsilon), & (5.18b) \\ u^\epsilon(x, 0) = u_0(x), v^\epsilon(x, 0) = v_0(x), & (5.18c) \\ B_u u^\epsilon(0, t) + B_v v^\epsilon(0, t) = 0. & (5.18d) \end{cases}$$

Here B_u and B_v are two constants that satisfy the so-called Stiff Kreiss Condition (SKC) [33]: $\frac{B_u}{B_v} \notin [-1, \frac{\lambda+|\lambda|}{2}]$. The Laplace Transform of the solution to this system can be written as:

$$\begin{aligned} \hat{U}_{IBVP}^\epsilon(x, \xi) &= e^{\mu+(\epsilon\xi)\frac{x}{\epsilon}} \Phi_+(\epsilon\xi) \left[\hat{U}_{IBVP}^\epsilon(0, \xi) + \int_0^x e^{-\mu+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y) dy \right] \\ &\quad + e^{\mu-(\epsilon\xi)\frac{x}{\epsilon}} \Phi_-(\epsilon\xi) \left[\hat{U}_{IBVP}^\epsilon(0, \xi) + \int_0^x e^{-\mu-(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y) dy \right], \end{aligned} \quad (5.19)$$

where $\hat{U}_{IBVP}^\epsilon(0, \xi) = \begin{pmatrix} \hat{u}_{IBVP}^\epsilon \\ \hat{v}_{IBVP}^\epsilon \end{pmatrix}$ satisfies

$$\begin{cases} B_u \hat{u}_{IBVP}^\epsilon(0, \xi) + B_v \hat{v}_{IBVP}^\epsilon(0, \xi) = 0, & (5.20a) \\ \Phi_+(\epsilon\xi) (\hat{U}_{IBVP}^\epsilon(0, \xi) + \int_0^\infty e^{-\mu+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y) dy) = 0. & (5.20b) \end{cases}$$

From definition (5.16), the second condition (5.20b) can be written as

$$g_+(\epsilon\xi) \hat{u}_{IBVP}^\epsilon(0, \xi) - \hat{v}_{IBVP}^\epsilon(0, \xi) = \hat{w}_{IBVP}^\epsilon(\epsilon\xi),$$

thus

$$\hat{U}_{IBVP}^\epsilon(0, \xi) = \frac{\hat{w}_{IBVP}^\epsilon(\epsilon\xi)}{B_u + B_v g_+(\epsilon\xi)} \begin{pmatrix} B_v \\ -B_u \end{pmatrix}. \quad (5.21)$$

Now the energy estimate can be used to get the upper bound of $\int_0^T |U_{IBVP}(0, t)|^2 dt$. Let $H = \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix}$, multiply (5.18) by $e^{-2\alpha t} U^T H$, and integrate over $[0, T] \times [0, \infty)$, one has (here we omit the subscription and superscription for a while)

$$\begin{aligned} &\frac{1}{2} \int_0^\infty (U, HU)(x, T) e^{-2\alpha T} dx + \alpha \int_0^T \int_0^\infty (U, HU)(x, t) e^{-2\alpha t} dx dt \\ &+ \frac{1}{\epsilon} \int_0^T \int_0^\infty (v - \lambda u)^2 e^{-2\alpha t} dx dt + \frac{1}{2} \int_0^T (\lambda u^2 - 2uv + \lambda v^2)(0, t) e^{-2\alpha t} dt \\ &= \frac{1}{2} \int_0^\infty (U_0(x), HU_0(x)) dx. \end{aligned}$$

One needs to choose the boundary condition such that $\lambda u(0, t)^2 - 2u(0, t)v(0, t) + \lambda v(0, t)^2 \geq c|U(0, t)|^2$, where c is a bounded constant. Later we will show that this kind of boundary condition exists and it is a subclass of SKC. Then one can get

$$\int_0^T |U_{IBVP}^\epsilon(0, t)|^2 e^{-2\alpha t} dt \leq O(1) \int_0^\infty |U_0(x)|^2 dx. \quad (5.22)$$

Let $T \rightarrow \infty$, then

$$\int_0^\infty |U_{IBVP}^\epsilon(0, t)|^2 e^{-2\alpha t} dt \leq O(1) \int_0^\infty |U_0(x)|^2 dx. \quad (5.23)$$

By Parseval's identity and (5.21) (5.23), one obtains (5.17). As for the boundary condition, there are plenty of choices. Any B_u and B_v that satisfy

$$\begin{aligned} \frac{B_u}{B_v} &> -\frac{1}{\lambda}(1 - \sqrt{1 - \lambda^2}) \quad \text{or} \quad \frac{B_u}{B_v} < -\frac{1}{\lambda}(1 + \sqrt{1 - \lambda^2}), \quad \text{for } \lambda > 0, \\ -\frac{1}{\lambda}(1 - \sqrt{1 - \lambda^2}) &< \frac{B_u}{B_v} < -\frac{1}{\lambda}(1 + \sqrt{1 - \lambda^2}), \quad \text{for } \lambda < 0, \\ \frac{B_u}{B_v} &> 0, \quad \text{for } \lambda = 0, \end{aligned}$$

will work, and it is not hard to see it is a subclass of the SKC. \square

Similarly, we have the following corollary.

Corollary 12. *Let*

$$\hat{w}_{IBVP2}(\xi) = \int_0^{-\infty} e^{-\mu - (\xi)y} (u_0(y) - v_0(y)g_-(\xi)) dy, \quad (5.24)$$

then

$$\int_{-\infty}^{\infty} |\hat{w}_{IBVP2}(\xi)|^2 d\beta \leq O(1) \int_{-\infty}^0 |U_0(x)|^2 dx. \quad (5.25)$$

Now we go back to the proof of Theorem 3 of the stiff well-posedness with nonzero initial data and when the problem is set in $(-\infty, \infty)$ instead of $[-L, L]$.

Proof. When $x > 0$, from the solution (5.10) one gets

$$\begin{aligned} &\int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x, \xi)|^2 d\beta \leq \int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}_{IVP}^\epsilon|^2 d\beta \\ &+ \int_0^{\infty} dx \int_{-\infty}^{\infty} \left| \begin{pmatrix} 1 \\ g_+(\epsilon\xi) \end{pmatrix} e^{\mu - (\epsilon\xi)\frac{x}{\epsilon}} (v_R - g_-(\epsilon\xi)u_R) \right|^2 \frac{1}{|g_+(\epsilon\xi) - g_-(\epsilon\xi)|^2} d\beta \\ &= I_1 + I_2. \end{aligned} \quad (5.26)$$

By (5.14) I_1 can be estimated as:

$$I_1 \leq O(1) \int_0^{\infty} |U_0(x)|^2 dx. \quad (5.27)$$

As for I_2 , since $\frac{1}{|g_+(\epsilon\xi) - g_-(\epsilon\xi)|}$ is uniformly bounded, one has

$$I_2 \leq O(1) \int_0^{\infty} dx \int_{-\infty}^{\infty} e^{2Re\mu - (\epsilon\xi)\frac{x}{\epsilon}} [|v_R|^2 + O(1)|u_R|^2] d\beta.$$

Then by (5.7) (5.8) (5.16) and (5.24), one obtains:

$$\begin{aligned} g_+(\epsilon\xi)u_R - v_R &= \hat{w}_{IBVP}, \\ -g_-(\xi)u_R - v_R &= \hat{w}_{IBVP2}. \end{aligned}$$

