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# HYPOCOERCIVITY AND UNIFORM REGULARITY FOR THE VLASOV–POISSON–FOKKER–PLANCK SYSTEM WITH UNCERTAINTY AND MULTIPLE SCALES*

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**Abstract.** We study the Vlasov–Poisson–Fokker–Planck system with uncertainty and multiple scales. Here the uncertainty, modeled by random variables, enters the solution through initial data, while the multiple scales lead the system to its high-field or parabolic regimes. With the help of proper Lyapunov-type inequalities, under some mild conditions on the initial data, the regularity of the solution in the random space, as well as exponential decay of the solution to the global Maxwellian, are established under Sobolev norms, which are *uniform* in terms of the scaling parameters. These are the first hypocoercivity results for a nonlinear kinetic system with random input, which are important for the understanding of the sensitivity of the system under random perturbations, and for the establishment of spectral convergence of popular numerical methods for uncertainty quantification based on (spectrally accurate) polynomial chaos expansions.

**Key words.** Vlasov–Poisson–Fokker–Planck system, uncertainty quantification, random input, hypocoercivity

**AMS subject classifications.** 35Q40, 35Q83, 35Q84

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In this paper we are interested in the Vlasov–Poisson–Fokker–Planck (VPFP) system with random inputs. The VPFP system describes the Brownian motion of a large system of particles in a surrounding bath. One of the applications is in electrostatic plasma, in which one considers the interactions between the electrons and a surrounding bath via the Coulomb force [4]. The uncertainty in a kinetic equation can arise from the initial and boundary data, the forcing term, collisional kernels, etc., due to modeling and measurement errors. In this paper we will mainly focus on the case in which the initial data contain random inputs, modeled by random variables with given probability density functions. The goal is to understand the regularity of the solution in the random space, as well as its long-time behavior. Such a study is important in order to understand the *sensitivity* of the system under random perturbations. It is also the basis to study the convergence of numerical schemes for such problems, for example, the popular methods for uncertainty quantification, such as polynomial chaos expansion based stochastic Galerkin or stochastic collocation methods [10, 13, 30, 29], which enjoy a spectral convergence, if the solution has the desired regularity in the random space.

While there have been many developments in the regularity of the solution to elliptic or parabolic equations with uncertainties [2, 5, 6], such study has been scarce

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for hyperbolic type equations [11, 27, 31, 3, 7] because of the poor regularity of the solution. The uncertainty quantification, while popular in many types of partial differential equations, has seldom been studied for kinetic equations until very recently [33, 14, 21, 20]. Typically kinetic equations possess multiple scales, leading to various different asymptotic regimes, demanding carefully designed numerical methods to handle different asymptotic behavior of the equations. For deterministic kinetic equations, one efficient multiscale paradigm is the *asymptotic-preserving* schemes, which mimic the asymptotic transitions from kinetic equations to their diffusion or hydrodynamic limits in the numerically discrete space [16, 17]. This concept was extended to random kinetic equations in [21], in the framework of *stochastic asymptotic-preserving* methods. Convergence study of these methods clearly requires the understanding of the regularity of the solution. Moreover, the correct asymptotic behavior of the numerical methods in various asymptotic regimes also requires the understanding of the long time behavior, and how the decay rates depend on the small scaling parameters. For the linear transport equation with random isotropic scattering in the diffusive regime, such regularity and asymptotic behavior were first studied in [18], in which the regularity of the solution was established, as well as its exponential decay toward the local equilibrium, all *uniformly* in the mean free path (or Knudsen number). Uniform regularity for the semiconductor Boltzmann equation, in which the scattering is anisotropic and random, was established in [19]. Called *hypocoercivity* by Villani [28], the property of uniform exponential decay toward the global equilibrium [8] was further explored in [22] for general linear kinetic equations with uncertainty. So far there has been no work on hypocoercivity for nonlinear kinetic equations with uncertainty with uniform (in small scaling parameters) estimate. The purpose of this paper is to conduct such a study for the nonlinear VPFP system with random initial input.

Depending on different scales, the VPFP system possesses two distinguished asymptotic limits, the high field limit and the parabolic limit. We will treat these different scalings in a unified framework. With the help of proper Lyapunov-type inequalities, we first develop two energy estimates for the microscopic (VPFP) and macroscopic (limiting) systems, which allows us to obtain the *uniform*—in terms of the scaling parameters—regularity in the random space of the perturbative solution of the nonlinear VPFP system near global Maxwellian. Under some mild conditions on the initial data, we found that the solution will decay exponentially to the global Maxwellian in a rate *independent* of the small scaling parameter. Our results also reveal that the initial random perturbation will die out exponentially in time, uniformly in the scaling parameter, and thus the solution is insensitive to the initial random perturbation, in all asymptotic regimes.

For the deterministic VPFP system, the regularity and convergence toward the global Maxwellian or asymptotic limits were conducted in, for example, [1, 9, 12, 23, 26, 15, 25]. Our energy estimates rely on the hypocoercivity results of [9] and on the energy estimates in [15] with suitable modification to effectively separate the microscopic and macroscopic scales in order to get better estimates in the asymptotic regimes. When the small scaling parameters are involved, which was not considered in [15], it is crucial to get rid of the bad dependence on these parameters in the initial condition and rate of convergence to the global equilibrium. Therefore we have not only extended the regularity results to the random space but also improved the micro-macro energy estimates by separating the microscopic energy from the macroscopic energy suitably, so when small scales are involved, we can get a uniform convergence rate toward the global equilibrium, and a milder initial condition at the same time. As a result, we get an exponential decay of the perturbative solution—independent of the small parameter—under some mild initial condition, which leads to a uniform regularity of the solution in random space for both high field and parabolic limits.

In this paper, for clarity of the presentation and notation, we carry out all analysis in one space dimension for all independent variables. Its extension to higher dimension in $x$, $v$, and $z$ is straightforward with some changes of the constants (see [32], for example).

This paper is organized as follows. Section 1 gives an introduction of the VPFP system with uncertainty and its two different asymptotic regimes. The main results are stated in section 2. Then in sections 3 and 4 we prove the energy estimates from microscopic and macroscopic systems, respectively. The uniform regularity of the perturbative solution is obtained in section 5.

## 1. The VPFP system with uncertainty and asymptotic scalings.

**1.1. The VPFP system with uncertainty.** In the dimensionless VPFP system with uncertainty, the time evolution of particle density distribution function $f(t, \mathbf{x}, \mathbf{v}, \mathbf{z})$ under the action of an electrical potential $\phi(t, x, z)$ satisfies

$$\begin{cases} \partial_t f + \frac{1}{\delta} v \partial_x f - \frac{1}{\epsilon} \partial_t x \phi \partial_v f = \frac{1}{\delta\epsilon} \mathcal{F} f, \\ -\partial_x^2 \phi = \rho - 1, \quad t > 0, \quad x, v \in \mathbb{R}, z \in I_z \subseteq \mathbb{R}, \end{cases} \tag{1.1}$$

with initial data:

$$f(0, x, v, z) = f^0(x, v, z), \quad x, v \in \mathbb{R}, z \in I_z \subseteq \mathbb{R}. \tag{1.2}$$

The distribution function $f(t, x, v, z)$ depends on time $t$, position $x$, velocity $v$, and random variable $z$. $\phi(t, x, z)$ is a self-consistent electrical potential and $\rho(t, x, z)$ is the density function defined as

$$\rho(t, x, z) = \int_{\mathbb{R}} f(t, x, v, z) dv. \tag{1.3}$$

In the VPFP system, $\mathcal{F}$ is a collision operator, describing the Brownian motion of the particles, which reads

$$\mathcal{F} f = \partial_v \left( M \partial_v \left( \frac{f}{M} \right) \right), \tag{1.4}$$

where $M$ is the *global equilibrium* or *global Maxwellian*,

$$M = \frac{1}{\sqrt{2\pi}} e^{-\frac{|v|^2}{2}}. \tag{1.5}$$

In the dimensionless system, $\delta$ is the reciprocal of the scaled thermal velocity, and $\epsilon$ represents the scaled thermal mean free path [26]. There are two different regimes for this system. One is the *high field regime*, where $\delta = 1$. As $\epsilon$ goes to zero, $f$ goes to the local Maxwellian $M_{\text{local}} = \frac{\rho}{\sqrt{2\pi}} e^{-\frac{|v - \partial_x \phi|^2}{2}}$, and the VPFP system converges to a hyperbolic limit [1, 12, 23]:

$$\begin{cases} \partial_t \rho + \partial_x (\rho \partial_x \phi) = 0, \\ -\partial_x^2 \phi = \rho - 1. \end{cases} \tag{1.6}$$

Another regime is the *parabolic regime*, where $\delta = \epsilon$. When $\epsilon$ goes to zero, $f$ goes to the global Maxwellian $M$, and the VPFP system converges to a parabolic limit [24]:

$$\begin{cases} \partial_t \rho - \partial_x (\partial_x \rho - \rho \partial_x \phi) = 0, \\ -\partial_x^2 \phi = \rho - 1. \end{cases} \tag{1.7}$$

In this paper, we are going to study both regimes together.

In the VPFP system with uncertainty, the random variable $z$ is in a properly defined probability space $(\Sigma, \mathbb{A}, \mathbb{P})$, whose event space is $\Sigma$ and is equipped with $\sigma$-algebra $\mathcal{A}$ and probability measure $\mathbb{P}$. Define $\pi(z) : I_z \longrightarrow \mathbb{R}^+$ as the probability density function of the random variable $z(\omega)$, $\omega \in \Sigma$. So one has a corresponding $L^2$ space in the measure of

$$d\mu = d\mu(x, v, z) = \pi(z) dx dv dz. \tag{1.8}$$

With this measure, one has the corresponding Hilbert space with the following inner product and norms:

$$\langle f, g\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{I_z} fg \, d\mu(x, v, z), \quad \text{or} \quad \langle \rho, j\rangle = \int_{\mathbb{R}} \int_{I_z} \rho j \, d\mu(x, z), \tag{1.9}$$

$$\text{with norm } \|f\|^2 = \langle f, f\rangle . \tag{1.10}$$

For the convenience of the readers, we list some elementary calculation on $M$ which will be used in later calculations:

$$\partial_v M = -vM, \quad \partial_v(\sqrt{M}) = -\frac{v}{2}\sqrt{M}; \tag{1.11}$$

$$\int_{\mathbb{R}} v^a \sqrt{M} \, dv = \int_{\mathbb{R}} v^a M \, dv = 0 \quad \text{for any odd } a; \tag{1.12}$$

$$\int_{\mathbb{R}} M \, dv = 1, \quad \int_{\mathbb{R}} v^2 M \, dv = 1, \quad \int_{\mathbb{R}} v^4 M dv = 3; \tag{1.13}$$

$$\int_{\mathbb{R}} |v|^3 M \, dv = \frac{4}{\sqrt{2\pi}} \le 2, \quad \int_{\mathbb{R}} \left(\partial_v (v\sqrt{M})\right)^2 \, dv = \frac{3}{4}. \tag{1.14}$$

**1.2. Notation.** Without loss of generality, we assume $\epsilon < 1$. In order to get the convergence rate of the solution to the global equilibrium, we define

$$h = \frac{f - M}{\sqrt{M}}, \quad \sigma = \int_{\mathbb{R}} h\sqrt{M} \, dv, \quad u = \int_{\mathbb{R}} h \, v\sqrt{M} \, dv, \tag{1.15}$$

where $h$ is the fluctuation around the equilibrium, $\sigma$ is the density fluctuation, and $u$ is the velocity fluctuation. Then the microscopic quantity $h$ satisfies

$$\begin{cases} \epsilon\delta \underbrace{\partial_t h}_{I} + \epsilon \underbrace{v\partial_x h}_{II} - \delta \underbrace{\partial_x \phi \partial_v h}_{III} + \delta \underbrace{\frac{v}{2}\partial_x \phi h}_{IV} + \delta \underbrace{v\sqrt{M}\partial_x \phi}_{V} = \underbrace{\mathcal{L}h}_{VI}, & (1.16) \\ \partial_x^2 \phi = -\sigma, & (1.17) \end{cases}$$

where $\mathcal{L}$ is the so-called linearized Fokker–Planck operator,

$$\mathcal{L}h = \frac{1}{\sqrt{M}} \mathcal{F}\left(M + \sqrt{M}h\right) = \frac{1}{\sqrt{M}} \partial_v \left(M \partial_v \left(\frac{h}{\sqrt{M}}\right)\right). \tag{1.18}$$

We give each term a number, in order to make it clear where the term comes from originally when doing the energy estimates later.

