EMERGENCE OF THE CONSENSUS AND SEPARATION IN AN AGENT-BASED MODEL WITH ATTRACTIVE AND SINGULAR REPULSIVE FORCES

JINWOOK JUNG AND SHI JIN

ABSTRACT. In this paper, we study an agent-based interacting particle system with attractive and singular repulsive forces. We prove the collision avoidance between particles from different groups due to repulsive forces. Moreover, we provide a sufficient condition for the emergence of asymptotic consensus in the same group and separation for different groups. We consider the one-dimensional and multi-dimensional cases separately since they exhibit different dynamics. Numerical simulations are performed to support our theoretical results.

CONTENTS

1. INTRODUCTION

During the last decade, there have been many studies on the consensus of multi-agent first order interacting particle systems [1, 2, 3, 4, 7, 8, 9, 12, 13, 14]. In this article, our interest lies in the dynamics of opposing groups [9], where particles in the same group attract each other while particles from different groups are repulsive. Here, the attraction force between particles from the same group is given by a communication weight function depending on the distance between particles and the repulsion force between particles from

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different groups is given by a singular kernel to ensure the collision avoidance between two groups. For these opposing groups, we study both the consensus in the same group and separation of inter-groups. Specifically, we consider the following Cauchy problem:

$$
\begin{aligned}
\dot{x}_i &= \frac{1}{N_1}\sum_{k=1}^{N_1}\phi(x_k-x_i)(x_k-x_i) - \frac{1}{N_2}\sum_{\ell=1}^{N_2}\frac{(y_\ell-x_i)}{|y_\ell-x_i|^\beta}, \quad 1\le i\le N_1, \quad x_i,y_j\in\mathbb{R}^d,\\
\dot{y}_j &= \frac{1}{N_2}\sum_{\ell=1}^{N_2}\psi(y_\ell-y_j)(y_\ell-y_j) - \frac{1}{N_1}\sum_{k=1}^{N_1}\frac{(x_k-y_j)}{|x_k-y_j|^\beta}, \quad 1\le j\le N_2,
\end{aligned}
\tag{1.1}
$$

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where ϕ and ψ are communication weight functions between particles of the same group satisfying

$$\phi(x)=\tilde{\phi}(\|x\|), \quad \psi(y)=\tilde{\psi}(\|y\|),$$

and $\tilde{\phi},\tilde{\psi}:[0,\infty)\to\mathbb{R}_+$ are bounded, monotonically decreasing and Lipschitz continuous: For $r,s\ge 0$,

$$
\begin{aligned}
&0<\tilde{\phi}(r)\le\tilde{\phi}(0), \quad (r-s)(\tilde{\phi}(r)-\tilde{\phi}(s))\le 0, \quad \sup_{r\ne s}\frac{|\tilde{\phi}(r)-\tilde{\phi}(s)|}{|r-s|}<\infty,\\
&0<\tilde{\psi}(r)\le\tilde{\psi}(0), \quad (r-s)(\tilde{\psi}(r)-\tilde{\psi}(s))\le 0, \quad \sup_{r\ne s}\frac{|\tilde{\psi}(r)-\tilde{\psi}(s)|}{|r-s|}<\infty.
\end{aligned}
$$

Note that our model (1.1) is a natural generalization of the aggregation model and was already considered in previous literature [1, 3, 4, 7]. However, we also note that these previous works considered the mean-field limit of (1.1) and did not fully address the emergent behaviors observed in (1.1). In [6], attractive and repulsive forces were considered but the consensus and separations were proved for special or simplified communication functions. For the emergent behavior in the first-order model with singular kernel, we refer to [8]. Now, our goal is to obtain the following estimates:

- Collision avoidance between two groups $\{x_i\}_{i=1}^{N_1}$ and $\{y_j\}_{j=1}^{N_2}$,
- Lower bound for minimal distances between two groups which is uniform in time,
- (Asymptotic) consensus and separation estimates,

where the notions for *asymptotic consensus* and *separation* are given in Definition 2.1.

The main results are two-fold. Since there is a noticeable difference between the dynamics of the system (1.1) in one dimension and multi-dimension, we separately provide the asymptotic estimates for system (1.1) depending on the dimension.

For the multi-dimensional case, we first consider the collision avoidance between two groups. If the singularity exponent β in the repulsive force between groups is greater than or equal to 2, we prove that particles from different groups can not collide with each other in a finite time. Moreover, under suitable conditions on communication weight functions, we can show that each group reaches an asymptotic consensus while the distance between the two groups grows to infinity. This can be fulfilled if we exploit the structure of system

(1.1) that is similar to a gradient flow (see Section 3 for detail).