Thus $u_R = O(1)\hat{w}_{IBVP}(\epsilon\xi) + O(1)\hat{w}_{IBVP2}(\xi)$, $v_R = O(1)\hat{w}_{IBVP}(\epsilon\xi) + O(1)\hat{w}_{IBVP2}(\xi)$. Finally by Lemma 11 and Corollary 12,

$$I_2 \leq -O(1)\frac{\epsilon}{2Re\mu_-(\epsilon\xi)} \int_{-\infty}^{\infty} |U_0(x)|^2 dx. \quad (5.28)$$

Then by (4.17)–(4.18), one sees that $\int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x, \xi)|^2 d\beta$ is uniformly bounded. In the same way, one can prove

$$\int_{-\infty}^0 dx \int_{-\infty}^{\infty} |\hat{U}^\epsilon(x, \xi)|^2 d\beta \leq O(1) \int_{-\infty}^{\infty} |U_0(x)|^2 dx. \quad (5.29)$$

Till now we have proved the stiff well-posedness of the original system stated in Theorem 3. \square

5.3 The asymptotic convergence and error estimates

Next we will prove the asymptotic convergence and error estimates. The first step is also using the Laplace Transform to represent the exact solution. We will consider the case $\lambda < 0$ first. Consider the domain decomposition system (3.1)–(3.2) with $L = \infty$. The case when L is finite is the same but with two more extra terms coming from the boundary which can be analyzed in the same way as follows.

First in comparing the solution to the domain decomposition system (3.1)–(3.2) with the original system (1.1), in order to avoid the difficulties caused by Laplace transform, we need the help of the following lemma, which compares the initial value problem (5.12) with its reduced system:

$$\begin{cases} u_t^0 + \lambda u_x^0 = 0, & (5.30a) \\ u^0(x, 0) = u_0(x). & (5.30b) \end{cases}$$

Here we assume $u_0(x)$ is supported on $[0, \infty)$.

Lemma 13. *Let U_{IVP}^ϵ and U_{IVP}^0 be the solution of relaxation problem (5.12) and equilibrium problem (5.30) respectively, then*

$$\int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}_{IVP}^\epsilon - \hat{U}_{IVP}^0|^2 d\beta \leq O(1)\epsilon^2 \|U_0\|_{H^2}^2 + O(1)\epsilon \|v_0 - \lambda u_0\|_{L^2[0, \infty)}^2. \quad (5.31)$$

Proof. The proof is based on the Fourier Transform, and one can refer to [33] for details. \square

Now we are ready to prove Theorem 4 about the asymptotic convergence of the domain decomposition system.

Proof. When $x > 0$, the solution is $u^r(x, t) = u_0(x - \lambda t)$. After the Laplace Transform, one gets:

$$\hat{u}^r(x, \xi) = -\frac{1}{\lambda} \int_x^{\infty} u_0(y) e^{-\frac{\xi}{\lambda}(x-y)} dy, \quad (5.32)$$

$$\hat{v}^r = \lambda \hat{u}^r. \quad (5.33)$$

For $x < 0$, the solution to (3.2) can be represented as

$$\begin{aligned}\hat{U}^l(x, \xi) &= e^{\mu+(\xi)x}\Phi_+(\xi)(\hat{D}(\xi) + \int_0^x e^{-\mu+(\xi)y}A^{-1}U_0(y)dy) \\ &\quad + e^{\mu-(\xi)x}\Phi_-(\xi)(\hat{D}(\xi) + \int_0^x e^{-\mu-(\xi)y}A^{-1}U_0(y)dy).\end{aligned}\quad (5.34)$$

Here $\hat{D}(\xi) = \begin{pmatrix} \hat{D}_u(\xi) \\ \hat{D}_v(\xi) \end{pmatrix}$ is determined by:

$$(-g_-(\xi) \ 1) \begin{pmatrix} \hat{D}_u(\xi) \\ \hat{D}_v(\xi) \end{pmatrix} + \int_0^{-\infty} e^{-\mu-(\xi)y}(-g_-(\xi) \ 1) \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} (y)dy = 0, \quad (5.35)$$

$$\frac{1}{g_+(\xi)-g_-(\xi)} \left[(\hat{D}_u(\xi)g_+(\xi) - \hat{D}_v(\xi))g_-(\xi) + (\hat{D}_v(\xi) - \hat{D}_u(\xi)g_-(\xi))g_+(\xi) \right] = - \int_0^{\infty} u_0(y)e^{\xi y}dy,$$

where the second equation is simplified as

$$\hat{D}_v(\xi) = - \int_0^{\infty} u_0(y)e^{\xi y}dy. \quad (5.36)$$

Now one can compare the difference of (5.32) and (5.10) on the right domain. Since the solution to (5.30) is (5.32), and part of (5.10) is (5.13), one has

$$\begin{aligned}& \int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{u}^\epsilon - \hat{u}^r|^2 d\beta \leq \int_0^{\infty} dx \int_{-\infty}^{\infty} |\hat{u}_{IVP}^\epsilon - \hat{u}_{IVP}^0|^2 d\beta \\ & + \int_0^{\infty} dx \int_{-\infty}^{\infty} \left| \frac{1}{g_+(\epsilon\xi) - g_-(\epsilon\xi)} \right|^2 (1 + |g_+(\epsilon\xi)|^2) |e^{\mu-(\epsilon\xi)\frac{x}{\epsilon}} (\hat{v}_R(\xi) - g_-(\epsilon\xi)\hat{u}_R(\xi))|^2 d\beta \\ & = \mathbb{I}_1 + \mathbb{I}_2,\end{aligned}$$

$$\mathbb{I}_1 \leq O(1)\epsilon^2 \|U_0\|_{H^2}^2 + O(1)\epsilon \|v_0 - \lambda u_0\|_{L^2[0, \infty)}^2, \quad (5.37)$$

$$\mathbb{I}_2 \leq O(1) \int_0^{\infty} e^{2\text{Re}\mu - (\epsilon\xi)\frac{x}{\epsilon}} dx \int_{-\infty}^{\infty} |\hat{v}_R(\xi) - g_-(\epsilon\xi)\hat{u}_R(\xi)|^2 d\beta \leq O(1)\epsilon \|U_0\|_{L^2}^2. \quad (5.38)$$

The calculation of the last inequality is the same as (5.28). Notice here that the term that contains $e^{2\text{Re}\mu - (\epsilon\xi)\frac{x}{\epsilon}}$ is due to the interface layer, since the initial data can induce an interface layer at the interface in this case.