We further introduce projections onto $\sqrt{M}$ and $v\sqrt{M}$,

$$\Pi_1 h = \sigma\sqrt{M}, \quad \Pi_2 h = uv\sqrt{M}, \quad \Pi h = \Pi_1 h + \Pi_2 h. \tag{1.19}$$

These projections have the following properties:

- $\partial_x\partial_z\Pi = \Pi\partial_x\partial_z$.
- Due to the mutual orthogonality of $\Pi_1 h$, $\Pi_2 h$, $(1-\Pi)h$ in $L_v^2$ space, let $\partial^{\mathbf{k}} = \partial_z^{k_1}\partial_x^{k_2}$,

$$\begin{aligned}\left\|\partial^{\mathbf{k}} h\right\|_{L_v^2}^2 &= \left\|\Pi_1\partial^{\mathbf{k}} h\right\|_{L_v^2}^2 + \left\|\Pi_2\partial^{\mathbf{k}} h\right\|_{L_v^2}^2 + \left\|(1-\Pi)\partial^{\mathbf{k}} h\right\|_{L_v^2}^2 \\ &= \left\|\partial^{\mathbf{k}}\sigma\right\|_{L_v^2}^2 + \left\|\partial^{\mathbf{k}} u\right\|_{L_v^2}^2 + \left\|(1-\Pi)\partial^{\mathbf{k}} h\right\|_{L_v^2}^2, \end{aligned} \tag{1.20}$$

which also implies

$$\left\|\partial^{\mathbf{k}}\sigma\right\|_{L_v^2}, \left\|\partial^{\mathbf{k}} u\right\|_{L_v^2}, \left\|(1-\Pi)\partial^{\mathbf{k}} h\right\|_{L_v^2} \le \left\|\partial^{\mathbf{k}} h\right\|_{L_v^2}. \tag{1.21}$$

Multiplying $\sqrt{M}$ and $v\sqrt{M}$ to (1.16), and integrating the equation over $v$, respectively, then one has the equations for the macroscopic quantities $\sigma$ and $u$,

$$\begin{cases} \delta\partial_t\sigma + \partial_x u = 0, & (1.22)\\ \epsilon\delta\underbrace{\partial_t u}_{I} + \epsilon\underbrace{\partial_x\sigma}_{II.1} + \epsilon\underbrace{\int v^2\sqrt{M}(1-\Pi)\partial_x h dv}_{II.2} + \delta\underbrace{\partial_x\phi\sigma}_{III} + \underbrace{u}_{VI} + \delta\underbrace{\partial_x\phi}_{V} = 0. & (1.23)\end{cases}$$

We call (1.16)–(1.17) the *microscopic system* and (1.22)–(1.23) the *macroscopic system*. Note (1.22)–(1.23) is not a closed system since it contains the microscopic quantities $h$.

We also define the following norms and energies:

- Norms:
  - $|h|^2 = \int_{\mathbb{R}} h^2\, dv$, $\|f\|$ and $\langle\cdot,\cdot\rangle$ is defined in (1.10).
  - $|h|_\nu^2 = \int_{\mathbb{R}}(1+|v|^2)h^2 + (\partial_v h)^2\, dv$, $\|h\|_\nu^2 = \int_{\mathbb{R}\times I_z} |h|_\nu\, d\mu(x,z)$,
  - $\|f\|_{H_z^m}^2 = \sum_{l=0}^m \left\|\partial_z^l f\right\|^2$, $\|f\|_{H_z^m(H_x^n)}^2 = \sum_{i\le n}\left\|\partial_x^i f\right\|_{H_z^m}^2$.
- Energy terms:
  - $E_h^{m,i} = \left\|\partial_x^i h\right\|_{H_z^m}^2$, $E_h^m = \|h\|_{H_z^m(H_x^1)} = E_h^{m,0} + E_h^{m,1}$,
  - $E_\phi^{m,i} = \left\|\partial_x^i\partial_x\phi\right\|_{H_z^m}^2$, $E_\phi^m = \|h\|_{H_z^m(H_x^1)} = E_\phi^{m,0} + E_\phi^{m,1}$.
- Dissipation terms:
  - $D_h^{m,i} = \sum_{l\le m}\left\|\partial_z^l\partial_x^i(1-\Pi)h\right\|_\nu^2$, $D_h^m = D_h^{m,0} + D_h^{m,1}$,
  - $D_\phi^{m,i} = \left\|\partial_x^i\partial_x\phi\right\|_{H_z^m}^2$, $D_\phi^m = D_\phi^{m,0} + D_\phi^{m,1}$;
  - $D_u^{m,i} = \left\|\partial_x^i u\right\|_{H_z^m}$, $D_u^m = D_u^{m,0} + D_u^{m,1}$;
  - $D_\sigma^{m,i} = \left\|\partial_x^i\sigma\right\|_{H_z^m}$, $D_\sigma^m = D_\sigma^{m,0} + D_\sigma^{m,1}$.

**2. Main results.** To get the regularity of the solution in the Hilbert space, one usually uses energy estimates. In order to balance the nonlinear term $\partial_x\phi\partial_v f$ and get a regularity independent of the small parameter $\epsilon$ (or depending on $\epsilon$ in a good way), one needs the coercivity property from the collision operator. The coercivity property one uses most commonly is

$$-\int_{\mathbb{R}} h\mathcal{L}h\, dv \ge C\,|(1-\Pi_1)h|^2; \tag{2.1}$$

see [8, 28]. However, this is not enough for the nonlinear case. We need stronger coercivity as listed in the following proposition; see [9] for the deterministic case. Here we extend the coercivity into random space.

PROPOSITION 2.1. *For $\mathcal{L}$ defined in* (1.4)*,*

(a) $-\langle \mathcal{L}h, h\rangle = -\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle + \|u\|^2$;

(b) $-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle = \|\partial_v(1-\Pi)h\|^2 + \frac{1}{4}\|v(1-\Pi)h\|^2 - \frac{1}{2}\|(1-\Pi)h\|^2$;

(c) $-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle \geq \|(1-\Pi)h\|^2$;

(d) *there exists a constant $\lambda_0 > 0$ such that the following hypocoercivity holds:*

$$-\langle \mathcal{L}h, h\rangle \geq \lambda_0 \|(1-\Pi)h\|_\nu^2 + \|u\|^2, \tag{2.2}$$

*and the largest $\lambda_0 = \frac{1}{7}$ in one dimension.*

*Proof.* Here we prove only (d); see [9] for (a), (b), (c). Since

$$\begin{aligned}
&-\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle \\
&\geq -a\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle - (1-a)\langle \mathcal{L}(1-\Pi)h, (1-\Pi)h\rangle \\
&\geq a\|\partial_v(1-\Pi)h\|^2 + \frac{a}{4}\|v(1-\Pi)h\|^2 - \frac{a}{2}\|(1-\Pi)h\|^2 + (1-a)\|(1-\Pi)h\|^2 \\
&\geq \min_{0<a<1}\left\{a, \frac{a}{4}, \left(1-\frac{3}{2}a\right)\right\}\|(1-\Pi)h\|_\nu^2,
\end{aligned} \tag{2.3}$$

for $a$ to be determined later, where the second inequality is according to (b) and (c). Then the largest $\lambda_0$ one can get is when $a = \frac{4}{7}$, $\lambda_0 = \frac{1}{7}$. Therefore,

$$-\langle \mathcal{L}h, h\rangle \geq \lambda_0 \|(1-\Pi)h\|_\nu^2 + \|u\|^2. \tag{2.4}$$

□

Before we go into technique details, we first summarize the main strategy of this paper here, which is mainly based on [15]. We omit $\epsilon$, $\delta$ to see the main structure of energy estimates first. We want to use energy estimates to analyze the energy $E^m = E_h^m + E_\phi^m$; the goal here is to obtain a Lyapunov-type inequality like

$$\partial_t E^m + D^m \leq \sqrt{E^m} D^m, \tag{2.5}$$

so that one can control the initial data to get a uniform regularity. However, if one only does energy estimates for (1.16)–(1.17), the dissipation from the linearized Fokker–Planck operator, $D_h^m + D_u^m$, cannot bound the nonlinear term $\sqrt{E_h^m + E_\phi^m}(D_h^m + D_u^m + D_\sigma^m + D_\phi^m)$. So we involve the microscopic system (1.22)–(1.23), where the dissipation terms $D_\sigma^m + D_\phi^m$ come from $II.1$ and $V$ in (1.22). Combining the microscopic and macroscopic energy estimates, one ends up with new energy estimates,

$$\partial_t \hat{E}^m + D^m \leq \sqrt{\hat{E}^m} D^m, \tag{2.6}$$

where $\hat{E}^m \sim E^m$, and $D^m = D_h^m + D_u^m + D_\sigma^m + D_\phi^m$, so the nonlinearity can be fully controlled by the dissipation terms, which gives what we want.

With $\epsilon$ and $\delta$ involved, one needs to bound the nonlinear term more carefully (see Lemma 2.7), which is the key difference from [15]. See Remark 5.1 for the importance of these careful estimates for the nonlinear term.

Based on the coercivity (2.2), we have the following two estimates for the microscopic and macroscopic systems, respectively.

LEMMA 2.2. *The solution to system* (1.16)–(1.17) *satisfies the following estimates for any* $m \geq 1$*:*

$$\begin{aligned} &\frac{1}{2}\partial_t \left[E_h^m + \frac{\delta}{\epsilon}E_\phi^m\right] + \frac{\lambda_0}{\delta\epsilon}D_h^m + \frac{1}{\delta\epsilon}D_u^m \\ &\leq \frac{AC_1^2}{a\epsilon}\sqrt{E_h^m}\left(3D_u^m + 2D_h^m\right) + \frac{2AC_1^2}{\epsilon}\sqrt{E_\phi^m}\left(\left(4 + \frac{1}{a}\right)D_u^m + 4D_h^m\right) \\ &\quad + \frac{aAC_1^2}{\epsilon}\sqrt{E_h^m}D_\phi^m + \frac{aAC_1^2}{\epsilon}\sqrt{E_\phi^m}D_\sigma^m \end{aligned} \tag{2.7}$$

*and*

$$\begin{aligned} &\partial_t \left[\sum_{l=0}^{m-1}\left\langle \partial_z^l u, \partial_z^l \partial_x \phi\right\rangle + \sum_{l=0}^{m}\left\langle \partial_z^l \partial_x u, \partial_z^l \partial_x^2 \phi\right\rangle + \frac{1}{2\epsilon}E_\phi^m\right] + \frac{1}{2\delta}D_\sigma^m + \frac{1}{\epsilon}D_\phi^m \\ &\leq \frac{1}{\delta}D_u^m + \frac{1}{2\delta}D_h^m + \frac{AC_1^2}{\epsilon}\sqrt{E_\phi^m}\left(3D_\sigma^m + 2D_\phi^m\right), \end{aligned} \tag{2.8}$$

*where*

$$A = 2\sqrt{m+1}\binom{m}{[\frac{m}{2}]} + \sqrt{m+1} \tag{2.9}$$

*is a constant only depending on* $m$ *and* $[m/2]$ *is the smallest integer larger than or equal to* $\frac{m}{2}$*, and* $C_1$ *is the Sobolev constant in one dimension defined in* (A.3)*.*

If one combines the above two inequalities, the "bad terms" on the right-hand side (RHS) can be controlled by the dissipation terms on the left-hand side (LHS) if the coefficients are carefully balanced. Hence, one can come to the conclusion that the solution exponentially decays to the global equilibrium.

*Remark* 2.3. The main difference between the energy estimates in Lemma 2.2 and the one obtained in [15] is that for both micro and macro systems, we separate the microscopic energy $E_h^m$ from the macroscopic energy $E_\phi^m$ for $D_\phi^m$ and $D_\sigma^m$, which gives us more flexibility to bound the energies, especially when small parameters are involved.

THEOREM 2.4. *For the high field regime* ($\delta = 1$)*, if*

$$E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0) \leq \frac{2\lambda_0^3}{(80AC_1)^2}, \tag{2.10}$$

*then*

$$E_h^m(t) \leq \frac{3}{\lambda_0}e^{-\frac{t}{\epsilon}}\left(E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0)\right), \quad E_\phi^m(t) \leq \frac{3}{\lambda_0}e^{-t}\left(\epsilon^2 E_h^m(0) + E_\phi^m(0)\right). \tag{2.11}$$

*For the parabolic regime* ($\delta = \epsilon$)*, if*

$$E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0) \leq \left(\frac{2\lambda_0^3}{(80AC_1)^2}\right)\frac{1}{\epsilon}, \tag{2.12}$$

*then*

$$E_h^m(t) \leq \frac{3}{\lambda_0}e^{-\frac{t}{\epsilon}}\left(E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0)\right), \quad E_\phi^m(t) \leq \frac{3}{\lambda_0}e^{-t}\left(\epsilon E_h^m(0) + E_\phi^m(0)\right). \tag{2.13}$$

*Here* $A$ *and* $C_1$ *are the same as in Lemma* 2.2*.*

*Remark* 2.5. Basically, Theorem 2.4 implies the following:

(a) For the high field regime, as long as initially the electric field $\partial_x \phi$ is $O(\epsilon)$ small, and the initial data $f$ is suitably bounded by (2.10), then the solution will converge to the global equilibrium exponentially, *uniformly in* $\epsilon$.

(b) For the parabolic regime, the initial conditions on both $f$ and $\partial_x \phi$ are independent of $\epsilon$. Furthermore, when $\epsilon$ become smaller, $f$ don't need to be near Maxwellian any more for the solution to converge to the global equilibrium exponentially.

(c) If one directly applies the conclusion of [15], then for the high field regime, $E_h^m(0)$ and $E_\phi^m(0)$ need to be $O(\epsilon)$ and $O(\epsilon^3)$ initially, while for the parabolic regime, $E_h^m(0)$ and $E_\phi^m(0)$ need to be $O(1)$ and $O(\epsilon)$ initially; see Remark 5.1 for details. Our result allows more general initial data for $f$ while keeping the optimal convergence rate at the same time, which is because of the new energy estimates we obtained in Lemma 2.2.

(d) One notices that the initial condition on the electric field for the high field regime is required to be $O(\sqrt{\epsilon})$; this is necessary because the limiting hyperbolic system won't preserve the regularity at a later time if the electric field doesn't vanish. On the other hand, for the parabolic regime, the condition on the electric field is $O(1)$, which is because when $\epsilon \to 0$, the VPFP system goes to a parabolic equation which enjoys better regularity compared to the high field regime.