Next, we discuss the one-dimensional case. First, we may relax the condition for the collision avoidance from $\beta \geq 2$ to $\beta > 1$. Since particles are in a real line, collision avoidance between two groups implies the following situation. If there is a particle that particles from the other group surround, particles in each group are separated into several clusters. Moreover, we can observe the preservation of the ordering among these clusters along time (see Lemma 4.1). Here, although each group can not reach an asymptotic consensus, nor asymptotic separation, we show that each cluster reaches a consensus at an exponential rate. If the singularity exponent β is greater than 2, we can use the estimates from multi-dimensional case to get the uniform lower and upper bounds for the distance between the two groups. Moreover, we show that each particle converges to its asymptotic limit.

The rest of this paper is organized as follows. In Section 2, we present basic estimates for system (1.1) and our main results. In Section 3, we consider the multi-dimensional case. First, we address the collision avoidance between two groups in system (1.1) to obtain the well-posedness. Second, we present a sufficient condition such that each group reaches a consensus while the distance between the two groups increases to infinite as time grows. In Section 4, we focus on the one-dimensional case. First, we prove the collision avoidance under a weaker condition on the parameter than the multi-dimensional case and consensus results in each group. Moreover, we provide a condition that yields the lower and upper bound estimates for distances between two groups and also the existence of asymptotic position for each particle. In Section 5, we present several results from numerical simulations regarding our system. Finally, in Section 6, we summarize our results and discuss some possible future works.

Notation We write $M_{m\times n}(\mathbb{R})$ as the set of $m\times n$ real matrices. Without further mention, a vector $v \in \mathbb{R}^m$ is sometimes regarded as $m \times 1$ matrix. We denote $\mathbb{I}_m$ by the $m \times m$ identity matrix and $\mathbb{1}_n$ be the vector in $\mathbb{R}^n$ whose components are all 1, i.e. $\mathbb{1}_n := (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. For $A = (a_{ij}) \in M_{m\times n}(\mathbb{R})$ and $B = (b_{rs}) \in M_{p\times q}(\mathbb{R})$, we write the *Kronecker product* of two matrices A and B as $A \otimes B$:

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}, \quad \text{or} \quad (A\otimes B)_{p(i-1)+r,q(j-1)+s} = a_{ij}b_{rs}.$$

We set the following set of indices

$$\mathcal{N}_1 := \{1, \cdots, N_1\}, \quad \mathcal{N}_2 := \{1, \cdots, N_2\},$$

and let X and Y be as follows:

$$X = (x_1, \cdots, x_{N_1})^T \in \mathbb{R}^{N_1 d}, \quad Y = (y_1, \cdots, y_{N_2})^T \in \mathbb{R}^{N_2 d}.$$

We use $\|\cdot\|$ to denote the standard ℓ^2-norm in the Euclidean space. We define the *diameter* of each group and *minimal ℓ^2-distance* between two groups as

$$\mathcal{D}(X) := \max_{i,i'\in\mathcal{N}_1} \|x_i - x_{i'}\|, \quad \mathcal{D}(Y) := \max_{j,j'\in\mathcal{N}_2} \|y_j - y_{j'}\|, \quad \delta(X,Y) := \min_{i\in\mathcal{N}_1, j\in\mathcal{N}_2} \|x_i - y_j\|,$$

and also the *mean position* for each group:

$$x_c := \frac{1}{N_1}\sum_{i=1}^{N_1} x_i, \quad y_c := \frac{1}{N_2}\sum_{j=1}^{N_2} y_j.$$

We denote $\nabla_X := (\nabla_{x_1}, \cdots, \nabla_{x_{N_1}})$, $\nabla_Y := (\nabla_{y_1}, \cdots, \nabla_{y_{N_2}})$ by gradient operators with respect to x_i's and y_j's, respectively, and let $\nabla := (\nabla_X, \nabla_Y)$.

2. Preliminaries

?⟨sec:2⟩? In this section, we provide some basic estimates for system (1.1) and present our main results. First, we present several notions to be used throughout the paper.

⟨D2.1⟩ **Definition 2.1.** *Let (X, Y) be a solution to system (1.1).*

(1) The pair (X, Y) reaches an asymptotic consensus if

$$\lim_{t\to\infty} \mathcal{D}(X) = \lim_{t\to\infty} \mathcal{D}(Y) = 0.$$

(2) The pair (X, Y) fulfills a separation if

$$\inf_{t\geq 0} \delta(X, Y) > 0,$$

and it fulfills an asymptotic separation if

$$\lim_{t\to\infty} \delta(X, Y) = \infty.$$

Next, we consider a conserved quantity in system (1.1).