Now compare the solution on the left domain, (5.34) with (5.6). The difference comes from the difference in coefficients, thus

$$\begin{aligned}& \int_{-\infty}^0 dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta \\ & \leq O(1) \int_{-\infty}^0 dx \int_{-\infty}^{\infty} |e^{\mu+(\xi)x}|^2 (1 + |g_-(\xi)|^2) [g_+(\xi)(\hat{D}_u - \hat{u}_L) - (\hat{D}_v - \hat{v}_L)]^2 d\beta \\ & \quad + \int_{-\infty}^0 dx \int_{-\infty}^{\infty} |e^{\mu-(\xi)x}|^2 (1 + |g_+(\xi)|^2) [-g_-(\xi)(\hat{D}_u - \hat{u}_L) + (\hat{D}_v - \hat{v}_L)]^2 d\beta.\end{aligned}$$

By boundary conditions (5.8) and (5.35), the second term vanishes, so

$$\int_{-\infty}^0 dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta \leq O(1) \int_{-\infty}^0 dx \int_{-\infty}^{\infty} e^{2\text{Re}\mu+(\xi)x} (|\hat{D}_u - \hat{u}_L|^2 + |\hat{D}_v - \hat{v}_L|^2) d\beta.$$

Next compare the parameters derived in the original system (5.7)–(5.9) with those of the domain decomposition method (5.35) (5.36), one gets

$$\begin{aligned} & \int_{-\infty}^0 dx \int_{-\infty}^{\infty} |\hat{U}^l - \hat{U}^\epsilon|^2 d\beta = O(1) \int_{-\infty}^{\infty} (|\hat{D}_u - \hat{u}_L|^2 + |\hat{D}_v - \hat{v}_L|^2) d\beta \\ &= O(1) \int_{-\infty}^{\infty} |\hat{D}_v - \hat{v}_L|^2 d\beta \\ &= O(1) \int_{-\infty}^{\infty} \left| - \int_0^{\infty} e^{\frac{\xi}{\lambda}y} u_0(y) dy - \frac{-\hat{w}_{IBVP} + \frac{g_+(\epsilon\xi)}{g_-(\xi)} \hat{w}_{IBVP2}}{1 - \frac{g_+(\epsilon\xi)}{g_-(\xi)}} \right|^2 d\beta \\ &\leq O(1) \int_{-\infty}^{\infty} \left| \int_0^{\infty} (u_0(y) - v_0(y)g_+(\epsilon\xi)) e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} dy - \int_0^{\infty} u_0(y) e^{\frac{\xi}{\lambda}y} dy \right|^2 d\beta \\ &\quad + O(1) \int_{-\infty}^{\infty} \left| g_+(\epsilon\xi) \int_0^{-\infty} (u_0(y) - v_0(y)g_-(\xi)) e^{-\mu_-(\xi)y} dy \right|^2 d\beta \\ &\quad + O(1) \int_{-\infty}^{\infty} \left| g_+(\epsilon\xi) \int_0^{\infty} u_0(y) e^{\frac{\xi}{\lambda}y} dy \right|^2 d\beta \\ &= \mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3. \end{aligned}$$

We begin with the simplest part \mathbb{J}_3 . First note that $g(\epsilon\xi)$ is uniformly bounded (see Lemma 7), and $\int_0^{\infty} u_0(y) e^{\frac{\xi}{\lambda}y} dy$ can be considered as Laplace transform of $u_0(y)$, so by Parseval's identity, the integral $\int_{-\infty}^{\infty} \left| \int_0^{\infty} u_0(y) e^{\frac{\xi}{\lambda}y} dy \right|^2 d\beta$ is uniformly bounded, then by dominated convergence theorem, $\mathbb{J}_3 \rightarrow 0$ as $\epsilon \rightarrow 0$. Moreover, since when $\lambda < 0$, $g_+(\epsilon\xi) = O(1)\epsilon\xi$, thus

$$\mathbb{J}_3 \leq O(1)\epsilon^2 \int_{-\infty}^{\infty} \left| \xi \int_0^{\infty} u_0(y) e^{\frac{\xi}{\lambda}y} dy \right|^2 d\beta.$$

If the compatibility condition on $u_0(y)$ is assumed such that $u_0(0) = 0$, $\int_0^{\infty} \xi u_0(y) e^{\frac{\xi}{\lambda}y} dy$ can be considered as the Laplace transform to $u'_0(y)$, so

$$\mathbb{J}_3 \leq O(1)\epsilon^2 \int_{-\infty}^{\infty} |\mathcal{L}(u'_0(y))(\xi)|^2 d\beta \leq O(1)\epsilon^2 \int_0^{\infty} |u'_0(y)|^2 dy. \quad (5.39)$$

Next we look at \mathbb{J}_2 . Similar to \mathbb{J}_3 , one will first get

$$\mathbb{J}_2 \leq O(1)\epsilon^2 \int_{-\infty}^{\infty} \left| \xi \int_0^{-\infty} (u_0(y) - v_0(y)g_-(\xi)) e^{-\mu_-(\xi)y} dy \right|^2 d\beta.$$

By recalling (5.24) and integration by parts, one gets

$$\hat{w}_{IBVP2} = -\frac{1}{\mu_-(\xi)} \int_0^{-\infty} e^{-\mu_-(\xi)y} (-u'_0 + g_-(\xi)v'_0) dy, \quad (5.40)$$

where the compatibility conditions $u_0(0) = 0$ and $v_0(0) = 0$ are used. Since $-\mu_-(\xi) = \mu_+(\xi) - 2\lambda$, one has

$$(\mu_+(\xi) - 2\lambda)\hat{w}_{IBVP2} = \int_0^{-\infty} e^{-\mu_-(\xi)y}(-u'_0 + g_-(\xi)v'_0)dy.$$

Notice when $\lambda < 0$, $\mu_+(\xi) = -\frac{\xi}{g_-(\xi)}$, thus

$$-\xi\hat{w}_{IBVP2} = 2\lambda g_-(\xi)\hat{w}_{IBVP2} + g_-(\xi) \int_0^{-\infty} (-u'_0(y) + g_-(\xi)v'_0(y))dy.$$

Therefore, the following estimate holds:

$$\int_{-\infty}^{\infty} |\xi\hat{w}_{IBVP2}|^2 d\beta \leq O(1) \int_{-\infty}^{\infty} \left| \int_0^{-\infty} e^{-\mu_-(\xi)y} (u'_0 - g_-(\xi)v'_0)(y) dy \right|^2 d\beta + O(1) \int_{-\infty}^{\infty} |\hat{w}_{IBVP2}(\xi)|^2 d\beta. \quad (5.41)$$

The integral with respect to y on the right hand side is similar to \hat{w}_{IBVP2} in (5.24), except to change u_0 and v_0 to u'_0 and v'_0 respectively. So one has

$$\int_{-\infty}^{\infty} |\xi\hat{w}_{IBVP2}|^2 d\beta \leq O(1) \int_0^{\infty} |U'_0(x)|^2 dx + O(1) \int_0^{\infty} |U_0(x)|^2 dx. \quad (5.42)$$

Therefore,

$$\mathbb{J}_2 \leq O(1)\epsilon^2 \int_0^{\infty} |U'_0(x)|^2 dx. \quad (5.43)$$

Now we turn to \mathbb{J}_1 . First using $g_+(\epsilon\xi) \sim O(1)\epsilon\xi$ gives

$$\mathbb{J}_1 \leq O(1) \int_{-\infty}^{\infty} \epsilon^2 \left| \xi \int_0^{\infty} v_0 e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} dy \right|^2 d\beta + O(1) \int_{-\infty}^{\infty} \left| \int_0^{\infty} u_0(y) (e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y}) dy \right|^2 d\beta. \quad (5.44)$$

Notice in (5.16) if one exchanges u_0 and v_0 and let $u_0 \equiv 0$, then use (5.16) (5.41), and similar to (5.42), one will get:

$$\int_{-\infty}^{\infty} \left| \xi \int_0^{\infty} v_0 e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} dy \right|^2 d\beta \leq O(1) \int_0^{\infty} |v'_0(x)|^2 dx. \quad (5.45)$$