(e) Notice here that although $\|\sigma(t)\|_{H_z^m}$ decays in time, the mass is still conserved, that is, $\int \sigma(t)dx = \int \sigma(0)dx$ holds for all $t > 0$. It is an interesting question to study the case when this conservation is not true for future research.

(f) Since $f = M + \sqrt{M}h$ and $M$ is the global Maxwellian without randomness, the regularity of the perturbative solution $h$ in random space implies the uniform regularity of the solution $f$. More specifically, one knows the regularity of the initial data in the random space is preserved in time. Furthermore, the bound is independent of the small parameter $\epsilon$.

One notices that the initial condition has a bad dependency on $m$. Actually this can be eliminated by defining a new energy norm. Since the main focus of this paper is uniform regularity in $\epsilon$, we just give a brief proof of the following theorem in the appendix.

THEOREM 2.6. *Define*

$$\left\|\partial_z^l h\right\|_l^2 = \left\|\frac{l+1}{l!}\partial_z^l h\right\|^2, \tag{2.14}$$

$$\tilde{E}_h^{m,i} = \sum_{l\le m}\left\|\partial_x^i \partial_z^l h\right\|_l^2, \quad \tilde{E}_h^m = \tilde{E}_h^{m,0} + \tilde{E}_h^{m,1}, \tag{2.15}$$

$$\tilde{E}_\phi^{m,i} = \sum_{l\le m}\left\|\partial_x^i \partial_z^l \partial_x \phi\right\|_l^2, \quad \tilde{E}_\phi^m = \tilde{E}_\phi^{m,0} + \tilde{E}_\phi^{m,1}. \tag{2.16}$$

*Theorem* 2.4 *still holds for the new energy norms with* $A=8\sqrt{\sum_i^\infty \frac{1}{(i+1)^2}}$.

*Proof.* See Appendix B for a proof. ∎

The proof of the main theorem requires some equalities and inequalities, which are given below.

LEMMA 2.7. *Let $\partial^{\mathbf{k}} = \partial_z^{k_1}\partial_x^{k_2}$, and similar for $\partial^{\mathbf{i}}, \partial^{\mathbf{l}}$:*

(a) $\langle \partial^{\mathbf{k}}\partial_x\phi, v\sqrt{M}\partial^{\mathbf{k}}h\rangle = \frac{\delta}{2}\partial_t \left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2$,

(b) $$\left\langle \partial^{\mathbf{k}}\partial_x\phi\, \partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\, \partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle \le C_1 \left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H_z^1(H_x^1)} \Big(a\left\|\partial^{\mathbf{i}}\sigma\right\|^2 + 2\left\|\partial^{\mathbf{i}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_\nu^2 + \left(2+\frac{1}{a}\right)\left\|\partial^{\mathbf{l}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\Big),$$

(c) $$\left\langle \partial^{\mathbf{k}}\partial_x\phi\, \partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\, \partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle \le C_1^2\sqrt{\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2 + \left\|\partial^{\mathbf{k}}\partial_x^2\phi\right\|^2}\left(a\sum_{i\le 1}\left\|\partial^{\mathbf{i}}\partial_z^i\sigma\right\|^2 + 2\sum_{i\le 1}\left\|\partial^{\mathbf{i}}\partial_z^i u\right\|^2 + 2\sum_{i\le 1}\left\|(1-\Pi)\partial^{\mathbf{i}}\partial_z^i h\right\|_\nu^2 + \left(2+\frac{1}{a}\right)\left\|\partial^{\mathbf{l}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right),$$

(d) $$\left\langle \partial^{\mathbf{k}}\partial_x\phi\, \partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle \le C_1\left\|\partial^{\mathbf{i}}h\right\|_{H_z^1(H_x^1)}\left(\frac{3}{a}\left\|\partial^{\mathbf{l}}u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2 + a\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2\right),$$

(e) $$\left\langle \partial^{\mathbf{k}}\partial_x\phi\, \partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\, \partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle \le C_1^2\sqrt{\left\|\partial^{\mathbf{i}}h\right\|^2 + \left\|\partial^{\mathbf{i}}\partial_x h\right\|^2}\left(a\sum_{i\le 1}\left\|\partial^{\mathbf{k}}\partial_z^i\partial_x\phi\right\|^2 + \frac{3}{a}\left\|\partial^{\mathbf{l}}u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right),$$

(f) $\left\|\partial^{\mathbf{k}}\partial_x\partial_t\phi\right\|^2 \le \frac{1}{\delta^2}\left\|\partial^{\mathbf{k}}u\right\|^2$.

*where $a$ can be any positive constant.*

*Proof.* See Appendix A for a proof. ∎

*Remark* 2.8. Notice that in the inequalities (b) and (c), the dissipations of $u$ and $(1-\Pi)h$ are related to both energies $h$ and $\partial_x\phi$. However, the dissipation of $\sigma$ is only related to the energy of $\partial_x\phi$ through (b), while the dissipation of $\partial_x\phi$ is only related to the energy of $h$ through (c). This is why we can get the separation of the micro and macro energies in Lemma 2.2 for $D_\sigma^m$ and $D_\phi^m$.

## 3. Energy estimates on the microscopic equations.

Now we prove the first part of Lemma 2.2, (2.7).

### 3.1. Energy estimates for $\left\|\partial_z^l h\right\|^2$.

Taking $\partial_z^l$ to (1.16), and multiplying by $\partial_z^l h$, then integrating it over $\mu(x,v,z)$, one has

$$\frac{\epsilon\delta}{2}\partial_t\left\|\partial_z^l h\right\|^2 + \delta\underbrace{\left\langle \partial_z^l\partial_x\phi, v\sqrt{M}\partial_z^l h\right\rangle}_{V}\underbrace{-\left\langle \mathcal{L}\partial_z^l h, \partial_z^l h\right\rangle}_{VI}$$

$$= \delta\sum_{i=0}^{l}\binom{l}{i}\left(\underbrace{\left\langle \partial_z^{l-i}\partial_x\phi\partial_v\partial_z^i h, \partial_z^l h\right\rangle}_{III} - \underbrace{\frac{1}{2}\left\langle v\partial_z^{l-i}\partial_x\phi\, \partial_z^i h, \partial_z^l h\right\rangle}_{IV}\right). \tag{3.1}$$

$V$ and $VI$ are "good terms", since by Lemma 2.7(a) and Proposition 2.1(d),

$$V = \frac{\delta}{2}\partial_t\left\|\partial_z^l\partial_x\phi\right\|^2, \quad VI \ge \lambda_0\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2 + \left\|\partial_z^l u\right\|^2. \tag{3.2}$$

However, $III$ and $IV$ are "bad terms" here, and one wants to control them by the dissapations.

For $i < l$, by Lemma 2.7(c),

$$III - IV \leq C_1 \left\|\partial_z^i h\right\|_{H_z^1(H_x^1)} \left(\frac{3}{a}\left\|\partial_z^l u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2 + a\left\|\partial_z^{l-i}\partial_x\phi\right\|^2\right). \tag{3.3}$$

For $i = l$, by Lemma 2.7(b),

$$\begin{aligned} III - IV \leq{}& aC_1\left\|\partial_x\phi\right\|_{H_z^1(H_x^1)}\left\|\partial_z^l\sigma\right\|^2 + C_1\left\|\partial_x\phi\right\|_{H_z^1(H_x^1)}\left(\left(4+\frac{1}{a}\right)\left\|\partial_z^l u\right\|^2\right.\\ &\left.+4\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2\right). \end{aligned} \tag{3.4}$$

Here if one treats the case of $i = l$ the same as the case of $i < l$, then the largest $i = m$ leads to $\|\partial_z^m h\|_{H_z^1(H_x^1)}$, which cannot be controlled by $\partial_t E_h^m$, so we treat $i = l$ differently from $i < l$. Therefore one has the energy estimate

$$\begin{aligned} &\frac{\delta}{2}\partial_t\left(\epsilon\left\|\partial_z^l h\right\|^2 + \delta\left\|\partial_z^l\partial_x\phi\right\|^2\right) + \lambda_0\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2 + \left\|\partial_z^l u\right\|^2\\ &\leq C_1\delta\sum_{l\neq 1, i=0}^{l-1}\binom{l}{i}\left\|\partial_z^i h\right\|_{H_z^1(H_x^1)}\left(\frac{3}{a}\left\|\partial_z^l u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2 + a\left\|\partial_z^{l-i}\partial_x\phi\right\|^2\right)\\ &\quad + aC_1\delta\sqrt{E_\phi^1}\left\|\partial_z^l\sigma\right\|^2 + C_1\delta\sqrt{E_\phi^1}\left(\left(4+\frac{1}{a}\right)\left\|\partial_z^l u\right\|^2 + 4\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2\right). \end{aligned} \tag{3.5}$$

Summing $l$ from 0 to $m$, one gets

$$\begin{aligned} &\frac{\delta}{2}\partial_t\left[\epsilon E_h^{m,0} + \delta E_\phi^{m,0}\right] + \lambda_0 D_h^{m,0} + D_u^{m,0}\\ &\leq C_1\delta\sum_{l=1}^{m}\sum_{i=0}^{l-1}\binom{l}{i}\left\|\partial_z^i h\right\|_{H_z^1(H_x^1)}\left(\frac{3}{a}\left\|\partial_z^l u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2\right)\\ &\quad + aC_1\delta\sum_{l=1}^{m}\sum_{i=1}^{m}\binom{l}{i}\left\|\partial_z^{l-i} h\right\|_{H_z^1(H_x^1)}\left\|\partial_z^i\partial_x\phi\right\|^2 + aC_1\delta\sqrt{E_\phi^1}D_\sigma^{m,0}\\ &\quad + C_1\delta\sqrt{E_\phi^1}\left(\left(4+\frac{1}{a}\right)D_u^{m,0} + 4D_h^{m,0}\right)\\ &= C_1\delta\sum_{l=1}^{m}\left(\sum_{i=0}^{l-1}\binom{l}{i}\left\|\partial_z^i h\right\|_{H_z^1(H_x^1)}\right)\left(\frac{3}{a}\left\|\partial_z^l u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2\right)\\ &\quad + aC_1\delta\sum_{i=1}^{m}\left(\sum_{l=i}^{m}\binom{l}{i}\left\|\partial_z^{l-i} h\right\|_{H_z^1(H_x^1)}\right)\left\|\partial_z^i\partial_x\phi\right\|^2\\ &\quad + aC_1\delta\sqrt{E_\phi^1}D_\sigma^{m,0} + C_1\delta\sqrt{E_\phi^1}\left(\left(4+\frac{1}{a}\right)D_u^{m,0} + 4D_h^{m,0}\right)\\ &\leq BC_1\delta\sqrt{E_h^m}\left(\frac{3}{a}D_u^{m,0} + \frac{2}{a}D_h^{m,0}\right) + aBC_1\delta\sqrt{E_h^m}D_\phi^{m,0} + aC_1\delta\sqrt{E_\phi^1}D_\sigma^{m,0}\\ &\quad + C_1\delta\sqrt{E_\phi^1}\left(\left(4+\frac{1}{a}\right)D_u^{m,0} + 4D_h^{m,0}\right) \end{aligned}$$

$$\leq \frac{BC_1\delta}{a}\sqrt{E_h^m}\left(3D_u^{m,0}+2D_h^{m,0}\right)+C_1\delta\sqrt{E_\phi^1}\left(\left(4+\frac{1}{a}\right)D_u^{m,0}+4D_h^{m,0}\right)$$
$$+aBC_1\delta\sqrt{E_h^m}D_\phi^{m,0}+aC_1\delta\sqrt{E_\phi^1}D_\sigma^{m,0}, \tag{3.6}$$

where $B=2\sqrt{m+1}\binom{m}{[\frac{m}{2}]}$, $[\frac{m}{2}]$ represent the smallest integer larger than $\frac{m}{2}$.

Before we move on to other estimates, let us first summarize what else we need. The goal of the energy estimates is to get an inequality like

$$\partial_t E + D \leq \sqrt{E}D, \tag{3.7}$$

so one can use the continuity argument to get the desired estimates. Therefore, one still needs $\partial_t E_h^{m,1}$, $D_\sigma^{m,0}$, $D_\phi^{m,0}$ on the LHS.