⟨L2.1⟩ **Lemma 2.1.** *Let (X, Y) be a solution to system (1.1). Then one has*

$$x_c(t) + y_c(t) = x_c(0) + y_c(0), \quad \forall t > 0.$$

Proof. Direct computation yields

$$\begin{aligned}\frac{d}{dt}x_c &= \frac{1}{(N_1)^2}\sum_{i,k\in\mathcal{N}_1} \phi(x_k - x_i)(x_k - x_i) - \frac{1}{N_1N_2}\sum_{i\in\mathcal{N}_1, j\in\mathcal{N}_2} \frac{(y_j - x_i)}{|y_j - x_i|^\beta} \\ &= -\frac{1}{N_1N_2}\sum_{i\in\mathcal{N}_1, j\in\mathcal{N}_2} \frac{(y_j - x_i)}{|y_j - x_i|^\beta},\end{aligned}$$

where we used the antisymmetry $i \leftrightarrow k$. Similarly,

$$\begin{aligned}\frac{d}{dt}y_c &= \frac{1}{(N_2)^2}\sum_{j,m\in\mathcal{N}_2} \psi(y_\ell - y_j)(y_\ell - y_j) - \frac{1}{N_1N_2}\sum_{i\in\mathcal{N}_1, j\in\mathcal{N}_2} \frac{(x_i - y_j)}{|x_i - y_j|^\beta} \\ &= -\frac{1}{N_1N_2}\sum_{i\in\mathcal{N}_1, j\in\mathcal{N}_2} \frac{(x_i - y_j)}{|x_i - y_j|^\beta}.\end{aligned}$$

Thus, combining two estimates gives the desired result.

□

Next, we introduce the Łojasiewicz inequality for later use, which was constructed by Łojasiewicz in the sixties [10, 11] (see also [5]) last century.

⟨L2.2⟩ **Lemma 2.2.** *Suppose that $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is analytic in the open set D. Let $\bar{x}$ be a critical point of f, i.e., $\nabla f(\bar{x}) = 0$. Then there exist $r > 0$, $c > 0$, and $\eta \in [\frac{1}{2}, 1)$ such that*

$$\|\nabla f(x)\| \ge c|f(x) - f(\bar{x})|^\eta, \quad \forall x \in B(\bar{x}, r),$$

where $B(\bar{x}, r)$ denotes the ball of radius r in $\mathbb{R}^n$ centered at $\bar{x}$.

Now, we provide our main results. First, we address the dynamics of system (1.1) for multi-dimensional cases. For this, we set several functionals that give a gradient flow-like structure to system (1.1):

$$\begin{aligned}
&W_X(t) := \frac{1}{2N_1^2} \sum_{i,k \in \mathcal{N}_1} \Phi(|x_i - x_k|), \quad W_Y(t) := \frac{1}{2N_2^2} \sum_{j,\ell \in \mathcal{N}_2} \Psi(|y_j - y_\ell|), \\
&\mathcal{D}^\beta(t) := \frac{1}{N_1 N_2} \begin{cases} \frac{1}{\beta - 2} \displaystyle\sum_{i \in \mathcal{N}_1, j \in \mathcal{N}_2} \frac{1}{|x_i - y_j|^{\beta - 2}}, & \text{if } \beta \neq 2, \\ \displaystyle\sum_{i \in \mathcal{N}_1, j \in \mathcal{N}_2} -\log|x_i - y_j|, & \text{if } \beta = 2, \end{cases} \\
&W = W(X, Y) := W_X + W_Y + \mathcal{D}^\beta,
\end{aligned}$$

where

$$\Phi(z) := \int_0^z \phi(r) r dr, \quad \text{and} \quad \Psi(z) := \int_0^z \psi(u) u\, du, \quad z \in \mathbb{R}_+.$$

Note that if $\beta > 2$, the existence of an upper bound to the functional $\mathcal{D}^\beta$ implies the distance between two groups is positive, while $\mathcal{D}^\beta \to 0$ implies the distance between two groups increases to infinity.

Then, our main result is as follows.

⟨T2.1⟩ **Theorem 2.1.** *Let (X, Y) be a solution to system (1.1) with $d \ge 2$. Assume that the initial configuration, the parameter β and communication weights ϕ and ψ satisfy*

($\mathcal{C}1$) $\delta(X, Y)(0) > 0$, $\beta > 2$.