On the other hand, for the term $\int_{-\infty}^{\infty} \left| \int_0^{\infty} u_0(y) (e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y}) dy \right|^2 d\beta$, integration by parts w.r.t. y three times, and assume the compatibility conditions $u_0(0) = u'_0(0) = u''_0(0)$, it

becomes

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left| \int_0^{\infty} u_0(y) (e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y}) dy \right|^2 d\beta \\
&= \int_{-\infty}^{\infty} \left| \left(e^{\frac{\xi}{\lambda}y} \left(\frac{\lambda}{\xi} \right)^3 - e^{-\frac{\mu_+(\epsilon\xi)}{\epsilon}y} \left(-\frac{\epsilon}{\mu_+(\epsilon\xi)} \right)^3 \right) u_0'''(y) dy \right|^2 d\beta \\
&= \int_{-\infty}^{\infty} \left| \int_0^{\infty} \left(e^{\frac{\xi}{\lambda}y} \left(\frac{\lambda}{\xi} \right)^3 - e^{\frac{\xi}{\lambda}y} \left(\frac{-\epsilon}{\mu_+(\epsilon\xi)} \right)^3 + e^{\frac{\xi}{\lambda}y} \left(\frac{-\epsilon}{\mu_+(\epsilon\xi)} \right)^3 - e^{-\frac{\mu_+(\epsilon\xi)}{\epsilon}y} \left(\frac{-\epsilon}{\mu_+(\epsilon\xi)} \right)^3 \right) u_0'''(y) dy \right|^2 d\beta \\
&\leq \int_{-\infty}^{\infty} \left| \left(\left(\frac{\lambda}{\xi} \right)^3 - \left(\frac{-\epsilon}{\mu_+(\epsilon\xi)} \right)^3 \right) \int_0^{\infty} e^{\frac{\xi}{\lambda}y} u_0'''(y) dy \right|^2 d\beta + \int_{-\infty}^{\infty} \left| \left(\frac{\epsilon}{\mu_+(\epsilon\xi)} \right)^3 \int_0^{\infty} (e^{\frac{\xi}{\lambda}y} - e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}) u_0'''(y) dy \right|^2 d\beta \\
&= \mathbb{L}_1 + \mathbb{L}_2
\end{aligned}$$

For \mathbb{L}_1 , notice that

$$\frac{\lambda}{\xi} - \frac{\epsilon}{\mu_+(\epsilon\xi)} = \frac{2\lambda\epsilon\xi + \lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}}{2\xi(1 + \epsilon\xi)} = \frac{\lambda\epsilon}{1 + \epsilon\xi} + \frac{\epsilon}{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1 + \epsilon\xi)}} = O(1)\epsilon$$

by Lemma 6, and

$$\frac{\epsilon}{\mu_+(\epsilon\xi)} = -\frac{1}{\xi g_-(\epsilon\xi)} = O(1)\frac{1}{\xi}, \tag{5.46}$$

by Lemma 7, one can estimate \mathbb{L}_1 as

$$\mathbb{L}_1 \leq O(1)\epsilon^2 \int_{-\infty}^{\infty} \left| \frac{1}{\xi} \int_0^{\infty} u_0'''(y) e^{\frac{\xi}{\lambda}y} dy \right|^2 d\beta \leq O(1)\epsilon^2 \|u_0'''\|_{L^2}^2. \tag{5.47}$$

In term \mathbb{L}_2 , use Cauchy-Schwartz inequality for integral w.r.t. y , one obtains

$$\begin{aligned}
\mathbb{L}_2 &\leq O(1) \int_{-\infty}^{\infty} \left| \frac{1}{\xi} \right|^6 \int_0^{\infty} \left| e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y} \right|^2 dy \int_0^{\infty} |u_0'''(y)|^2 dy d\beta \\
&\leq O(1) \|u_0'''\|_{L^2}^2 \int_{-\infty}^{\infty} \epsilon^2 \left| \frac{1}{\xi} \right|^2 d\beta \|u_0'''\|_{L^2}^2 = O(1)\epsilon^2,
\end{aligned} \tag{5.48}$$

where the second inequality using the fact in (4.27) and (4.29). Therefore, one arrives at the estimation for \mathbb{J}_3 :

$$\mathbb{J}_3 \leq O(1)\epsilon^2 \|u_0(y)\|_{H^3}^2 + O(1)\epsilon^2 \|v_0(y)\|_{H^1}^2, \tag{5.49}$$

In summary,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |\hat{U}^\epsilon - \hat{U}|^2 d\beta \leq O(1)\epsilon \|v_0 - \lambda u_0\|_{L^2}^2 + O(1)\epsilon \|U_0\|_{L^2[0,\infty)}^2 + O(1)\epsilon^2 \|U_0\|_{H^3}^2. \tag{5.50}$$

The case with $\lambda > 0$ is rather similar, but there is no interface layer at $x = 0$, so one will find the term that contains $\|U_0\|_{L^2}^2$ will have a convergence rate $O(1)\epsilon^2$ instead of $O(1)\epsilon$. \square

6 Domain-decomposition based numerical schemes and numerical experiments

We use Δt and Δx to represent the time step and mesh size respectively, u_j^n to denote u at time $n\Delta t$ and position $j\Delta x$. Let $M = T/\Delta t$, and $N = L/\Delta x$. We use the upwind scheme to the Riemann invariants $u \pm v$ to solve the left part (3.2) or (3.3), and use the Godunov scheme to solve the equilibrium equation in (3.1) or (3.4).

6.1 The numerical scheme

Case I: $f'(u_0^n) < 0, \forall n \geq 0$

- **Step 1. Discretization of (3.1) on the right domain.**

For $j = 0, 1, \dots, N, n = 0, 1, \dots, M$, solve

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0, \quad (6.1)$$

$$v_j^{n+1} = f(u_j^{n+1}), \quad (6.2)$$

$$u_j^0 = u_0(x_j), \quad v_j^0 = v_0(x_j), \quad (6.3)$$

$$u_N^n = b_R(t^n); \quad (6.4)$$

where $F_{j+\frac{1}{2}}^n = f(R(0, u_j^n, u_{j+1}^n))$, $F_{j-\frac{1}{2}}^n = f(R(0, u_{j-1}^n, u_j^n))$, and $R(0, \zeta, \eta)$, the Riemann solver, is defined as:

$$R(0, \zeta, \eta) = \begin{cases} \zeta, & \text{if } f'(\zeta), f'(\eta) \leq 0, \\ \eta, & \text{if } f'(\zeta), f'(\eta) \geq 0, \\ \zeta, & \text{if } f'(\zeta) > 0 > f'(\eta), s > 0, \\ \eta, & \text{if } f'(\zeta) > 0 > f'(\eta), s < 0, \\ f'^{-1}(0), & \text{otherwise.} \end{cases}$$

where $s = \frac{f(\zeta) - f(\eta)}{\zeta - \eta}$ is the shock speed.