**3.2. Energy estimates for $\left\|\partial_z^l\partial_x h\right\|^2$.** Taking $\partial_z^l\partial_x$ to (1.16), and multiplying by $\partial_z^l\partial_x h$, then integrating it over $\mu(x,v,z)$,

$$\frac{\epsilon\delta}{2}\partial_t\left\|\partial_z^l\partial_x h\right\|^2+\delta\underbrace{\left\langle\partial_z^l\partial_x^2\phi, v\sqrt{M}\partial_z^l\partial_x h\right\rangle}_{V}\underbrace{-\left\langle\mathcal{L}\partial_z^l\partial_x h,\partial_z^l\partial_x h\right\rangle}_{VI}$$
$$=\delta\sum_{i=0}^{l}\binom{l}{i}\Bigg\langle\underbrace{\partial_z^{l-i}\partial_x^2\phi\partial_v\partial_z^i h}_{III.1}+\underbrace{\partial_z^{l-i}\partial_x\phi\partial_v\partial_x\partial_z^i h}_{III.2}-\underbrace{\frac{v}{2}\partial_z^{l-i}\partial_x^2\phi\partial_z^i h}_{IV.1}$$
$$\underbrace{-\frac{v}{2}\partial_z^{l-i}\partial_x\phi\partial_z^i\partial_x h}_{IV.2},\partial_z^l\partial_x h\Bigg\rangle. \tag{3.8}$$

Similar to (3.2), for $V$ and $VI$, one has

$$V=\frac{\delta}{2}\partial_t\left\|\partial_z^l\partial_x^2\phi\right\|^2,\quad VI\geq\lambda_0\left\|(1-\Pi)\partial_z^l\partial_x h\right\|^2+\left\|\partial_z^l\partial_x u\right\|^2. \tag{3.9}$$

For the bad terms on the RHS, for $i<l$, by Lemma 2.7(d),

$$III.1-IV.1\leq C_1\left\|\partial_z^i h\right\|_{H_z^1(H_x^1)}\left(\frac{3}{a}\left\|\partial_z^l\partial_x u\right\|^2+\frac{2}{a}\left\|(1-\Pi)\partial_z^l\partial_x h\right\|^2+a\left\|\partial_z^{l-i}\partial_x^2\phi\right\|^2\right). \tag{3.10}$$

For $i=l$, by Lemma 2.7(e),

$$III.1-IV.1\leq C_1^2\sqrt{\|\partial_z^l h\|^2+\|\partial_z^l\partial_x h\|^2}\Bigg(\frac{3}{a}\left\|\partial_z^l\partial_x u\right\|^2+\frac{2}{a}\left\|(1-\Pi)\partial_z^l\partial_x h\right\|^2$$
$$+a\sum_{i\leq 1}\left\|\partial_z^i\partial_x^2\phi\right\|^2\Bigg). \tag{3.11}$$

*Remark* 3.1. If one treats $i=l$ same as $i<l$, then the term $\|\partial_z^m h\|^2_{H_z^1(H_x^1)}$ cannot be controlled by $E_h^m$, because the term $\left\|\partial_z^{m+1}\partial_x h\right\|^2$ is not included in the energy term $E_h^m$. That is why we treat all four estimates differently in (3.10)–(3.13).

For $i > 0$, by Lemma 2.7(b),

$$III.2 - IV.2 \le C_1 \left\|\partial_z^{l-i}\partial_x\phi\right\|_{H_z^1(H_x^1)} \left( a\left\|\partial_z^i\partial_x\sigma\right\|^2 + 2\left\|\partial_z^i\partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^i\partial_x h\right\|_\nu^2 \right.$$

$$\left. + \left(2+\frac{1}{a}\right)\left\|\partial_z^l\partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_\nu^2 \right). \tag{3.12}$$

For $i = 0$, by Lemma 2.7(c),

$$III.2 - IV.2$$

$$\le C_1^2\sqrt{\|\partial_z^l\partial_x\phi\|^2 + \|\partial_z^l\partial_x^2\phi\|^2} \left( a\sum_{i\le 1}\left\|\partial_z^i\partial_x\sigma\right\|^2 + 2\sum_{i\le 1}\left\|\partial_z^i\partial_x u\right\|^2 \right.$$

$$\left. + 2\sum_{i\le 1}\left\|(1-\Pi)\partial_z^i\partial_x h\right\|_\nu^2 + \left(2+\frac{1}{a}\right)\left\|\partial_z^l\partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_\nu^2 \right). \tag{3.13}$$

Using all of the above estimates in (3.8), summing the resulting estimates for $l$ from 0 to $m$, gives

(3.14)

$$\frac{\delta}{2}\partial_t\left[\epsilon E_h^{m,1} + \delta E_\phi^{m,1}\right] + \lambda_0 D_h^{m,1} + D_u^{m,1}$$

$$\le C_1\delta\sum_{l=1}^m\left(\sum_{i=0}^{l-1}\binom{l}{i}\left\|\partial_z^i h\right\|_{H_z^1(H_x^1)}\right)\left(\frac{3}{a}\left\|\partial_z^l\partial_x u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_\nu^2\right.$$

$$\left. + \ a\left\|\partial_z^{l-i}\partial_x^2\phi\right\|^2\right) + C_1\delta\sum_{l=1}^m\left(\sum_{i=1}^{l}\binom{l}{i}\left\|\partial_z^{l-i}\partial_x\phi\right\|_{H_z^1(H_x^1)}\right)\left(a\left\|\partial_z^i\partial_x\sigma\right\|^2 + 2\left\|\partial_z^i\partial_x u\right\|^2\right.$$

$$\left. + 2\left\|(1-\Pi)\partial_z^i\partial_x h\right\|_\nu^2 + \left(2+\frac{1}{a}\right)\left\|\partial_z^l\partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_\nu^2\right)$$

$$+ \ C_1^2\delta\sum_{l=0}^m\sqrt{\|\partial_z^l h\|^2 + \|\partial_z^l\partial_x h\|^2}\left(\frac{3}{a}\left\|\partial_z^l\partial_x u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial_z^l\partial_x h\right\|^2 + a\sum_{i\le 1}\left\|\partial_z^i\partial_x^2\phi\right\|^2\right)$$

$$+ \ C_1^2\delta\sum_{l=0}^m\sqrt{\|\partial_z^l\partial_x\phi\|^2 + \|\partial_z^l\partial_x^2\phi\|^2}\left(a\sum_{i\le 1}\left\|\partial_z^i\partial_x\sigma\right\|^2 + 2\sum_{i\le 1}\left\|\partial_z^i\partial_x u\right\|^2\right.$$

$$\left. + 2\sum_{i\le 1}\left\|(1-\Pi)\partial_z^i\partial_x h\right\|_\nu^2 + \left(2+\frac{1}{a}\right)\left\|\partial_z^l\partial_x u\right\|^2 + 2\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_\nu^2\right)$$

$$\le BC_1\delta\sqrt{E_h^m}\left(\frac{3}{a}D_u^{m,1} + \frac{2}{a}D_h^{m,1} + aD_\phi^{m,1}\right)$$

$$+ \ BC_1\delta\sqrt{E_\phi^m}\left(aD_\sigma^{m,1} + 2D_u^{m,1} + 2D_h^{m,1} + \left(2+\frac{1}{a}\right)D_u^{m,1} + 2D_h^{m,1}\right)$$

$$+ \ C_1^2\delta\sqrt{E_h^m}\left(\frac{3}{a}D_u^{m,1} + \frac{2}{a}D_h^{m,1}\right) + aC_1^2\delta\sqrt{m+1}\sqrt{E_h^m}D_\phi^{1,1}$$

$$+ \ C_1^2\delta\sqrt{m+1}\sqrt{E_\phi^m}\left(aD_\sigma^{1,1} + 2D_u^{1,1} + 2D_h^{1,1}\right)$$

$$+ \ C_1^2\delta\sqrt{E_\phi^m}\left(\left(2+\frac{1}{a}\right)D_u^{m,1} + 2D_h^{m,1}\right)$$

$$\leq \frac{(B+1)C_1^2\delta}{a}\sqrt{E_h^m}\left(3D_u^{m,1}+2D_h^{m,1}\right)+(B+1)\,C_1^2\delta\sqrt{E_\phi^m}\left(\left(4+\frac{1}{a}\right)D_u^{m,1}\right.$$
$$\left.+4D_h^{m,1}\right)+a\left(B+\sqrt{m+1}\right)C_1^2\delta\sqrt{E_h^m}D_\phi^{m,1}+a\left(B+\sqrt{m+1}\right)C_1^2\delta\sqrt{E_\phi^m}D_\sigma^{m,1},$$

where $A$ is defined as (2.9). Now combining (3.6) and (3.14) completes the energy estimates for the microscopic system,

$$\frac{\delta}{2}\partial_t\left[\epsilon\underbrace{E_h^m}_{I}+\delta\underbrace{E_\phi^m}_{V}\right]+\underbrace{\lambda_0 D_h^m+D_u^m}_{VI}$$
$$\leq\underbrace{\frac{AC_1^2\delta}{a}\sqrt{E_h^m}\left(3D_u^m+2D_h^m\right)+2AC_1^2\delta\sqrt{E_\phi^m}\left(\left(4+\frac{1}{a}\right)D_u^m+4D_h^m\right)}_{III+IV}$$
$$\underbrace{+aAC_1^2\delta\sqrt{E_h^m}D_\phi^m+aAC_1^2\delta\sqrt{E_\phi^m}D_\sigma^m}_{III+IV}. \tag{3.15}$$

Up to now, one still needs the dissipations $D_\sigma^m$ and $D_\phi^m$ on the LHS to balance the bad terms on the RHS. So next we turn to the macroscopic system.

## 4. Energy estimates on the macroscopic system.

We now prove (2.8) in Lemma 2.2.

### 4.1. Dissipation terms $\left\|\partial_z^l\sigma\right\|^2$ and $\left\|\partial_z^l\partial_x\phi\right\|^2$.

Taking $\partial_z^l$ to (1.23) and multiplying by $\partial_z^l\partial_x\phi$, then integrating it over $\mu(x,z)$, one has

$$\epsilon\delta\underbrace{\left\langle\partial_t\partial_z^l u,\partial_z^l\partial_x\phi\right\rangle}_{I}+\epsilon\underbrace{\left\langle\partial_z^l\partial_x\sigma,\partial_z^l\partial_x\phi\right\rangle}_{II.1}+\underbrace{\left\langle\partial_z^l u,\partial_z^l\partial_x\phi\right\rangle}_{VI}+\delta\underbrace{\left\|\partial_z^l\partial_x\phi\right\|^2}_{V}$$
$$=-\epsilon\underbrace{\left\langle(1-\Pi)\partial_z^l\partial_x h,v^2\sqrt{M}\partial_z^l\partial_x\phi\right\rangle}_{II.2}-\delta\sum_{i=0}^{l}\binom{l}{i}\underbrace{\left\langle\partial_z^{l-i}\partial_x\phi\partial_z^i\sigma,\partial_z^l\partial_x\phi\right\rangle}_{III}. \tag{4.1}$$

First one has

$$I=\partial_t\left\langle\partial_z^l u,\partial_z^l\partial_x\phi\right\rangle-\left\langle\partial_z^l u,\partial_t\partial_z^l\partial_x\phi\right\rangle, \tag{4.2}$$

then by Lemma 2.7(d),

$$\left\langle\partial_z^l u,\partial_z^l\partial_x\partial_t\phi\right\rangle=\delta\left\langle\partial_z^l\partial_t\sigma,\partial_z^l\partial_t\phi\right\rangle=\delta\left\|\partial_z^l\partial_x\partial_t\phi\right\|^2\leq\frac{1}{\delta}\left\|\partial_z^l u\right\|^2. \tag{4.3}$$

$II.1$ and $VI$ are "good terms" here, since

$$II.1=\left\langle\partial_z^l\sigma,-\partial_z^l\partial_x^2\phi\right\rangle=\left\|\partial_z^l\sigma\right\|^2, \tag{4.4}$$

$$VI=-\left\langle\partial_z^l\partial_x u,\partial_z^l\phi\right\rangle=\delta\left\langle\partial_z^l\partial_t\sigma,\partial_z^l\phi\right\rangle=\delta\left\langle\partial_z^l\partial_x\partial_t\phi,\partial_z^l\partial_x\phi\right\rangle=\frac{\delta}{2}\partial_t\left\|\partial_z^l\partial_x\phi\right\|^2, \tag{4.5}$$

while $II.2$ and $III$ are "bad terms,"

$$-II.2=\left\langle(1-\Pi)\partial_z^l h,v^2\sqrt{M}\partial_z^l\partial_x^2\phi\right\rangle\leq\frac{1}{2}\left\|v\sqrt{M}\partial_z^l\sigma\right\|^2+\frac{1}{2}\left\|v(1-\Pi)\partial_z^l h\right\|^2$$
$$\leq\frac{1}{2}\left\|\partial_z^l\sigma\right\|^2+\frac{1}{2}\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2. \tag{4.6}$$

Note, for $l=0$,

$$-III = \langle \partial_x\phi\sigma, \partial_x\phi\rangle = -\left\langle \partial_x^2\phi, (\partial_x\phi)^2\right\rangle = \left\langle \partial_x\phi, 2\partial_x\phi\partial_x^2\phi\right\rangle = -2\left\langle \partial_x\phi\sigma, \partial_x\phi\right\rangle, \tag{4.7}$$

which implies

$$-III = 0. \tag{4.8}$$

For $l>0$, and $i=0$,

$$\begin{aligned} -III &= \left\langle \sigma, \left(\partial_z^l\partial_x\phi\right)^2\right\rangle = \left\langle \partial_x\phi, \partial_x\left(\partial_z^l\partial_x\phi\right)^2\right\rangle = -2\left\langle \partial_x\phi\partial_z^l\sigma, \partial_z^l\partial_x\phi\right\rangle \\ &\le C_1\left\|\partial_x\phi\right\|_{H_z^1(H_x^1)}\left(\left\|\partial_z^l\sigma\right\|^2 + \left\|\partial_z^l\partial_x\phi\right\|^2\right), \end{aligned} \tag{4.9}$$

and for $0<i\le l$,

$$-III \le \frac{C_1}{2}\left\|\partial_z^{l-i}\partial_x\phi\right\|_{H_z^1(H_x^1)}\left(\left\|\partial_z^i\sigma\right\|^2 + \left\|\partial_z^l\partial_x\phi\right\|^2\right). \tag{4.10}$$

Combining all terms in (4.1), one has

$$\begin{aligned} &\delta\partial_t\left[\epsilon\left\langle \partial_z^l u, \partial_z^l\partial_x\phi\right\rangle + \frac{1}{2}\left\|\partial_z^l\partial_x\phi\right\|^2\right] + \frac{\epsilon}{2}\left\|\partial_z^l\sigma\right\|^2 + \delta\left\|\partial_z^l\partial_x\phi\right\|^2 \\ &\le \epsilon\left\|\partial_z^l u\right\|^2 + \frac{\epsilon}{2}\left\|(1-\Pi)\partial_z^l h\right\|_\nu^2 + 2C_1\delta\sum_{l\neq 0, i=1}^{l}\binom{l}{i}\left\|\partial_z^{l-i}\partial_x\phi\right\|_{H_z^1(H_x^1)}\left(\left\|\partial_z^i\sigma\right\|^2 + \left\|\partial_z^l\partial_x\phi\right\|^2\right). \end{aligned} \tag{4.11}$$