($\mathcal{C}2$) *There exist $\mathcal{D}_X^\infty$ and $\mathcal{D}_Y^\infty > 0$ which depend on the initial configuration such that*

$$W(0) = \frac{1}{N_1^2} \int_0^{\mathcal{D}_X^\infty} \phi(r) r\, dr = \frac{1}{N_2^2} \int_0^{\mathcal{D}_Y^\infty} \psi(u) u\, du.$$

($\mathcal{C}3$) *There exist $\kappa_1, \kappa_2 > 0$ such that*

$$\begin{aligned}
&\max\{(\kappa_1 - \phi_m)^2,\ (\phi(0) - \kappa_1)^2\} < 4\kappa_1 \phi_m, \\
&\max\{(\kappa_2 - \psi_m)^2,\ (\psi(0) - \kappa_2)^2\} < 4\kappa_2 \psi_m,
\end{aligned}$$

where $\phi_m := \phi(\mathcal{D}_X^\infty)$ and $\psi_m := \psi(\mathcal{D}_Y^\infty)$.

Then, we can observe both the asymptotic consensus and separation:

$$\mathcal{D}(X(t)),\ \mathcal{D}(Y(t)) \to 0, \quad \delta(X, Y)(t) \to \infty, \quad \text{as} \quad t \to \infty.$$

Remark 2.1. *The condition ($\mathcal{C}3$) requires ϕ and ψ to be small perturbations of certain constants κ_1 and κ_2, respectively, i.e. $\phi \approx \kappa_1$ and $\psi \approx \kappa_2$. We also remark that if $\phi \equiv \kappa_1$ and $\psi \equiv \kappa_2$, then the conditions ($\mathcal{C}2$) and ($\mathcal{C}3$) automatically hold.*

Next, we consider the one-dimensional case. Since the particles lie in the real line, the collision avoidance between two groups implies the preservation of ordering between two groups. To be precise, if we consider non-collisional initial data for system (1.1), then the initial configuration should be one of the following four types:

(2.1) init_1d

$$
\begin{aligned}
&(1)\ x_1^0 < \cdots < x_{i_1}^0 < y_1^0 < \cdots < y_{j_1}^0 < x_{i_1+1}^0 < \cdots < x_{i_2}^0 < \cdots < y_{j_p}^0 < x_{i_p+1}^0 < \cdots < x_{i_{p+1}}^0, \\
&(2)\ x_1^0 < \cdots < x_{i_1}^0 < y_1^0 < \cdots < y_{j_1}^0 < x_{i_1+1}^0 < \cdots < x_{i_2}^0 < \cdots < y_{j_p}^0, \\
&(3)\ y_1^0 < \cdots < y_{j_1}^0 < x_1^0 < \cdots < x_{i_1}^0 < y_{j_1+1}^0 < \cdots < y_{j_2}^0 < \cdots < x_{i_p}^0 < y_{j_p+1}^0 < \cdots < y_{j_{p+1}}^0, \\
&(4)\ y_1^0 < \cdots < y_{j_1}^0 < x_1^0 < \cdots < x_{i_1}^0 < y_{j_1+1}^0 < \cdots < y_{j_2}^0 < \cdots < x_{i_p}^0.
\end{aligned}
\tag{2.1}
$$

Here, we let $i_0 = 1$, $j_0 = 1$ and define sets of indices as

$$\mathcal{X}_r := \{i_{r-1}+1,\ \cdots, i_r\}, \quad \mathcal{Y}_r := \{j_{r-1}+1, \cdots, j_r\},$$

which denote the sets of indices of the r-th cluster of the X group and the Y group, respectively. In addition, we let

$$X_r := \{x_i \mid i \in \mathcal{X}_r\}, \quad Y_r := \{y_j \mid j \in \mathcal{Y}_r\}.$$

Once the collision avoidance is guaranteed, each cluster contains the same particles for any finite time, i.e. the sets of indices $\mathcal{X}_r$ and $\mathcal{Y}_r$ remain unchanged for any finite time. Now, we are ready to present our results on one-dimensional case.

⟨T2.2⟩ **Theorem 2.2.** *Let (X, Y) be a solution to system (1.1) with $d = 1$ and communication weights ϕ and ψ and initial configuration satisfy*

$$(\mathcal{A})\ \delta(X,Y)(0) > 0, \quad \text{and} \quad \phi(x)x,\ \psi(y)y \quad \text{are monotonically increasing on} \quad \mathbb{R}.$$

(1) If $\beta > 1$, we have $\delta(X, Y) > 0$ for all $t \geq 0$ and each sub-group in (1)-(4) of (2.1) accumulate to one point.