- **Step 2. Discretization of (3.2) on the left domain.**

For $j = -N, \dots, -1, 0, n = 0, 1, \dots, M$, let the Riemann invariants $P_j^n = u_j^n + v_j^n, Q_j^n = u_j^n - v_j^n$, and solve

$$\frac{P_j^{n+1} - P_j^n}{\Delta t} + \frac{P_j^n - P_{j-1}^n}{\Delta x} = -(v_j^n - f(u_j^n)), \quad (6.5)$$

$$\frac{Q_j^{n+1} - Q_j^n}{\Delta t} - \frac{Q_{j+1}^n - Q_j^n}{\Delta x} = (v_j^n - f(u_j^n)), \quad (6.6)$$

$$P_j^0 = u_0(x_j) + v_0(x_j), \quad Q_j^0 = u_0(x_j) - v_0(x_j), \quad (6.7)$$

$$u_{-N}^{n+1} = b_L(t^{n+1}), \quad v_0^{n+1} \text{ obtained from right by (6.2).} \quad (6.8)$$

Case II: $f'(u_0^n) > 0, \forall n \geq 0$

- **Step 1. Discretization of (3.3) on the left domain.**

For $j = -N, \dots, -1, 0, n = 0, 1, \dots, M$, let the Riemann invariants $P_j^n = u_j^n + v_j^n, Q_j^n = u_j^n - v_j^n$, then solve

$$\frac{P_j^{n+1} - P_j^n}{\Delta t} + \frac{P_j^n - P_{j-1}^n}{\Delta x} = -(v_j^n - f(u_j^n)), \quad (6.9)$$

$$\frac{Q_j^{n+1} - Q_j^n}{\Delta t} - \frac{Q_{j+1}^n - Q_j^n}{\Delta x} = (v_j^n - f(u_j^n)) \quad (6.10)$$

$$P_j^0 = u_0(x_j) + v_0(x_j), \quad Q_j^0 = u_0(x_j) - v_0(x_j), \quad (6.11)$$

$$u_{-N}^{n+1} = b_L(t^{n+1}), \quad (6.12)$$

$$P_0^{n+1} = u_0^{n+1} + f(u_0^{n+1}); \quad (6.13)$$

- **Step 2. Discretization of (3.1) on the left domain.**

For $j = 1, \dots, N, n = 0, 1, \dots, M$, solve

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0, \quad (6.14)$$

$$u_j^0 = u_0(x_j), \quad v_j^0 = v_0(x_j), \quad (6.15)$$

$$u_0^{N+1} \text{ obtained from left,} \quad (6.16)$$

where $F_{j+\frac{1}{2}}^n$ and $F_{j-\frac{1}{2}}^n$ are defined as in *Case I*. To solve for u_0^{n+1} , since (6.9) is an explicit scheme for P_0^{n+1} , we first use it to get P_0^{n+1} , and then use Newton iteration for (6.13) to get u_0^{n+1} .

6.2 Coupling of multiple regions

The previous method for two regions can be easily extended to three or more regions with different scales. For example, consider the coupling that consists of equilibrium (where $\epsilon(x)$ is small) region on the left, relaxation (where $\epsilon(x)$ is of $O(1)$) in the middle, and equilibrium region on the right, that is,

$$\epsilon(x) = \epsilon, \quad x \in [-L, x_1]; \quad \epsilon(x) = 1, \quad x \in [x_1, x_2]; \quad \epsilon(x) = \epsilon, \quad x \in [x_2, L],$$

where $x_1 < x_2$. Consider the case $f'(u(x_1, t)) < 0$ and $f'(u(x_2, t)) > 0$ for $t \leq T$. The other cases can be treated similarly. Our algorithm will solve the middle region $[x_1, x_2]$ first with interface condition $v = f(u)$ at x_1 and x_2 , and then solve the left and right regions. To be more specific, one can follow the following steps.

- **Step 1.** For $j = N_1 + 1, \dots, N_2$, $n = 0, 1, \dots, M$ that correspond to the middle region $[x_1, x_2)$, solve the equations (6.9)–(6.11) for Riemann invariants $P_j^n = u_j^n + v_j^n$, $Q_j^n = u_j^n - v_j^n$ with boundary conditions at x_1 and x_2 respectively given by

$$P_{N_1+1}^{n+1} = u_{N_1+1}^{n+1} + f(u_{N_1+1}^{n+1}); \quad P_{N_2}^{n+1} = u_{N_2}^{n+1} + f(u_{N_2}^{n+1}). \quad (6.17)$$

Notice here one needs to use Newton's iteration at both boundary points to get $u_{N_1+1}^{n+1}$ and $u_{N_2}^{n+1}$ from $P_{N_1+1}^{n+1}$ and $P_{N_2}^{n+1}$ respectively using (6.17).

- **Step 2.** For $j = 0, \dots, N_1$, $n = 0, 1, \dots, M$, one is in the left region $[-L, x_1)$, solve (6.1) and (6.3) with boundary value $u_{N_1+1}^{n+1}$ got from the previous step.
- **Step 3.** For $j = N_2 + 1, \dots, N$, $n = 0, 1, \dots, M$, solve (6.14) and (6.15) with the boundary value $u_{N_2}^{n+1}$ obtained from step 1.

In summary, near the interface, if there is a boundary layer in the equilibrium region, then solve the equilibrium equation first and then pass on to the relaxation regions through the value of v ; on the other hand, if there is no boundary layer, then one can always take $v = f(u)$ as the interface condition and solve the relaxation region first. In any situation, the system can be completely decoupled in different regions, and one can always find an appropriate order to solve them.

6.3 More general cases

- If $f'(u)$ changes sign at interface, one can check the sign of $f'(u)$ at the current time step, and then use either (6.1)–(6.8) or (6.9)–(6.16) to continue to the next step.
- If ϵ also depends on t , so the interface may be dynamic, then one needs to adaptively adjust the interface location (see for example [8]) and then use the domain decomposition method.
- In higher space dimension, if the interface is a curve or surface, we simply use the Cartesian grids and extend the $1d$ method to higher dimensions using dimension-by-dimension technique. This will result a first order error due to the grid effect. A more sophisticated method would use an interface aligned mesh or immersed interface method [21]. We will not elaborate on this since it is out of the scope of the paper.

6.4 Numerical examples

The first two examples are given to validate our domain decomposition system numerically. Therefore we focus on the behavior of l^1 error with a changing ϵ (we only change ϵ for $x > 0$, while for $x < 0$, let $\epsilon = 1$). Here we use $\Delta x = 10^{-3}$, $\Delta t = 2.5 \times 10^{-4}$ in the regime $\Delta x, \Delta t \ll \epsilon$, and run the algorithm to $T = 0.2$. We change ϵ from 0.05 to 0.0025, then calculate the error

$$U_{l^1} = \max_{0 \leq n \leq M} \sum_{j=0}^N |(u^\epsilon)_j^n - u_j^n| \Delta x, \quad V_{l^1} = \max_{0 \leq n \leq M} \sum_{j=0}^N |(v^\epsilon)_j^n - v_j^n| \Delta x.$$

Here $(u^\epsilon)_j^n$ and $(v^\epsilon)_j^n$ are obtained by directly solving the original system (1.1) - (1.5).

Example 1. Let $f(u^\epsilon) = \frac{1}{4}(e^{-u^\epsilon} - 1)$ in (1.1), with initial condition $u^\epsilon(x, 0) = \sin(\pi x)^3$, and boundary condition $u(-1, t) = u(1, t) = 0$. In this case, $f'(u) < 0$, so there will be an interface layer at the interface $x = 0$. Figure 1 gives the $\log(\text{error})$ versus $\log(\epsilon)$. One can see that the convergence rate is $O(\epsilon)$.

Example 2. Now we consider the case $f'(u) > 0$. Let $f(u^\epsilon) = \frac{1}{4}(e^{u^\epsilon} - 1)$, initial condition $u^\epsilon(x, 0) = \sin(\pi x)^3$, and boundary condition $u(-1, t) = u(1, t) = 0$. Still one sees that the convergence rate is $O(\epsilon)$, as shown in Figure 2.