Summing $l$ from 0 to $m$ gives

$$\begin{aligned} &\delta\partial_t\left[\epsilon\sum_{l=0}^{m}\left\langle \partial_z^l u, \partial_z^l\partial_x\phi\right\rangle + \frac{1}{2}E_\phi^{m,0}\right] + \frac{\epsilon}{2}D_\sigma^{m,0} + \delta D_\phi^{m,0} \\ &\le \epsilon D_u^{m,0} + \frac{\epsilon}{2}D_h^{m,0} + 2AC_1\delta\sqrt{E_\phi^m}\left(D_\sigma^{m,0} + D_\phi^{m,0}\right). \end{aligned} \tag{4.12}$$

**4.2. Dissipation terms $\left\|\partial_z^l\partial_x\sigma\right\|^2$ and $\left\|\partial_z^l\partial_x^2\phi\right\|^2$ .** Taking $\partial_z^l$ to (1.22) and multiplying by $\partial_z^l\partial_x\sigma$, then integrating it over $\mu(x,z)$,

$$\begin{aligned} &\epsilon\delta\underbrace{\left\langle \partial_t\partial_z^l u, \partial_z^l\partial_x\sigma\right\rangle}_{I} + \epsilon\underbrace{\left\|\partial_z^l\partial_x\sigma\right\|^2}_{II.1} + \underbrace{\left\langle \partial_z^l u, \partial_z^l\partial_x\sigma\right\rangle}_{VI} + \delta\underbrace{\left\langle \partial_z^l\partial_x\phi, \partial_z^l\partial_x\sigma\right\rangle}_{V} \\ &= -\epsilon\underbrace{\left\langle (1-\Pi)\partial_x\partial_z^l h, v^2\sqrt{M}\partial_z^l\partial_x\sigma\right\rangle}_{II.2} - \delta\sum_{i=0}^{l}\binom{l}{i}\underbrace{\left\langle \partial_z^{l-i}\partial_x\phi\partial_z^i\sigma, \partial_z^l\partial_x\sigma\right\rangle}_{III}. \end{aligned} \tag{4.13}$$

Note that

$$I = \partial_t\left\langle \partial_z^l u, \partial_x\partial_z^l\sigma\right\rangle - \left\langle \partial_z^l u, \partial_z^l\partial_x\partial_t\sigma\right\rangle = \partial_t\left\langle \partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\right\rangle - \frac{1}{\delta}\left\|\partial_z^l\partial_x u\right\|^2, \tag{4.14}$$

$$VI = \left\langle \partial_z^l u, \partial_z^l\partial_x\sigma\right\rangle = \delta\left\langle \partial_z^l\partial_t\sigma, \partial_z^l\sigma\right\rangle = \frac{\delta}{2}\partial_t\left\|\partial_z^l\sigma\right\|^2 = \frac{\delta}{2}\partial_t\left\|\partial_z^l\partial_x^2\phi\right\|^2, \tag{4.15}$$

$$V = -\left\langle \partial_z^l\partial_x^2\phi, \partial_z^l\sigma\right\rangle = \left\|\partial_z^l\partial_x^2\phi\right\|^2, \tag{4.16}$$

$$-II.2 \le \frac{1}{2}\left\|\partial_z^l\partial_x\sigma\right\|^2 + \frac{1}{2}\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_\nu^2, \tag{4.17}$$

For $i \neq 0$

$$-III \leq \frac{C_1}{2} \left\|\partial_z^{l-i}\partial_x\phi\right\|_{H_z^1(H_x^1)} (\left\|\partial_z^i\sigma\right\|^2 + \left\|\partial_z^l\partial_x\sigma\right\|^2), \tag{4.18}$$

For $i = 0$

$$-III \leq \frac{C_1^2}{2}\sqrt{\|\partial_z^l\partial_x\phi\|^2 + \|\partial_z^l\partial_x^2\phi\|^2}\left(\sum_{i\leq 1}\left\|\partial_z^i\sigma\right\|^2 + \left\|\partial_z^l\partial_x\sigma\right\|^2\right). \tag{4.19}$$

Using (4.14)–(4.19) in (4.13) implies

$$\begin{aligned}
&\delta\partial_t\left[\epsilon\left\langle\partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\right\rangle + \frac{1}{2}\left\|\partial_z^l\partial_x^2\phi\right\|^2\right] + \frac{\epsilon}{2}\left\|\partial_z^l\partial_x\sigma\right\|^2 + \delta\left\|\partial_z^l\partial_x^2\phi\right\|^2 \\
&\leq \frac{\epsilon}{2}\left\|(1-\Pi)\partial_z^l\partial_x h\right\|_\nu^2 + \epsilon\left\|\partial_z^l\partial_x u\right\|^2 + \frac{C_1\delta}{2}\sum_{i=1}^{l}\tbinom{l}{i}\left\|\partial_z^{l-i}\partial_x\phi\right\|_{H_z^1(H_x^1)}\Big(\left\|\partial_z^i\sigma\right\|^2 \\
&\quad + \left\|\partial_z^l\partial_x\sigma\right\|^2\Big) + \frac{C_1^2}{2}\sqrt{\|\partial_z^l\partial_x\phi\|^2 + \|\partial_z^l\partial_x^2\phi\|^2}\left(\sum_{i\leq 1}\left\|\partial_z^i\sigma\right\|^2 + \left\|\partial_z^l\partial_x\sigma\right\|^2\right).
\end{aligned} \tag{4.20}$$

Summing $l$ from 0 to $m-1$, one has

$$\begin{aligned}
&\delta\partial_t\left[\epsilon\sum_{l=0}^{m}\left\langle\partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\right\rangle + \frac{1}{2}E_\phi^{m,1}\right] + \frac{\epsilon}{2}D_\sigma^{m,1} + \delta D_\phi^{m,1} \\
&\leq \epsilon D_u^{m,1} + \frac{\epsilon}{2}D_h^{m,1} + AC_1^2\delta\sqrt{E_\phi^m}D_\sigma^m.
\end{aligned} \tag{4.21}$$

Combining (4.12) and (4.21), one finishes the energy estimates for the microscopic system,

$$\begin{aligned}
&\delta\partial_t\Bigg[\underbrace{\epsilon\sum_{l=0}^{m}\left\langle\partial_z^l u, \partial_z^l\partial_x\phi\right\rangle + \epsilon\sum_{l=0}^{m}\left\langle\partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\right\rangle}_{I} + \underbrace{\frac{1}{2}E_\phi^m}_{VI}\Bigg] + \frac{\epsilon}{2}\underbrace{D_\sigma^m}_{II.1} + \delta\underbrace{D_\phi^m}_{V} \\
&\leq \epsilon\underbrace{D_u^m}_{I} + \frac{\epsilon}{2}\underbrace{D_h^m}_{II.2} + \underbrace{AC_1^2\delta\sqrt{E_\phi^m}\left(3D_\sigma^m + 2D_\phi^m\right)}_{III}.
\end{aligned} \tag{4.22}$$

## 5. Exponential decay to the Maxwellian.

**5.1. Heuristics for the proof of exponential decay.** Before we do the analysis for the two energy estimates, we first go through the process in a more general framework. Consider that one has the energy estimate

$$\frac{1}{2}\partial_t\hat{E} + \alpha D \leq \beta\sqrt{\hat{E}}D \tag{5.1}$$

for constants $\alpha, \beta > 0$, energy term $\hat{E}$, and dissipation term $D$. If one wants to get an exponential decay for energy $E$, then one requires

$$\text{REQUIREMENT 1:} \quad \exists \text{ constants } c, C > 0, \quad \text{such that} \quad cE \leq \hat{E} \leq CE. \tag{5.2}$$

Since (5.1) is equivalent to

$$\partial_t \sqrt{\hat{E}} \leq \frac{1}{\sqrt{\hat{E}}} \left( \beta \sqrt{\hat{E}} - \alpha \right) D, \tag{5.3}$$

one requires the RHS to be negative initially to make the energy decay,

$$\text{REQUIREMENT 2:} \quad \sqrt{\hat{E}(0)} \leq \frac{\alpha}{2\beta}; \tag{5.4}$$

then by the standard continuity argument, since $\sqrt{\hat{E}}$ is decreasing, for $t > 0$,

$$\partial_t \sqrt{\hat{E}} \leq -\frac{\alpha}{2\sqrt{\hat{E}}} D. \tag{5.5}$$

Futhermore, one needs

$$\text{REQUIREMENT 3:} \quad \exists \text{ constants } \eta > 0 \quad \text{such that} \quad D \geq \eta \hat{E} \tag{5.6}$$

to get exponential decay of $\hat{E}$ from (5.5), which is

$$\hat{E}(t) \leq e^{-\alpha\eta t} \hat{E}(0). \tag{5.7}$$

By Requirement 1, this also implies the exponential decay of the energy term $E$,

$$E(t) \leq \frac{C}{c} e^{-\alpha\eta t} E(0). \tag{5.8}$$

From the above heuristics of the proof, one notes that in order to get the optimal convergence rate with least restriction on initial data, one needs both $\frac{\alpha}{2\beta}$ in Requirement 2 and $\alpha\eta$ in (5.7) to be as large as possible.

*Remark* 5.1. Without uncertainty, if one directly uses the energy estimates from [15], when the small parameter $\epsilon$ are put in, for the high field regime, where $\delta = 1$, the energy estimates reads

$$\begin{aligned} &\frac{1}{2}\partial_t \left[ E_h^m + \frac{1}{\epsilon} E_\phi^m \right] + \frac{1}{\epsilon} \left( D_h^m + D_u^m \right) \\ &\leq \frac{C_1}{\epsilon} \sqrt{E_h^m + E_\phi^m} (D_u^m + D_h^m + D_\phi^m + D_\sigma^m) \end{aligned} \tag{5.9}$$

and,

$$\begin{aligned} &\partial_t \left[ \sum_{l=0}^{m} \left\langle \partial_z^l u, \partial_z^l \partial_x \phi \right\rangle + \sum_{l=0}^{m} \left\langle \partial_z^l \partial_x u, \partial_z^l \partial_x^2 \phi \right\rangle + \frac{1}{2\epsilon} E_\phi^m \right] + \frac{1}{2} D_\sigma^m + \frac{1}{\epsilon} D_\phi^m \\ &\leq D_u^m + D_h^m + \frac{C_1}{\epsilon} \sqrt{E_h^m + E_\phi^m} (D_\phi^m + D_\sigma^m). \end{aligned} \tag{5.10}$$

In Remark 5.1, $C_1$, $C_2$, $C_3$, $C_4$ represent different constants independent of $\epsilon$. Let $G^m = \sum_{l=0}^{m-1} \left\langle \partial_z^l u, \partial_z^l \partial_x \phi \right\rangle + \sum_{l=0}^{m} \left\langle \partial_z^l \partial_x u, \partial_z^l \partial_x^2 \phi \right\rangle + \frac{1}{2\epsilon} E_\phi^m$. Since $-\epsilon E_h^m + \frac{1}{4\epsilon} E_\phi^m \leq G^m \leq \epsilon E_h^m + \frac{3}{4\epsilon} E_\phi^m$, if one combines the microscopic and macroscopic energy estimates (5.9) + $\gamma$ (5.10), one needs $\gamma \leq O(\frac{1}{\epsilon})$ to satisfy Requirement 1. Furthermore, if one wants to get the optimal convergence rate based on this energy estimate, then one needs the dissipation terms to be as large as possible, that is, $\gamma$ as large as possible, which means $\gamma = O(\frac{1}{\epsilon})$. Therefore one derives

$$\frac{1}{2}\partial_t\hat{E}^m + \frac{1}{\epsilon}(D_h^m + D_u^m) + \frac{1}{\epsilon}D_\sigma^m + \frac{1}{\epsilon^2}D_\phi^m$$
$$\le C_2\left(\frac{1}{\epsilon}\sqrt{E_h^m+E_\phi^m}(D_u^m+D_h^m)+\frac{1}{\epsilon^2}\sqrt{E_h^m+E_\phi^m}D_\sigma^m+\frac{1}{\epsilon^2}\sqrt{E_h^m+E_\phi^m}D_\phi^m\right), \tag{5.11}$$

where $\hat{E}^m = (E_h^m + \frac{1}{\epsilon}E_\phi^m) + \frac{1}{\epsilon}G^m$. Letting $E^m = E_h^m + \frac{1}{\epsilon^2}E_\phi^m$, one can check $\hat{E}^m$ and $E^m$ satisfies Requirement 1. So (5.11) leads to

$$\frac{1}{2}\partial_t\hat{E}^m + \frac{1}{\epsilon}(D_h^m + D_u^m) + \frac{1}{\epsilon}D_\sigma^m + \frac{1}{\epsilon^2}D_\phi^m \le C_3\left(\frac{1}{\epsilon}\sqrt{\hat{E}^m}(D_h^m + D_u^m) + \frac{1}{\epsilon^2}\sqrt{\hat{E}^m}D_\sigma^m + \frac{1}{\epsilon^2}\sqrt{\hat{E}^m}D_u^m\right). \tag{5.12}$$

Comparing the term $D_\sigma^m$ on both sides, one notes that $\sqrt{\hat{E}^m}$ needs to be $O(\epsilon)$ such that the bad term on the RHS can be controlled by the $O(1)$ dissipation on the LHS. That is, one requires

$$E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0) \le O(\epsilon) \tag{5.13}$$

to obtain the exponential decay

$$E_h^m + \frac{1}{\epsilon}E_\phi^m \le C_4e^{-O(1)t}\left(E_h^m(0) + \frac{1}{\epsilon}E_\phi^m\right). \tag{5.14}$$

This means the initial data $E_h^m = O(\epsilon)$, $E_\phi^m = O(\epsilon^3)$. These conditions are much stronger than the one in (2.10) of Theorem 2.4.