(2) Moreover, if $\beta > 2$ and two groups are not initially separated(see Remark 4.1), we get

$$\inf_{t\geq 0} \delta(X,Y)(t) > 0, \quad \sup_{t\geq 0} \mathcal{D}(X)(t) + \sup_{t\geq 0} \mathcal{D}(Y)(t) < \infty,$$

and there exists a pair (X^∞, Y^∞) such that

$$\lim_{t\to\infty} X(t) = X^\infty, \quad \lim_{t\to\infty} Y(t) = Y^\infty.$$

3. Dynamics of the opposing groups : multi-dimensional case

In this section, we provide the estimates regarding the dynamics of system (1.1) with $d \geq 2$. First, we show the collision avoidance in system (1.1) and then, we use a gradient flow-like structure in the system to show the asymptotic dynamics. Here, we note that several estimates can also be applied to one-dimensional case.

3.1. **Collision avoidance in the system.** Here, we provide the collision avoidance between two groups involved in the dyanmics of system (1.1), which also implies the well-posedness of the Cauchy problem.

⟨L3.1⟩ **Lemma 3.1.** *Let* (X, Y) *be a solution to system* (1.1) *satisfying*

$$\delta(X,Y)(0) > 0, \quad \beta \geq 2.$$

Then for any $t > 0$*, there is no collision between two groups* X *and* Y*, i.e.*

$$x_i(t) \neq y_j(t), \quad \textit{for any} \quad (i,j) \in \mathcal{N}_1 \times \mathcal{N}_2, \quad \forall t > 0.$$

Proof. We argue by contradiction, i.e. assume that there exists $T^* > 0$ such that

$$x_i(T^*) = y_j(T^*), \quad \text{for some} \quad (i,j) \in \mathcal{N}_1 \times \mathcal{N}_2.$$

Without loss of generality, we let T^* be the first collision time, i.e.

$$x_i(t) \neq y_j(t), \quad \text{for any} \quad (i,j) \in \mathcal{N}_1 \times \mathcal{N}_2, \quad \forall t < T^*.$$

Then, we fix a particle x_{i^*} that is involved in the collision at $t = T^*$ and consider the indices of particles in X and Y that collides with x_{i^*} at position $x_{i^*}(T^*)$ and time $t = T^*$:

$$\begin{gathered} I := \{i_1^*, \cdots, i_r^*\}, \quad J := \{j_1^*, \cdots, j_s^*\}, \\ \lim_{t \to (T^*)^-} x_{i^*}(t) = \lim_{t \to (T^*)^-} x_i(t) = \lim_{t \to (T^*)^-} y_j(t), \quad i \in I, \ j \in J, \\ \lim_{t \to (T^*)^-} x_{i^*}(t) \neq \lim_{t \to (T^*)^-} x_i(t), \quad \lim_{t \to (T^*)^-} x_{i^*}(t) \neq \lim_{t \to (T^*)^-} y_j(t), \quad i \in (\mathcal{N}_1 \setminus I), \ j \in (\mathcal{N}_2 \setminus J). \end{gathered}$$

Then, one can find two positive constants η_1 and η_2 satisfying

$$\begin{aligned} &\inf_{t<T^*} \min\{|x_i(t) - y_j(t)| : i \in I, \ j \in \mathcal{N}_2 \setminus J\} =: \eta_1, \\ &\inf_{t<T^*} \min\{|x_i(t) - y_j(t)| : i \in \mathcal{N}_1 \setminus I, \ j \in J\} =: \eta_2. \end{aligned} \tag{3.1}$$

C-0

Here, we set the functional $\mathcal{X}$ that measures the distance between particles involved in the collision at $x = x_{i^*}(T^*)$:

$$\mathcal{X}(t) := \left(\frac{N_2}{4N_1} \sum_{i,i' \in I} |(x_i - x_{i'})(t)|^2 + \frac{N_1}{4N_2} \sum_{j,j' \in J} |(y_j - y_{j'})(t)|^2 + \frac{1}{2} \sum_{i \in I, j \in J} |(x_i - y_j)(t)|^2 \right)^{1/2}.$$

Clearly, by definition, $\mathcal{X}(t) > 0$ for $t < T^*$. Before estimating $\mathcal{X}$, note that there exists a constant $C = C(T^*)$ such that

$$\sup_{0 \leq t < T^*} \max_{i \in \mathcal{N}_1} |x_i(t)| + \sup_{0 \leq t < T^*} \max_{j \in \mathcal{N}_2} |y_j(t)| \leq C. \tag{3.2}$$