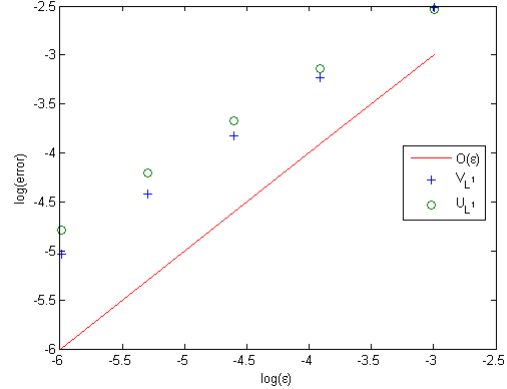
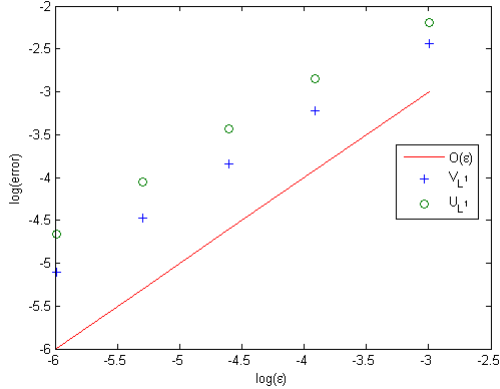


Figure 1: convergence rate for Example 1

Figure 2: convergence rate for Example 2

Next we will compare our domain decomposition method using the underresolved mesh with the original relaxation system. Let $\epsilon = 0.002$ be fixed for $x > 0$. The relaxation system is solved by fine mesh ($\Delta x, \Delta t \ll \epsilon$) to serve as the reference solution to (1.1) - (1.5), which are referred to as "analytical" solutions in the Figs 3 - 6.

Example 3. The set up is the same as Example 1. The solutions are plotted at $T = 0.5$. In this case, there is an interface layer in u at $x = 0$, as one can see from Figures 3 and 4. In comparison, one can see that the relaxation system solved with a relatively large mesh size ($\Delta x, \Delta t \gg \epsilon$), referred to as "under-relax" in Figs 3 - 6, gives poor results at the interface which results in larger numerical errors away from the interface. The error becomes smaller if the mesh size is reduced (yet still underresolved). On the other hand, the domain decomposition method gives more accurate approximation even when the mesh size is large ($\Delta x, \Delta t \gg \epsilon$).

Example 4. The set up is the same as Example 2. The results at $T = 0.6$ are plotted in Figure 5 and 6. Similar to Example 3, one can find that the relaxation system is better approximated with the decreasing of the mesh size, while the domain decomposition method gives good approximation even with the large mesh size compared to ϵ .

Example 5. Let $f(u^\epsilon)$ be the same as in Example 2, but consider the Riemann initial data:

$$u^\epsilon(x, 0) = \begin{cases} -1, & \text{if } -1 \leq x \leq -0.2; \\ 1, & \text{if } -0.2 < x \leq 1. \end{cases}$$

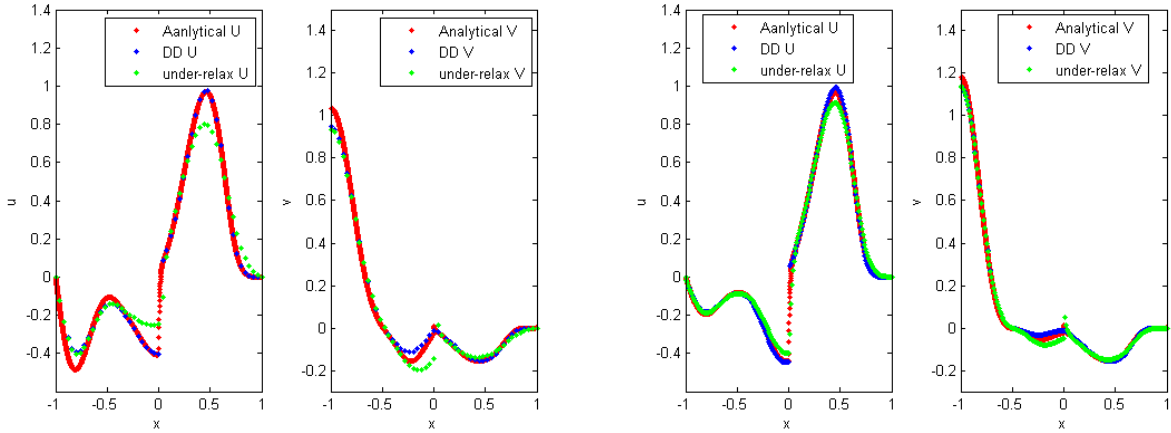


Figure 3: Example 3, $\Delta x = 0.04$, $\Delta t = 0.02$. Figure 4: Example 3, $\Delta x = 0.01$, $\Delta t = 0.005$.

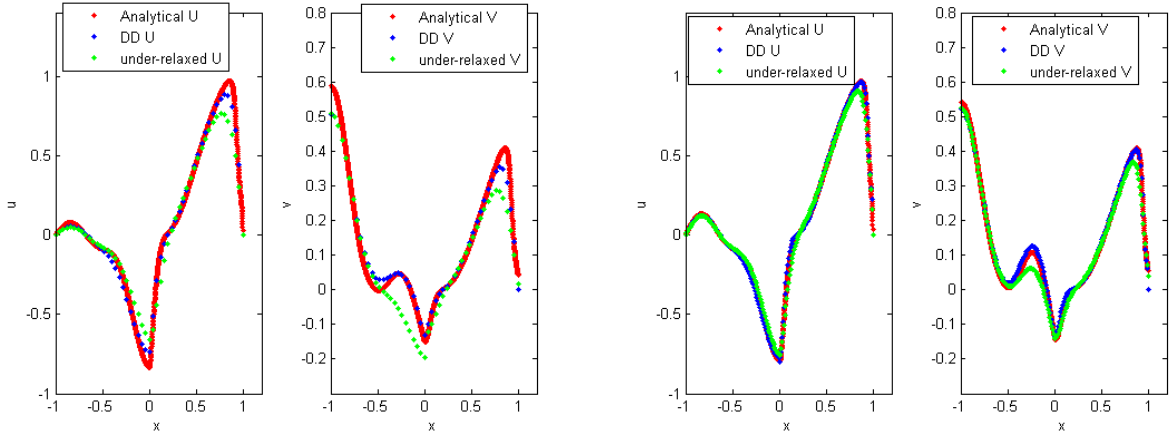


Figure 5: Example 4, $\Delta x = 0.04$, $\Delta t = 0.02$. Figure 6: Example 4, $\Delta x = 0.01$, $\Delta t = 0.005$.

In this case a contact discontinuity formed at the left hand side will propagate across the interface to the right. Let $\Delta x = 0.02$, $\Delta t = 0.01$. From Figure 7, one will see that, before the contact discontinuity passes through the interface, there is not much difference between the under-resolved solution of the relaxation system and the domain decomposition solution, but after that the domain decomposition method has an obvious advantage in producing more accurate results. The results are given at different times to show the dynamics of the solution.