However, if the coefficient of the $D_\sigma$ only depends on $E_\phi^m$, like the estimates we obtained in Lemma 2.2, then (5.11) becomes

$$\frac{1}{2}\partial_t\hat{E}^m + \frac{1}{\epsilon}(D_h^m + D_u^m) + \frac{1}{\epsilon}D_\sigma^m + \frac{1}{\epsilon^2}D_\phi^m$$
$$\le C\left(\frac{1}{\epsilon}\sqrt{E_h^m + E_\phi^m}(D_u^m + D_h^m) + \frac{1}{\epsilon}\sqrt{\frac{1}{\epsilon^2}E_\phi^m}D_\sigma^m + \frac{1}{\epsilon^2}\sqrt{E_h^m}D_\phi^m\right)$$
$$\le C\left(\frac{1}{\epsilon}\sqrt{\hat{E}^m}(D_h^m + D_u^m) + \frac{1}{\epsilon}\sqrt{\hat{E}^m}D_\sigma^m + \frac{1}{\epsilon^2}\sqrt{\hat{E}^m}D_u^m\right). \tag{5.15}$$

Now the bad terms and good terms can be well balanced even if the initial data of $\hat{E}^m$ is $O(1)$.

**5.2. The high field regime.** For the high field regime, where $\delta = 1$, set

$$F^m = \epsilon E_h^m+E_\phi^m, \quad G^m=\epsilon\sum_{l=0}^{m-1}\langle\partial_z^l u, \partial_z^l\partial_x\phi\rangle+\epsilon\sum_{l=0}^{m}\langle\partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\rangle+\frac{1}{2}E_\phi^m,$$
$$\hat{E}^m = F^m + \frac{2\lambda_0}{\epsilon}G^m, \quad E^m = \epsilon E_h^m + \frac{1}{\epsilon}E_\phi^m, \quad a = 1, \tag{5.16}$$

where $F^m$ is the term inside $\partial_t$ in (2.7) and $G^m$ is that in (2.8).

By (1.21) and Young's inequality, one can bound $G^m$ by

$$-\epsilon^2E_h^m + \left(\frac{1}{2} - \frac{1}{4}\right)E_\phi^m \le G^m \le \epsilon^2E_h^m + \left(\frac{1}{2} + \frac{1}{4}\right)E_\phi^m,$$
$$-\epsilon^2E_h^m + \frac{1}{4}E_\phi^m \le G^m \le \epsilon^2E_h^m + \frac{3}{4}E_\phi^m. \tag{5.17}$$

Since $\lambda_0 \leq \frac{1}{4}$, one obtains

$$(1-2\lambda_0)\epsilon E_h^m + \left(\epsilon + \frac{\lambda_0}{2}\right)\frac{1}{\epsilon}E_\phi^m \leq \hat{E}^m \leq (1+2\lambda_0)\,\epsilon E_h^m + \left(\epsilon + \frac{3\lambda_0}{2}\right)\frac{1}{\epsilon}E_\phi^m,$$

$$\frac{\lambda_0}{2}E^m \leq \hat{E}^m \leq \frac{3}{2}E^m, \tag{5.18}$$

or equivalently

$$\frac{3}{2}\sqrt{\hat{E}^m} \leq \sqrt{\epsilon E_h^m} + \sqrt{\frac{1}{\epsilon}E_\phi^m} \leq \frac{2}{\lambda_0}\sqrt{\hat{E}^m}. \tag{5.19}$$

So one has the equivalence between the energies $E^m$ and $\hat{E}^m$, and the dissipation terms can be lower bounded by $E^m$,

$$E^m \leq \epsilon\,(D_u^m + D_h^m + D_\sigma^m) + \frac{1}{\epsilon}D_\phi^m. \tag{5.20}$$

By (2.7) + $\frac{\lambda_0}{\epsilon}$(2.8), one has the energy estimates

$$\begin{aligned}
&\frac{1}{2}\partial_t\hat{E}^m + \left(\lambda_0 - \frac{\lambda_0}{2}\right)D_h^m + (1-\lambda_0)\,D_u^m + \frac{\lambda_0}{2}D_\sigma^m + \frac{\lambda_0}{\epsilon}D_\phi^m \\
&\leq AC_1^2\left[\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\epsilon E_h^m} + 2\sqrt{\epsilon}\sqrt{\frac{1}{\epsilon}E_\phi^m}\right)(5D_u^m + 4D_h^m) + \sqrt{\epsilon}\left(1+\frac{1}{\epsilon}\right)\sqrt{\frac{1}{\epsilon}E_\phi^m}D_\sigma^m\right. \\
&\quad \left. + \frac{1}{\sqrt{\epsilon}}\left(1+\frac{1}{\epsilon}\right)\sqrt{\epsilon E_h^m}D_\phi^m\right] \\
&\leq \frac{10}{\sqrt{\epsilon}}AC_1^2\left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)(D_h^m + D_u^m) + \frac{2}{\sqrt{\epsilon}}AC_1^2\left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)D_\sigma^m \\
&\quad + \frac{2}{\epsilon^{3/2}}AC_1^2\left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)D_\phi^m,
\end{aligned} \tag{5.21}$$

which implies

$$\begin{aligned}
&\frac{1}{2}\partial_t\hat{E}^m + \frac{\lambda_0}{2}(D_h^m + D_u^m) + \frac{\lambda_0}{2}D_\sigma^m + \frac{\lambda_0}{\epsilon}D_\phi^m \\
&\leq \frac{20AC_1^2}{\lambda_0\sqrt{\epsilon}}\sqrt{\hat{E}^m}(D_h^m + D_u^m) + \frac{4AC_1^2}{\lambda_0\sqrt{\epsilon}}\sqrt{\hat{E}^m}D_\sigma^m + \frac{4AC_1^2}{\lambda_0\epsilon^{3/2}}\sqrt{\hat{E}^m}D_\psi^m.
\end{aligned} \tag{5.22}$$

Therefore, by a standard continuity argument, under the condition of

$$\sqrt{\hat{E}^m(0)} \leq \min\left\{\frac{\frac{\lambda_0}{4}}{\frac{20AC_1^2}{\lambda_0\epsilon^{1/2}}},\ \frac{\frac{\lambda_0}{4}}{\frac{4AC_1^2}{\lambda_0\epsilon^{1/2}}},\ \frac{\frac{\lambda_0}{2\epsilon}}{\frac{4AC_1^2}{\lambda_0\epsilon^{3/2}}}\right\},$$

which holds if

$$\hat{E}^m(0) \leq \left(\frac{\lambda_0^2\epsilon^{1/2}}{80AC_1^2}\right)^2,$$

or equivalently

$$E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0) \leq \frac{2\lambda_0^3}{(80AC_1^2)^2}, \tag{5.23}$$

one then has the estimate

$$\frac{1}{2}\partial_t \hat{E}^m + \frac{\lambda_0}{4}\left(D_h^m + D_u^m + D_\sigma^m + \frac{1}{\epsilon}D_\phi^m\right) \leq 0, \tag{5.24}$$

which implies

$$\begin{aligned}
\frac{1}{2}\hat{E}^m(t) - \frac{1}{2}\hat{E}^m(0) &\leq -\frac{\lambda_0}{4}\int_0^t \left(E_h^m(s) + \frac{1}{\epsilon}E_\phi^m(s)\right) ds \\
\epsilon E_h^m(t) + \frac{1}{\epsilon}E_\phi^m(t) &\leq -\int_0^t \left(E_h^m(s) + \frac{1}{\epsilon}E_\phi^m(s)\right) ds + 3\lambda_0\left(\epsilon E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0)\right) \\
E_h^m(t) &\leq -\frac{1}{\epsilon}\int_0^t E_h^m(s)ds + \frac{3}{\epsilon\lambda_0}\left(\epsilon E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0)\right) \\
E_h^m(t) &\leq \frac{3}{\lambda_0}e^{-\frac{t}{\epsilon}}\left(E_h^m(0) + \frac{1}{\epsilon^2}E_\phi^m(0)\right). \qquad (5.25)
\end{aligned}$$

Similarly, for $E_\phi^m(t)$,

$$\begin{aligned}
E_\phi^m(t) &\leq -\int_0^t E_\phi^m(s)ds + \frac{3\epsilon}{\lambda_0}\left(\epsilon E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0)\right), \\
E_\phi^m(t) &\leq \frac{3}{\lambda_0}e^{-t}\left(\epsilon^2 E_h^m(0) + E_\phi^m(0)\right). \qquad (5.26)
\end{aligned}$$

This completes the proof of (2.11) in Theorem 2.4.

**5.3. The parabolic regime.** For the parabolic regime, where $\delta = \epsilon$, set

$$F^m = \epsilon E_h^m + \epsilon E_\phi^m, \quad G^m = \epsilon\sum_{l=0}^{m-1}\left\langle \partial_z^l u, \partial_z^l\partial_x\phi\right\rangle + \epsilon\sum_{l=0}^{m}\left\langle \partial_z^l\partial_x u, \partial_z^l\partial_x^2\phi\right\rangle + \frac{1}{2}E_\phi^m, \tag{5.27}$$

$$\hat{E}^m = F^m + 2\lambda_0 G^m, \quad E^m = \epsilon E_h^m + E_\phi^m, \quad a = \sqrt{\epsilon}. \tag{5.28}$$

Similar to (5.17), the bound of $G^m$ is

$$\begin{aligned}
-\epsilon E_h^m + \left(\frac{1}{2} - \frac{\epsilon}{4}\right)E_\phi^m \leq G^m &\leq \epsilon E_h^m + \left(\frac{1}{2} + \frac{\epsilon}{4}\right)E_\phi^m \leq \epsilon E_h^m + \frac{3}{4}E_\phi^m, \\
-\epsilon E_h^m + \frac{1}{4}E_\phi^m \leq G^m &\leq \epsilon E_h^m + \frac{3}{4}E_\phi^m \qquad (5.29)
\end{aligned}$$

and $\lambda_0 \leq \frac{1}{4}$, so one obtains

$$\begin{aligned}
(1 - 2\lambda_0)\epsilon E_h^m + \left(\epsilon + \frac{\lambda_0}{2}\right)E_\phi^m \leq \hat{E}^m &\leq (1 + 2\lambda_0)\epsilon E_h^m + \left(\epsilon + \frac{3\lambda_0}{2}\right), \\
\frac{\lambda_0}{2}E^m \leq \hat{E}^m &\leq \frac{3}{2}E^m, \qquad (5.30)
\end{aligned}$$

or equivalently

$$\frac{2}{3}\sqrt{\hat{E}^m} \leq \epsilon^{1/2}\sqrt{E_h^m} + \sqrt{E_\phi^m} \leq \frac{2}{\lambda_0}\sqrt{\hat{E}^m}. \tag{5.31}$$

By (2.7) + $\lambda_0$(2.8), one has energy estimates

$$\frac{1}{2}\partial_t \hat{E}^m + \left(\frac{\lambda_0}{\epsilon} - \frac{\lambda_0}{2}\right) D_h^m + \left(\frac{1}{\epsilon} - \lambda_0\right) D_u^m + \frac{\lambda_0}{2} D_\sigma^m + \lambda_0 D_\phi^m$$
$$\leq \frac{AC_1^2}{\sqrt{\epsilon}} \left(\sqrt{E_h^m} + 2\sqrt{E_\phi^m}\right)(5D_u^m + 4D_h^m) + AC_1^2 (1+\lambda_0)\sqrt{E_\phi^m} D_\sigma^m$$
$$+ \sqrt{\epsilon} AC_1^2 \sqrt{E_h^m} D_\phi^m + 2\lambda_0 AC_1^2 \sqrt{E_\phi^m} D_\phi^m$$
$$\leq 10AC_1^2 \left(\frac{1}{\epsilon^{1/2}} \frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right)(D_u^m + D_h^m) + 2AC_1^2 \left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right) D_\sigma^m$$
$$+ 2AC_1^2 \left(\frac{2}{\lambda_0}\sqrt{\hat{E}^m}\right) D_\phi^m, \tag{5.32}$$

which implies

$$\frac{1}{2}\partial_t \hat{E}^m + \frac{\lambda_0}{2\epsilon^2}(\epsilon D_h^m + \epsilon D_u^m)) + \frac{\lambda_0}{2\epsilon}(\epsilon D_\sigma^m) + \lambda_0 D_\phi^m$$
$$\leq \frac{20AC_1^2}{\lambda_0 \epsilon^{3/2}}\sqrt{\hat{E}^m}(\epsilon D_h^m + \epsilon D_u^m)) + \frac{4AC_1^2}{\lambda_0 \epsilon}\sqrt{\hat{E}^m}(\epsilon D_\sigma^m) + \frac{4AC_1^2}{\lambda_0}\sqrt{\hat{E}^m} D_\phi^m. \tag{5.33}$$

So if the initial data satisfies the condition

$$\sqrt{\hat{E}^m(0)} \leq \min\left\{\frac{\frac{\lambda_0}{4\epsilon^2}}{\frac{20AC_1^2}{\lambda_0\epsilon^{3/2}}}, \frac{\frac{\lambda_0}{4\epsilon}}{\frac{4AC_1^2}{\lambda_0\epsilon}}, \frac{\frac{\lambda_0}{2}}{\frac{4AC_1^2}{\lambda_0}}\right\},$$
$$\hat{E}^m(0) \leq \left(\frac{\lambda_0^2}{80AC_1^2}\right)^2,$$
$$\text{or equivalently,} \quad E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0) \leq \frac{2\lambda_0^3}{(80AC_1)^2\epsilon}, \tag{5.34}$$

then similar to (5.24)–(5.26), one has

$$\frac{1}{2}\hat{E}^m + \frac{\lambda_0}{4}\left(E_h^m + E_\phi^m\right) \leq 0,$$
$$\epsilon E_h^m(t) + E_\phi^m(t) \leq -\int_0^t E_h^m(s) + E_\phi^m(s)ds + \frac{3}{\lambda_0}\left(\epsilon E_h^m(0) + E_\phi^m(0)\right),$$
$$E_h^m(t) \leq \frac{3}{\lambda_0}e^{-\frac{t}{\epsilon}}\left(E_h^m(0) + \frac{1}{\epsilon}E_\phi^m(0)\right), \quad E_\phi^m(t) \leq \frac{3}{\lambda_0}e^{-t}\left(\epsilon E_h^m(0) + E_\phi^m(0)\right). \tag{5.35}$$

This completes the proof of (2.13) in Theorem 2.4.