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since the solution (X, Y) is continuous up to $t = T^*$. Now, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \sum_{i,i' \in I} |x_i - x_{i'}|^2 \\ &\quad = \frac{1}{N_1} \sum_{\substack{i,i' \in I \\ k \in \mathcal{N}_1}} \phi(x_k - x_i)(x_k - x_i)(x_i - x_{i'}) - \frac{1}{N_2} \sum_{\substack{i,i' \in I \\ \ell \in \mathcal{N}_2}} \frac{(y_\ell - x_i)}{|y_\ell - x_i|^\beta}(x_i - x_{i'}) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{N_1}\sum_{\substack{i,i'\in I\\ k\in\mathcal{N}_1}}\phi(x_k-x_{i'})(x_k-x_{i'})(x_i-x_{i'})+\frac{1}{N_2}\sum_{\substack{i,i'\in I\\ \ell\in\mathcal{N}_2}}\frac{(y_\ell-x_{i'})}{|y_\ell-x_{i'}|^\beta}(x_i-x_{i'})\\
&=\frac{2}{N_1}\sum_{\substack{i,i'\in I\\ k\in\mathcal{N}_1}}\phi(x_k-x_i)(x_k-x_i)(x_i-x_{i'})-\frac{2}{N_2}\sum_{\substack{i,i'\in I\\ \ell\in\mathcal{N}_2}}\frac{(y_\ell-x_i)}{|y_\ell-x_i|^\beta}(x_i-x_{i'})\\
&\ge-\frac{2\phi(0)C}{N_1}\sum_{\substack{i,i'\in I\\ k\in\mathcal{N}_1}}|x_i-x_{i'}|-\frac{2}{N_2}\sum_{\substack{i,i'\in I\\ \ell\in(\mathcal{N}_2\setminus J)}}\frac{|x_i-x_{i'}|}{\eta_1^{\beta-1}}-\frac{2}{N_2}\sum_{\substack{i,i'\in I\\ \ell\in J}}\frac{(y_\ell-x_i)}{|y_\ell-x_i|^\beta}(x_i-x_{i'})\\
&\ge-C(T^*)\mathcal{X}-\frac{2}{N_2}\sum_{\substack{i,i'\in I\\ \ell\in J}}\frac{(y_\ell-x_i)}{|y_\ell-x_i|^\beta}(x_i-x_{i'}),
\end{aligned}
$$

where we used (3.1), (3.2), the boundedness of ϕ and the change of index $i\leftrightarrow i'$. Similarly, one obtains

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j,j'\in J}|y_j-y_{j'}|^2\\
&\quad=\frac{1}{N_2}\sum_{\substack{j,j'\in J\\ \ell\in\mathcal{N}_2}}\psi(y_\ell-y_j)(y_\ell-y_j)(y_j-y_{j'})-\frac{1}{N_1}\sum_{\substack{j,j'\in J\\ k\in\mathcal{N}_1}}\frac{(x_k-y_j)}{|x_k-y_j|^\beta}(y_j-y_{j'})\\
&\qquad-\frac{1}{N_2}\sum_{\substack{j,j'\in J\\ \ell\in\mathcal{N}_2}}\psi(y_\ell-y_{j'})(y_\ell-y_{j'})(y_j-y_{j'})+\frac{1}{N_1}\sum_{\substack{j,j'\in J\\ k\in\mathcal{N}_1}}\frac{(x_k-y_{j'})}{|x_k-y_{j'}|^\beta}(y_j-y_{j'})\\
&\quad=\frac{2}{N_2}\sum_{\substack{j,j'\in J\\ \ell\in\mathcal{N}_2}}\psi(y_\ell-y_j)(y_\ell-y_j)(y_j-y_{j'})-\frac{2}{N_1}\sum_{\substack{j,j'\in J\\ k\in\mathcal{N}_1}}\frac{(x_k-y_j)}{|x_k-y_j|^\beta}(y_j-y_{j'})\\
&\quad\ge-C(T^*)\mathcal{X}-\frac{2}{N_1}\sum_{\substack{k\in I\\ j,j'\in J}}\frac{(x_k-y_j)}{|x_k-y_j|^\beta}(y_j-y_{j'}).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i\in I,j\in J}|x_i-y_j|^2\\
&\quad=\frac{1}{N_1}\sum_{\substack{i\in I,j\in J\\ k\in\mathcal{N}_1}}\phi(x_k-x_i)(x_k-x_i)(x_i-y_j)-\frac{1}{N_2}\sum_{\substack{i\in I,j\in J\\ \ell\in\mathcal{N}_2}}\frac{(y_\ell-x_i)}{|y_\ell-x_i|^\beta}(x_i-y_j)\\
&\qquad-\frac{1}{N_2}\sum_{\substack{i\in I,j\in J\\ \ell\in\mathcal{N}_2}}\psi(y_\ell-y_j)(y_\ell-y_j)(x_i-y_j)+\frac{1}{N_1}\sum_{\substack{i\in I,j\in J\\ k\in\mathcal{N}_1}}\frac{(x_k-y_j)}{|x_k-y_j|^\beta}(x_i-y_j)\\
&\quad\ge-C(T^*)\mathcal{X}-\frac{1}{N_2}\sum_{\substack{i\in I,j\in J\\ \ell\in\mathcal{N}_2}}\frac{(y_\ell-x_i)}{|y_\ell-x_i|^\beta}(x_i-y_j)+\frac{1}{N_1}\sum_{\substack{i\in I,j\in J\\ k\in\mathcal{N}_1}}\frac{(x_k-y_j)}{|x_k-y_j|^\beta}(x_i-y_j)
\end{aligned}
$$