Example 6. Let $f(u^\epsilon)$ be the same as in Example 1, and consider the following Riemann initial data:

$$u^\epsilon(x, 0) = \begin{cases} 1, & \text{if } -1 \leq x \leq 0.2; \\ -1, & \text{if } 0.2 < x \leq 1. \end{cases}$$

Here we use $\Delta x = 0.02$ and $\Delta t = 0.01$. In this case, a shock forms at the right region and propagates to the left region. From Figure 8, one can see that, when the shock crosses the interface, the domain decomposition method gives spurious solution at the interface. This

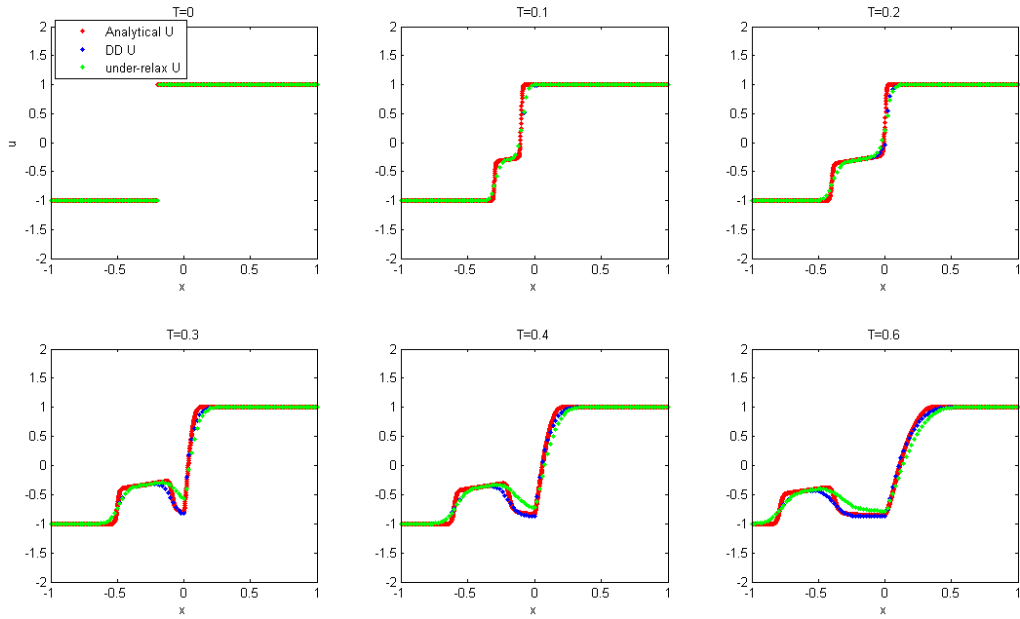


Figure 7: Example 5, a contact discontinuity passing through the interface.

is because our interface layer analysis assumes that the solution is smooth, yet here the interaction between the interface layer and shock complicates the problem, thus our domain decomposition system may not be valid here.

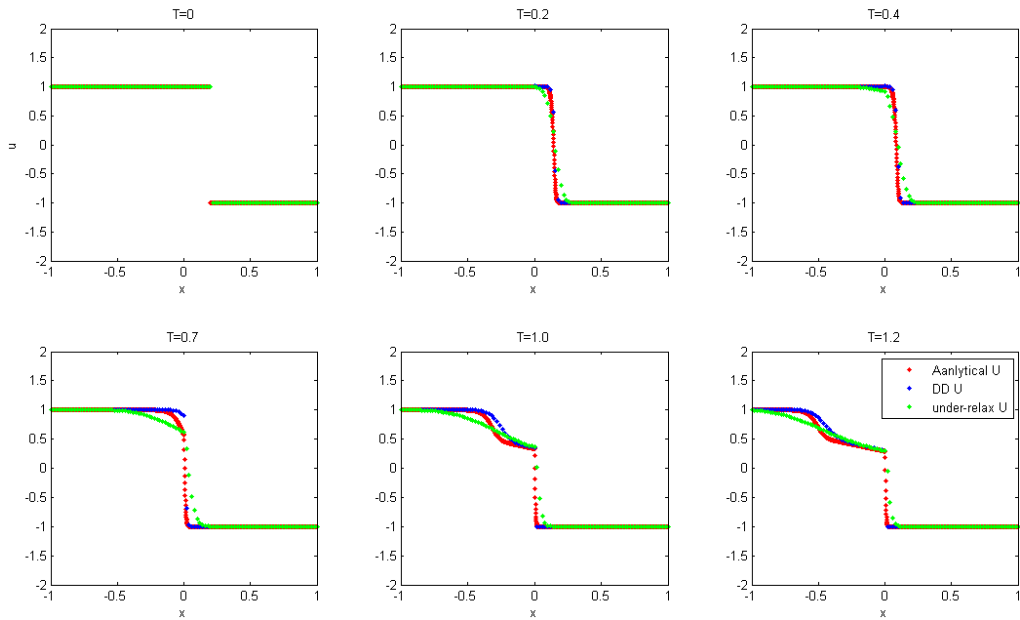


Figure 8: Example 6, a shock from the right region passing through the interface.

Example 7. Let $f(u^\epsilon)$ be the same as in Example 1, and consider the following Riemann initial data:

$$u^\epsilon(x, 0) = \begin{cases} -1, & \text{if } -1 \leq x \leq 0.2; \\ 1, & \text{if } 0.2 < x \leq 1. \end{cases}$$

With this initial data, a rarefaction wave forms in the right region, and propagates across the interface to the left. We still let $\Delta x = 0.02$ and $\Delta t = 0.01$, and the solutions are plotted at different times in Figure 9. One can see that, unlike a shock, the domain decomposition method gives a good approximation when the rarefaction wave crosses the interface.

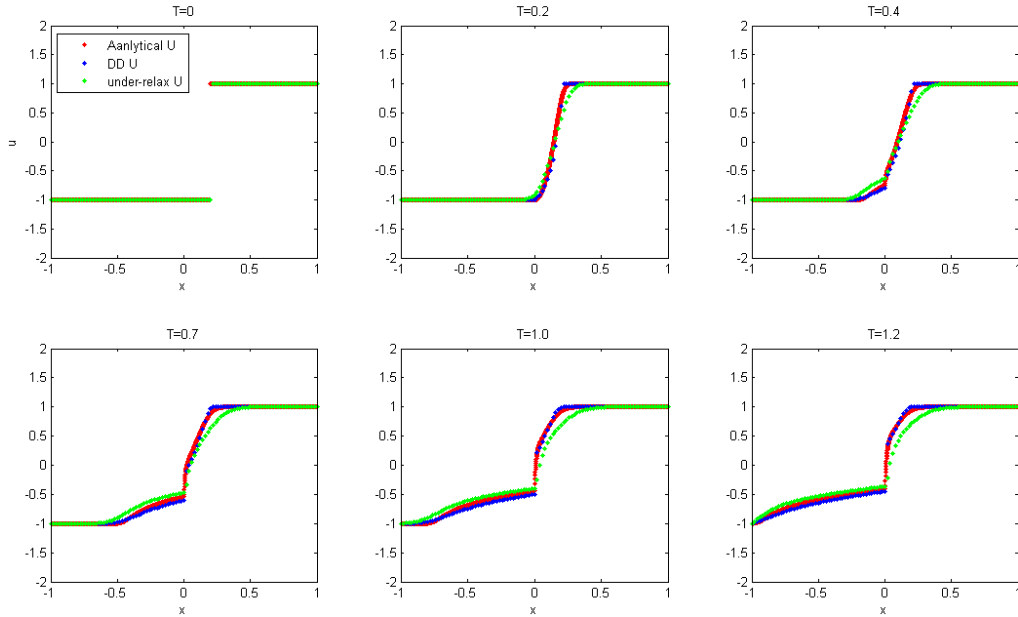


Figure 9: Example 7, rarefaction wave

7 Conclusion

In this paper, a domain decomposition method is presented and analyzed on a semilinear hyperbolic system with multiple relaxation times. In the region where the relaxation time is small, an asymptotic equilibrium equation is used for computational efficiency which is coupled with the original relaxation system on the other part of the region through an interface condition. A rigorous analysis establishes the well-posedness and error estimate in terms of the relaxation time on this domain decomposition method, and numerical results are presented to study the performance of this method.