## Appendix A. The proof of Lemma 2.7.

*Proof.*

(a) By the definition of $u$ in (1.15), and (1.17), (1.22),

(A.1)
$$\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\sqrt{M}\partial^{\mathbf{k}}h\right\rangle = \left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{k}}u\right\rangle = -\left\langle \partial^{\mathbf{k}}\phi, \partial^{\mathbf{k}}\partial_x u\right\rangle = \delta\left\langle \partial^{\mathbf{k}}\phi, \partial^{\mathbf{k}}\partial_t\sigma\right\rangle$$
$$= -\delta\left\langle \partial^{\mathbf{k}}\phi, \partial^{\mathbf{k}}\partial_x^2\partial_t\phi\right\rangle = \delta\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{k}}\partial_x\partial_t\phi\right\rangle = \frac{\delta}{2}\partial_t\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2,$$

where the last equality of the first line is because of (1.22), and the first equality of the second line is because of (1.17).

(b) First break $\partial^{\mathbf{k}} h = \partial^{\mathbf{k}}\sigma\sqrt{M} + (\partial^{\mathbf{k}} h - \partial^{\mathbf{k}}\sigma\sqrt{M})$, and then use $\partial^{\mathbf{k}} h - \partial^{\mathbf{k}}\sigma\sqrt{M} = \partial^{\mathbf{k}} u\, v\sqrt{M} + (1-\Pi)\partial^{\mathbf{k}} h$; one has

(A.2)
$$
\begin{aligned}
&\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}} h), \partial^{\mathbf{l}} h\right\rangle\\
&= \left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}\sigma\sqrt{M}), \partial^{\mathbf{l}} h\right\rangle + \left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}} h - \partial^{\mathbf{i}}\sigma\sqrt{M}), \partial^{\mathbf{l}}\sigma\sqrt{M}\right.\\
&\quad \left. + \left(\partial^{\mathbf{l}} h - \partial^{\mathbf{l}}\sigma\sqrt{M}\right)\right\rangle\\
&= -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi\partial^{\mathbf{i}}\sigma, \partial^{\mathbf{l}} h v\sqrt{M}\right\rangle + \left\langle \partial^{\mathbf{k}}\partial_x\phi\partial_v(\partial^{\mathbf{i}} h), \partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle - \left\langle \partial^{\mathbf{k}}\partial_x\phi\partial_v(\partial^{\mathbf{i}}\sigma\sqrt{M}),\right.\\
&\quad \left.\partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle + \left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial_v\left(\partial^{\mathbf{i}} u v\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}} h\right)\left(\partial^{\mathbf{l}} u v\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}} h\right)\right\rangle\\
&\le -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}} u\right\rangle - \left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}} h, \partial^{\mathbf{l}}\sigma\partial_v(\sqrt{M})\right\rangle + \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}\sigma v M\right\rangle\\
&\quad + \left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{L^\infty_{x,z}}\left(\left\|\partial^{\mathbf{i}} u\partial_v(v\sqrt{M})\right\|^2 + \left\|\partial_v(1-\Pi)\partial^{\mathbf{i}} h\right\|^2\right.\\
&\quad \left. + \left\|\partial^{\mathbf{l}} u(v\sqrt{M})\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{l}} h\right\|^2\right)\\
&\le -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}} u\right\rangle + \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}} u\right\rangle + 0\\
&\quad + C_1\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1_z(H^1_x)}\left(\frac{3}{4}\left\|\partial^{\mathbf{i}} u\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{i}} h\right\|_\nu^2 + \left\|\partial^{\mathbf{l}} u\right\|^2 + \left\|(1-\Pi)\partial^{\mathbf{l}} h\right\|_\nu^2\right),
\end{aligned}
$$

where the last inequality comes from the Sobolev embedding for one dimension, for all f $\in H^1_z(H^1_x)$,

(A.3)
$$
\|f\|_{C^0_x} \le C_1^2\|f\|_{H^1_x}, \quad \|f\|_{C^0_x} \le C_1\|f\|_{H^1_z}, \quad \|f\|_{C^0_{x,z}} \le C_1\|f\|_{H^1_z(H^1_x)},
$$

for some constant $C_1 \ge 1$.
Next,

(A.4)
$$
\begin{aligned}
&-\frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}} h, \partial^{\mathbf{l}} h\right\rangle\\
&= -\frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}\sigma\sqrt{M}, \partial^{\mathbf{l}} h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\left(\partial^{\mathbf{i}} h - \partial^{\mathbf{i}}\sigma\sqrt{M}\right), \partial^{\mathbf{l}}\sigma\sqrt{M}\right.\\
&\quad \left. + \left(\partial^{\mathbf{l}} h - \partial^{\mathbf{l}}\sigma\sqrt{M}\right)\right\rangle\\
&= -\frac{1}{2}\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}} u\rangle - \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\partial^{\mathbf{i}} h\partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle + \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\partial^{\mathbf{i}}\sigma\sqrt{M}\,\partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle\\
&\quad -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\left(\partial^{\mathbf{i}} u v\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}} h\right)\left(\partial^{\mathbf{l}} u v\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}} h\right)\right\rangle
\end{aligned}
$$

$$\leq -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle - \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}u\partial^{\mathbf{l}}\sigma\right\rangle + 0$$
$$+ \frac{1}{2}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{L^\infty_{x,z}}\left(\int |v|(\partial^{\mathbf{i}}uv\sqrt{M})^2 d\mu + \int |v|\left((1-\Pi)\partial^{\mathbf{i}}h\right)^2 d\mu\right.$$
$$\left. + \int |v|(\partial^{\mathbf{l}}uv\sqrt{M})^2 d\mu + \int |v|((1-\Pi)\partial^{\mathbf{l}}h)^2 d\mu\right)$$
$$\leq -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle - \frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}u\partial^{\mathbf{l}}\sigma\right\rangle + \frac{1}{2}C_1\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1_z(H^1_x)}\left(2\left\|\partial^{\mathbf{i}}u\right\|^2\right.$$
$$\left. + \frac{1}{2}\left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_\nu^2 + 2\left\|\partial^{\mathbf{l}}u\right\|^2 + \frac{1}{2}\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right).$$

Therefore (A.2) + (A.4) gives

$$\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle$$
$$\leq -\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle + C_1\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1_z(H^1_x)}\left(2\left\|\partial^{\mathbf{i}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_\nu^2 + 2\left\|\partial^{\mathbf{l}}u\right\|^2\right.$$
$$\left. + 2\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right)$$
$$\leq C_1\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{H^1_z(H^1_x)}\left(a\left\|\partial^{\mathbf{i}}\sigma\right\|^2 + 2\left\|\partial^{\mathbf{i}}u\right\|^2 + 2\left\|(1-\Pi)\partial^{\mathbf{i}}h\right\|_\nu^2 + \left(2+\frac{1}{a}\right)\left\|\partial^{\mathbf{l}}u\right\|^2\right.$$
$$\left. + 2\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right)$$

for $a$ to be determined later.

(c) For the term $\langle \partial^{\mathbf{k}}\partial_x\phi, \partial_v(\partial^{\mathbf{i}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}}h)(\partial^{\mathbf{l}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}}h)\rangle$, one can also bound by

(A.5)
$$\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial_v\left(\partial^{\mathbf{i}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}}h\right)\left(\partial^{\mathbf{l}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}}h\right)\right\rangle$$
$$\leq \int_{I_z}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|_{L^\infty_x}\left\|\partial_v\left(\partial^{\mathbf{i}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}}h\right)\right\|_{L^2_{x,v}}$$
$$\times\left\|\partial^{\mathbf{l}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}}h\right\|_{L^2_{x,v}} d\mu(z)$$
$$\leq C_1\left\|\left\|\partial_v\left(\partial^{\mathbf{i}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}}h\right)\right\|_{L^2_{x,v}}\right\|_{L^\infty_z}$$
$$\times\sqrt{\|\partial^{\mathbf{k}}\partial_x\phi\|^2 + \|\partial^{\mathbf{k}}\partial_x^2\phi\|^2}\left\|\partial^{\mathbf{l}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}}h\right\|$$
$$\leq C_1^2\sqrt{\|\partial^{\mathbf{k}}\partial_x\phi\|^2 + \|\partial^{\mathbf{k}}\partial_x^2\phi\|^2}\left(\frac{1}{2}\sum_{i\leq 1}\left\|\partial_v\left(\partial^{\mathbf{i}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{i}}\partial_z^i h\right)\right\|^2\right.$$
$$\left. + \frac{1}{2}\left\|\partial^{\mathbf{l}}uv\sqrt{M} + (1-\Pi)\partial^{\mathbf{l}}h\right\|^2\right)$$
$$\leq C_1^2\sqrt{\|\partial^{\mathbf{k}}\partial_x\phi\|^2 + \|\partial^{\mathbf{k}}\partial_x^2\phi\|^2}\left(\frac{3}{4}\sum_{i\leq 1}\left\|\partial^{\mathbf{i}}\partial_z^i u\right\|^2 + \sum_{i\leq 1}\left\|(1-\Pi)\partial^{\mathbf{i}}\partial_z^i h\right\|_\nu^2 + \left\|\partial^{\mathbf{l}}u\right\|^2\right.$$
$$\left. + \left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right).$$

Similarly,

$$-\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\left(\partial^{\mathbf{i}}uv\sqrt{M}+(1-\Pi)\partial^{\mathbf{i}}h\right)\left(\partial^{\mathbf{l}}uv\sqrt{M}+(1-\Pi)\partial^{\mathbf{l}}h\right)\right\rangle$$

$$\le \frac{C_1^2}{2}\sqrt{\|\partial^{\mathbf{k}}\partial_x\phi\|^2+\|\partial^{\mathbf{k}}\partial_x^2\phi\|^2}\left(2\sum_{i\le 1}\left\|\partial^{\mathbf{i}}\partial_z^i u\right\|^2+\frac{1}{2}\sum_{i\le 1}\left\|(1-\Pi)\partial^{\mathbf{i}}\partial_z^i h\right\|_\nu^2\right.$$

$$\left.+2\left\|\partial^{\mathbf{l}}u\right\|^2+\frac{1}{2}\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right), \tag{A.6}$$

(A.7)

$$-\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}\sigma\partial^{\mathbf{l}}u\right\rangle \le C_1^2\sqrt{\|\partial^{\mathbf{k}}\partial_x\phi\|^2+\|\partial^{\mathbf{k}}\partial_x^2\phi\|^2}\left(a\sum_{i\le 1}\left\|\partial^{\mathbf{i}}\partial_z^i\sigma\right\|^2+\frac{1}{a}\left\|\partial^{\mathbf{l}}u\right\|^2\right),$$

which gives

$$\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle-\frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle$$

$$\le C_1^2\sqrt{\|\partial^{\mathbf{k}}\partial_x\phi\|^2+\|\partial^{\mathbf{k}}\partial_x^2\phi\|^2}\left(a\sum_{i\le 1}\left\|\partial^{\mathbf{i}}\partial_z^i\sigma\right\|^2+2\sum_{i\le 1}\left\|\partial^{\mathbf{i}}\partial_z^i u\right\|^2\right.$$

$$\left.+2\sum_{i\le 1}\left\|(1-\Pi)\partial^{\mathbf{i}}\partial_z^i h\right\|_\nu^2+\left(2+\frac{1}{a}\right)\left\|\partial^{\mathbf{l}}u\right\|^2+2\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right)$$

for $a$ to be determined later.