$$= -C(T^*)\mathcal{X} + \mathcal{I}_1 + \mathcal{I}_2.$$

For $\mathcal{I}_1$, we have

$$\begin{aligned}
\mathcal{I}_1 &= -\frac{1}{N_2}\sum_{\substack{i\in I\\ j,\ell\in J}} \frac{(y_\ell - x_i)}{|y_\ell - x_i|^\beta}(x_i - y_j) - \frac{1}{N_2}\sum_{\substack{i\in I, j\in J\\ \ell\in(\mathcal{N}_2\setminus J)}} \frac{(y_\ell - x_i)}{|y_\ell - x_i|^\beta}(x_i - y_j)\\
&\geq -C(T^*)\mathcal{X} - \frac{1}{N_2}\sum_{\substack{i\in I\\ j,\ell\in J}} \frac{(y_\ell - x_i)}{|y_\ell - x_i|^\beta}(x_i - y_j)\\
&= -C(T^*)\mathcal{X} + \frac{1}{N_2}\sum_{\substack{i\in I\\ j,\ell\in J}} \frac{1}{|y_\ell - x_i|^{\beta-2}} - \frac{1}{N_2}\sum_{\substack{i\in I\\ j,\ell\in J}} \frac{(y_\ell - x_i)}{|y_\ell - x_i|^\beta}(y_\ell - y_j).
\end{aligned}$$

For $\mathcal{I}_2$, one obtains

$$\begin{aligned}
\mathcal{I}_2 &= \frac{1}{N_1}\sum_{\substack{i,k\in I\\ j\in J}} \frac{(x_k - y_j)}{|x_k - y_j|^\beta}(x_i - y_j) + \frac{1}{N_1}\sum_{\substack{i\in I, j\in J\\ k\in(\mathcal{N}_1\setminus I)}} \frac{(x_k - y_j)}{|x_k - y_j|^\beta}(x_i - y_j)\\
&\geq -C(T^*)\mathcal{X} + \frac{1}{N_1}\sum_{\substack{i,k\in I\\ j\in J}} \frac{(x_k - y_j)}{|x_k - y_j|^\beta}(x_i - y_j)\\
&= -C(T^*)\mathcal{X} + \frac{1}{N_1}\sum_{\substack{i,k\in I\\ j\in J}} \frac{1}{|x_k - y_j|^{\beta-2}} + \frac{1}{N_1}\sum_{\substack{i,k\in I\\ j\in J}} \frac{(x_k - y_j)}{|x_k - y_j|^\beta}(x_i - x_k).
\end{aligned}$$

Hence, we combine all the previous estimates to get

$$\begin{aligned}
&\frac{d}{dt}\mathcal{X}^2(t)\\
&= \frac{N_2}{4N_1}\frac{d}{dt}\sum_{i,i'\in I}|x_i - x_{i'}|^2 + \frac{N_1}{4N_2}\frac{d}{dt}\sum_{j,j'\in J}|y_j - y_{j'}|^2 + \frac{1}{2}\frac{d}{dt}\sum_{i\in I, j\in J}|x_i - y_j|^2\\
&\geq -\hat{C}\mathcal{X} + \frac{|J|}{N_2}\sum_{i\in I, j\in J}\frac{1}{|y_j - x_i|^{\beta-2}} + \frac{|I|}{N_1}\sum_{i\in I, j\in J}\frac{1}{|x_i - y_j|^{\beta-2}},
\end{aligned}$$

where $\hat{C} = \hat{C}(T^*)$ is a positive constant. Since $\beta \geq 2$ and $\mathcal{X} \to 0$ as $t \to (T^*)^-$, there exists $t_* < T^*$ such that

$$\frac{|J|}{N_2}\sum_{i\in I, j\in J}\frac{1}{|y_j - x_i|^{\beta-2}} + \frac{|I|}{N_1}\sum_{i\in I, j\in J}\frac{1}{|x_i - y_j|^{\beta-2}} \geq \hat{C}\mathcal{X} + \mathcal{X}^2, \quad t \in (t_*, T^*).$$