This is a prototype model for the more general coupling of kinetic and hydrodynamic equations which are competitive multiscale computational methods using multi-physics, thus a deep mathematical understanding of this simpler model problem will shed light on the more general physical problems.

There are still remaining problems to be studied. Among them we mention the problem of shock passing through the interface, nonlinear hyperbolic systems with relaxation, and the error estimate on the numerical schemes based on such a domain decomposition method.

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References

- [1] G. Bal, Y. Maday, *Coupling of transport and diffusion models in linear transport theory*, Math. Model. Numer. Anal. 36, no. 1, 69 - 86, 2002.
- [2] S. Bianchini, *Hyperbolic limit of the Jin-Xin relaxation model*, Comm. Pure Applied Math. 59, 688-753, 2006.
- [3] J. F. Bourgat, P. Le Tallec, B. Perthame, Y. Qiu, *Coupling Boltzmann and Euler equations without overlapping, in domain decomposition methods in science and engineering (Como, 1992)*, Contemp. Math. **157**, Amer. Math. Soc. Providence, RI, 377 - 398, 1994.
- [4] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*, Oxford University Press, 2003.
- [5] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer-Verlag, New York, 1988.
- [6] A. Chalabi, D. Seghir, *Convergence of relaxation schemes for initial boundary value problems for conservation laws*, Computers and Mathematics with Applications 43, no. 8 - 9, 1079 - 1093, 2002.
- [7] G.Q. Chen, C.D. Levermore and T.P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. 47, 787-830, 1994.
- [8] P. Degond, G. Dimarco and L. Mieussens, *A Multiscale Kinetic-Fluid Solver with Dynamic Localization of Kinetic Effects*, J. Comput. Phys., 229, 4907-4933, 2010.
- [9] P. Degond, S. Jin, *A smooth transition model between kinetic and diffusion equations*, SIAM J. Num. Anal. 42, 2671 - 2687, 2005
- [10] P. Degond, S. Jin and L. Mieussens, *A Smooth Transition Model Between Kinetic and Hydrodynamic Equations*, J. Comp. Phys. 209, 665-694, 2005.
- [11] P. Degond, J.-G. Liu and L. Mieussens, *Macroscopic fluid modes with localized kinetic upscaling effects* Multiscale Model. Simul. 5, 695-1043, 2006.
- [12] P. Degond, C. Schmeiser, *Kinetic boundary layers and fluid-kinetic coupling in semi-conductors*, Transport Theory Statist. Phys. 28, no. 1, 31 - 55, 1999.
- [13] F. Golse, S. Jin, C.D. Levermore, *A domain decomposition analysis for a two-scale linear transport problem*, Math. Model Num. Anal. 37, no. 6, 869 - 892, 2003.

- [14] R. L. Higdon, *Initial-boundary value problems for linear hyperbolic systems*, SIAM Review, vol 28, no. 2, 177 - 217, 1986.
- [15] S. Jin, Z. P. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. 48, no.3, 235 - 276, 1995.
- [16] H. O. Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math. 23, 277 - 298, 1970.
- [17] A. Klar, *Convergence of alternating domain decomposition schemes for kinetic and aerodynamic equations*, Math. Methods Appl. Sci.18, no. 8, 649 - 670, 1995.
- [18] A. Klar, H. Neunzert, J. Struckmeier, *Transition from kinetic theory to macroscopic fluid equations: a problem for domain decomposition and a source for new algorithm*, Transp. Theory and Stat. Phys. 29, 93 - 106, 2000.
- [19] S. N. Kruzkov, *First order quasilinear equations with several independent variables*, Mat. Sb. (N.S.) 81(123), 228 - 255, 1970.
- [20] L. D. Landau, E. M. Lifschitz, *Statistical Physics*, Elsevier (Singapore) Pte Ltd, 1980.
- [21] R. J. Leveque, Z. Li, *The immersed interface method for elliptic equations with discontinuous coefficients and singular sources*, SIAM J. Numer. Anal. 31, 1019-1044, 1994
- [22] C. D. Levermore, *Moment closure hierarchies for kinetic theories*, J. Statist. Phys. 83, no. 5 - 6, 1021 - 1065, 1996.
- [23] A. Majda, S. Osher, *Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math. 28, no. 7 - 8, pp. 607 - 675, 1975.
- [24] R. Natalini, *Convergence to equilibrium for the relaxation approximation of conservation laws*, Comm. Pure Appl. Math. 49, no. 8, 795 - 823, 1996.
- [25] R. Natalini, *Recent mathematical results on hyperbolic relaxation problems*, Analysis of Systems of Conservation Laws (Aachen 1997), Chapman Hall/CRC, Boca Raton, 128 - 198, 1999.
- [26] R. Natalini, B. Hanouzet, *Weakly coupled system of quasilinear hyperbolic equations*, Differential Integral Equations 9, no. 6, 1279 - 1292, 1996.
- [27] J. V. Ralston, *Note on a paper of Kreiss*, Comm. Pure Appl. Math. 24, pp. 759 - 762, 1971.
- [28] Z. H. Teng, *First-order L^1 convergence for relaxation approximations to conservation laws*, Comm. Pure Appl. Math., Vol. LI, 0875 - 0895, 1998.
- [29] M. Tidriri, *New models for the solution of intermediate regimes in transport theory and radiative transfer: existence theory, positivity, asymptotic analysis, and approximations*, J. Stat. Phys. 104, 291 - 325, 2001.

- [30] W.G. Vincenti, C.H. Kruger, *Introduction to Physical Gas Dynamics*, Wiley, New York, 1965.
- [31] W. C. Wang, Z. P. Xin, *Asymptotic limit of initial boundary value problems for conservation laws with relaxational extensions*, Comm. Pure Appl. Math., Vol. LI, 0505 - 0535, 1998.
- [32] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.
- [33] Z. P. Xin, W. Q. Xu, *Stiff well-posedness and asymptotic convergence for a class of linear relaxation systems in a quarter plane*, Journal of Differential Equations 167, 388 - 437, 2000.
- [34] Z. P. Xin, W. Q. Xu, *Initial-boundary value problem to systems of conservation laws with relaxation*, Quarterly of applied mathematics 60, no.2, 251 - 281, 2002.
- [35] W. Q. Xu, *Boundary Conditions for Multi-dimensional Hyperbolic Relaxation Problems*, Discrete Contin. Dyn. Syst., 916 - 925, 2003.
- [36] X. Yang, F. Golse, Z. Y. Huang, S. Jin, *Numerical study of a domain decomposition method for a two-scale linear transport equation*, Netw. Heterog. Media 1, no. 1, 143 - 166, 2006.