(d) Since

(A.8)

$$\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle$$

$$=\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle+\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h-\partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle$$

$$=-\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}\sigma\partial_v(\sqrt{M})\right\rangle-\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial_v(\partial^{\mathbf{l}}uv\sqrt{M}+(1-\Pi)\partial^{\mathbf{l}}h)\right\rangle$$

$$\le\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{i}}u\partial^{\mathbf{l}}\sigma\right\rangle+\left\langle\left|\partial^{\mathbf{i}}h\right|, \left|\partial^{\mathbf{k}}\partial_x\phi\left(\partial^{\mathbf{l}}u\partial_v(v\sqrt{M})+\partial_v(1-\Pi)\partial^{\mathbf{l}}h\right)\right|\right\rangle$$

$$\le\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}}u\right\rangle+\left\|\left|\partial^{\mathbf{i}}h\right|\right\|_{L^\infty_{x,z}}\left(\frac{a}{2}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2+\frac{1}{a}\left\|\partial^{\mathbf{l}}u\partial_v(v\sqrt{M})\right\|^2\right.$$

$$\left.+\frac{1}{a}\left\|\partial_v(1-\Pi)\partial^{\mathbf{l}}h\right\|^2\right)$$

$$\le\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}}u\right\rangle+C_1\left\|\partial^{\mathbf{i}}h\right\|_{H_z^1(H_x^1)}\left(\frac{a}{2}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2+\frac{3}{4a}\left\|\partial^{\mathbf{l}}u\right\|^2\right.$$

$$\left.+\frac{1}{a}\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right).$$

Next, similar to (A.8),

(A.9)

$$-\frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle=-\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\partial^{\mathbf{i}}h\,\partial^{\mathbf{l}}h\right\rangle$$

$$=-\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\partial^{\mathbf{i}}h\,\partial^{\mathbf{l}}\sigma\sqrt{M}\right\rangle-\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, v\partial^{\mathbf{i}}h\left(\partial^{\mathbf{l}}uv\sqrt{M}+(1-\Pi)\partial^{\mathbf{l}}h\right)\right\rangle$$

$$
\begin{aligned}
&\le -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}}u\right\rangle + \frac{1}{2}\left\langle \left|\partial^{\mathbf{i}}h\right|, \left|\partial^{\mathbf{k}}\partial_x\phi\left(\partial^{\mathbf{l}}uv^2\sqrt{M} + v(1-\Pi)\partial^{\mathbf{l}}h\right)\right|\right\rangle \\
&\le -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}}u\right\rangle + \frac{C_1}{2}\left\|\partial^{\mathbf{i}}h\right\|_{H_z^1(H_x^1)}\left(\frac{a}{2}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2 + \frac{1}{a}\left\|\partial^{\mathbf{l}}u(v^2\sqrt{M})\right\|^2\right. \\
&\quad \left. + \frac{1}{a}\left\|v(1-\Pi)\partial^{\mathbf{l}}h\right\|^2\right) \\
&\le -\frac{1}{2}\left\langle \partial^{\mathbf{k}}\partial_x\phi, \partial^{\mathbf{l}}\sigma\partial^{\mathbf{i}}u\right\rangle + \frac{C_1}{2}\left\|\partial^{\mathbf{i}}h\right\|_{H_z^1(H_x^1)}\left(\frac{a}{2}\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2 + \frac{3}{a}\left\|\partial^{\mathbf{l}}u\right\|^2\right. \\
&\quad \left. + \frac{1}{a}\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right).
\end{aligned}
$$

Therefore (A.8) + (A.9) gives

$$
\begin{aligned}
&\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle \\
&\le C_1\left\|\partial^{\mathbf{i}}h\right\|_{H_z^1(H_x^1)}\left(a\left\|\partial^{\mathbf{k}}\partial_x\phi\right\|^2 + \frac{3}{a}\left\|\partial^{\mathbf{l}}u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right).
\end{aligned} \tag{A.10}
$$

(e) Similar to the proof in (c), based on the estimates in (d), one can bound the same term by

(A.11)

$$
\begin{aligned}
&\left\langle \partial^{\mathbf{k}}\partial_x\phi\,\partial_v(\partial^{\mathbf{i}}h), \partial^{\mathbf{l}}h\right\rangle - \frac{1}{2}\left\langle v\partial^{\mathbf{k}}\partial_x\phi\,\partial^{\mathbf{i}}h, \partial^{\mathbf{l}}h\right\rangle \\
&\le C_1^2\sqrt{\left\|\partial^{\mathbf{i}}h\right\|^2 + \left\|\partial^{\mathbf{i}}\partial_x h\right\|^2}\left(a\sum_{i\le 1}\left\|\partial^{\mathbf{k}}\partial_z^i\partial_x\phi\right\|^2 + \frac{3}{a}\left\|\partial^{\mathbf{l}}u\right\|^2 + \frac{2}{a}\left\|(1-\Pi)\partial^{\mathbf{l}}h\right\|_\nu^2\right).
\end{aligned}
$$

(f) By (1.22) and (1.16) one derives

$$
\partial_x(\partial^{\mathbf{k}}\partial_x\partial_t\phi) = -\partial^{\mathbf{k}}\partial_t\sigma = \partial_x\left(\frac{1}{\delta}\partial^{\mathbf{k}}u\right), \tag{A.12}
$$

and integrating it from $-\infty$ to $x$ implies

$$
\partial^{\mathbf{k}}\partial_x\partial_t\phi(x) = \frac{1}{\epsilon}\partial^{\mathbf{k}}u(x). \tag{A.13}
$$

Hence

$$
\left\|\partial^{\mathbf{k}}\partial_x\partial_t\phi\right\|^2 \le \frac{1}{\delta^2}\left\|\partial^{\mathbf{k}}u\right\|^2. \tag{A.14}
$$

∎

## Appendix B. The proof of Theorem 2.6.

*Proof.* The proof of Theorem 2.6 is almost the same as the proof of Theorem 2.4, expect for the estimates on the nonlinear term $\sum_{j=0}^{1}\sum_{l=0}^{m}\partial_x^j\partial_z^l\left(\partial_x\phi\partial_v h - \frac{v}{2}\partial_x\phi h\right)$ and $\sum_{j=0}^{1}\sum_{l=0}^{m}\partial_x^j\partial_z^l\left(\partial_x\phi\sigma\right)$. We first estimate the case of $j=0$.

Taking $\partial_z^l$ on (1.16), and multiplying by $\left(\frac{l+1}{l!}\right)^2\partial_z^l h$, then integrating it over $\mu(x,v,z)$, one has

(B.1)

$$
\begin{aligned}
&\frac{\delta}{2}\partial_t\left(\epsilon\left\|\frac{l+1}{l!}\partial_z^l h\right\|^2+\delta\left\|\frac{l+1}{l!}\partial_z^l\partial_x\phi\right\|^2\right)+\lambda_0\left\|\frac{l+1}{l!}(1-\Pi)\partial_z^l h\right\|_\nu^2+\left\|\frac{l+1}{l!}\partial_z^l u\right\|^2\\
&\leq\delta\sum_{i=0}^{l}\frac{l!}{i!(l-i)!}\frac{l+1}{l!}\left\langle\partial_z^{l-i}\partial_x\phi\left(\partial_v-\frac{v}{2}\right)\partial_z^i h,\frac{l+1}{l!}\partial_z^l h\right\rangle\\
&=\delta\sum_{i=0}^{l}\frac{l+1}{(l-i+1)(i+1)}\left\langle\left(\frac{l-i+1}{(l-i)!}\partial_z^{l-i}\partial_x\phi\right)\left(\frac{i+1}{i!}\partial_z^i h\right),\frac{l+1}{l!}\partial_z^l h\right\rangle\\
&\leq\delta\sum_{i=0}^{l}\left(\frac{1}{i+1}+\frac{1}{l-i+1}\right)\left\langle\left(\frac{l-i+1}{(l-i)!}\partial_z^{l-i}\partial_x\phi\right)\left(\frac{i+1}{i!}\partial_z^i h\right),\frac{l+1}{l!}\partial_z^l h\right\rangle\\
&\leq\delta C_1\sum_{i=0}^{l-1}\frac{1}{i+1}\left\|\frac{i+1}{i!}\partial_z^i h\right\|_{H_z^1(H_x^1)}\left(\frac{3}{a}\left\|\frac{l+1}{l!}\partial_z^l u\right\|^2+\frac{2}{a}\left\|\frac{l+1}{l!}(1-\Pi)\partial_z^l h\right\|_\nu^2\right.\\
&\qquad\left.+a\left\|\frac{l-i+1}{(l-i)!}\partial_z^{l-i}\partial_x\phi\right\|^2\right)\\
&\quad+\ \delta C_1\sum_{i=0}^{l-1}\frac{1}{l-i+1}\left\|\frac{l-i+1}{(l-i)!}\partial_z^{l-i}\partial_x\phi\right\|_{H_z^1(H_x^1)}\left(\left(2+\frac{1}{a}\right)\left\|\frac{l+1}{l!}\partial_z^l u\right\|^2\right.\\
&\quad+2\left\|\frac{l+1}{l!}(1-\Pi)\partial_z^l h\right\|_\nu^2+a\left\|\frac{i+1}{i!}\partial_z^i\sigma\right\|^2+2\left\|\frac{i+1}{i!}\partial_z^i u\right\|^2\\
&\quad\left.+2\left\|\frac{i+1}{i!}(1-\Pi)\partial_z^i h\right\|^2\right)+\delta C_1^2\sqrt{\left\|\frac{l+1}{l!}\partial_z^l h\right\|^2+\left\|\frac{l+1}{l!}\partial_z^l\partial_x h\right\|^2}\left(a\sum_{j\leq 1}\left\|\partial_x^j\partial_x\phi\right\|^2\right.\\
&\quad\left.+\frac{3}{a}\left\|\frac{l+1}{l!}\partial_z^l u\right\|^2+\frac{2}{a}\left\|\frac{l+1}{l!}(1-\Pi)\partial_z^l h\right\|_\nu^2\right),
\end{aligned}
$$

where the last inequality comes from Lemma 2.7(d), (b), (e). Notice

$$
\begin{aligned}
\left(\sum_{i=0}^{l-1}\frac{a_i}{i+1}\right)^2&=\sum_{i,j=0}^{l-1}\frac{a_ia_j}{(i+1)(j+1)}\leq\frac{1}{2}\sum_{i,j=0}^{l-1}\left(\frac{(i+1)^2}{(j+1)^2}\frac{a_i^2}{(i+1)^2}+\frac{(j+1)^2}{(i+1)^2}\frac{a_j^2}{(j+1)^2}\right)\\
&=\sum_{i=0}^{l-1}\left(\sum_{j=0}^{l-1}\frac{1}{(j+1)^2}\right)a_i^2\leq(A')^2\sum_{i=0}^{l-1}a_i^2,
\end{aligned}
\tag{B.2}
$$

where $A'=\sqrt{\sum_{i=0}^{\infty}\frac{1}{(i+1)^2}}$. Then one can bound

$$
\sum_{i=0}^{l-1}\frac{1}{i+1}\left\|\frac{i+1}{i!}\partial_z^i h\right\|_{H_z^1(H_x^1)}\leq 2A'\sqrt{\tilde{E}_h^l}\leq 2A'\sqrt{\tilde{E}_h^m}
\tag{B.3}
$$

and

$$\sum_{l=1}^{m}\sum_{i=0}^{l-1}\frac{1}{i+1}\left\|\frac{i+1}{i!}\partial_z^i h\right\|_{H_z^1(H_x^1)}\left\|\frac{l-i+1}{(l-i)!}\partial_z^{l-i}\partial_x\phi\right\|^2$$

$$\leq\left(\sum_{i=0}^{m-1}\frac{1}{i+1}\left\|\frac{i+1}{i!}\partial_z^i h\right\|_{H_z^1(H_x^1)}\right)\left(\sum_{i=0}^{m}\left\|\frac{i+1}{i!}\partial_z^i\partial_x\phi\right\|^2\right)\leq 2A'\sqrt{\tilde{E}_h^m}\tilde{D}_\phi^m. \tag{B.4}$$

Here $\tilde{D}_\phi^m$ is the corresponding new dissipation of $\partial_x\phi$ in the new norm, and similarly for$\tilde{D}_\sigma^m$, $\tilde{D}_u^m$. All other terms in (B.1) can be similarly bounded. Thus $\sum_{l=0}^m (B.1)$ gives

$$\begin{aligned}
&\frac{\delta}{2}\partial_t\left[\epsilon\tilde{E}_h^{m,0}+\delta\tilde{E}_\phi^{m,0}\right]+\lambda_0+\tilde{D}_u^{m,0}\\
\leq\, & 2\delta A'C_1\sqrt{\tilde{E}_h^m}\left(\frac{3}{a}\tilde{D}_u^{m,0}+\frac{2}{a}\tilde{D}_h^{m,0}+a\tilde{D}_\phi^{m,0}\right)+2\delta A'C_1\sqrt{\tilde{E}_\phi^m}\left(\left(4+\frac{1}{a}\right)\tilde{D}_u^{m,0}\right.\\
&\left.+4\tilde{D}_h^{m,0}+a\tilde{D}_\sigma^{m,0}\right)+2\delta A'C_1^2\sqrt{\tilde{E}_h^m}\left(a\tilde{D}_\phi^{m,0}+\frac{3}{a}\tilde{D}_u^{m,0}+\frac{2}{a}\tilde{D}_h^{m,0}\right)\\
\leq\, & 4\delta A'C_1\sqrt{\tilde{E}_h^m}\left(\frac{3}{a}\tilde{D}_u^{m,0}+\frac{2}{a}\tilde{D}_h^{m,0}+a\tilde{D}_\phi^{m,0}\right)+2\delta A'C_1\sqrt{\tilde{E}_\phi^m}\left(\left(4+\frac{1}{a}\right)\tilde{D}_u^{m,0}\right.\\
&\left.+4\tilde{D}_h^{m,0}+a\tilde{D}_\sigma^{m,0}\right).
\end{aligned} \tag{B.5}$$

□

One can use a similar method to bound other nonlinear terms. We omit the details here. Now one can see that the constant $A'$ is independent of m, which leads to the independence of $m$ in the initial condition.

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