Thus, we get

$$\frac{d}{dt}\mathcal{X}^2(t) \geq \mathcal{X}^2, \quad t \in (t_*, T^*),$$

and hence,

$$\frac{d}{dt}(e^{-t}\mathcal{X}^2) \geq 0, \quad t \in (t_*, T^*).$$

This yields

$$e^{-T^*}\mathcal{X}^2(T^*) - e^{-t_*}\mathcal{X}^2(t_*) = -e^{-t_*}\mathcal{X}^2(t_*) \geq 0,$$

which is a contradiction to our assumption that $\mathcal{X}^2(t) > 0$ for $t < T^*$.

□

3.2. **Asymptotic consensus and separation for $\beta > 2$.** In this subsection, we provide the proof for Theorem 2.1. First, we prove that particles in each group stay within a certan distance from the mean positions in each group, while the distance between two groups has a positive lower bound when $\beta > 2$.

⟨L3.2⟩ **Lemma 3.2.** *Let (X, Y) be a solution to system (1.1). Assume that the conditions ($\mathcal{C}1$) and ($\mathcal{C}2$) hold. Then,*

$$\sup_{t\geq 0} \mathcal{D}(X)(t) \leq \mathcal{D}_X^\infty, \qquad \sup_{t\geq 0} \mathcal{D}(Y)(t) \leq \mathcal{D}_Y^\infty,$$

$$\inf_{t\geq 0} \delta(X,Y)(t) \geq \left((\beta-2)N_1N_2W(0)\right)^{1/(\beta-2)}.$$

Proof. Direct computation gives

$$\frac{1}{N_1}\frac{dX}{dt} = -\nabla_X W, \quad \frac{1}{N_2}\frac{dY}{dt} = -\nabla_Y W. \tag{3.3}$$

?C-1?

Thus, we directly get

$$\begin{aligned} \frac{d}{dt}W &= \nabla_X W \cdot \frac{dX}{dt} + \nabla_Y W \cdot \frac{dY}{dt} \\ &= -N_1\|\nabla_X W\|^2 - N_2\|\nabla_Y W\|^2 \leq 0, \end{aligned} \tag{3.4}$$

C-1-1

which implies

$$\sup_{t\geq 0} W(t) \leq W(0).$$

Since functionals W_X, W_Y and $\mathcal{D}^\beta$ are non-negative for $\beta > 2$, one may obtain

$$\begin{aligned} \frac{1}{N_1^2}\int_0^{\mathcal{D}_X^\infty} \phi(r)r\,dr = W(0) \\ &\geq W(t) \geq \frac{1}{2N_1^2}\sum_{i,k\in\mathcal{N}_1}\int_0^{|x_i-x_k|}\phi(r)r\,dr \\ &\geq \frac{1}{N_1^2}\int_0^{\mathcal{D}(X)(t)}\phi(r)r\,dr, \end{aligned}$$

and since ϕ is positive, this implies the uniform boundedness of $\mathcal{D}(X)$ and the similar analysis can be applied to $\mathcal{D}(Y)$.

For the uniform lower bound estimate for δ, we have

$$
\begin{aligned}
W(0) \geq W(t) &\geq \frac{1}{\beta-2}\frac{1}{N_1N_2}\sum_{i\in\mathcal{N}_1, j\in\mathcal{N}_2}|x_i-y_j|^{2-\beta} \\
&\geq \frac{1}{\beta-2}\frac{1}{N_1N_2}\delta^{2-\beta},
\end{aligned}
$$

and this directly implies our desired result.

□

Remark 3.1. *Note that relation* (3.4) *still holds when* $\beta \geq 2$. *However, the functional* W *may not be positive if* $\beta = 2$ *and hence, we can not obtain any useful information from* (3.4).

As a corollary, we can obtain the asymptotic velocity alignment in each group from the estimates in Lemma 3.2.

?⟨C3.1⟩? **Corollary 3.1.** *Let* (X, Y) *be a solution to system* (1.1) *satisfying the conditions* $(\mathcal{C}1)$ *and* $(\mathcal{C}2)$. *Then, one gets*

$$
\lim_{t\to\infty}\frac{dX}{dt} = \lim_{t\to\infty}\frac{dY}{dt} = 0.
$$

Proof. From (3.4), we get

$$
\frac{d}{dt}W + \|\nabla W\|^2 \leq 0,
$$

and we integrate this relation with respect to time to get

(3.5) C-1-2
$$
W(t) + \int_0^t \|\nabla W\|^2 ds \leq W(0).
$$

Here, for each $i \in \mathcal{N}_1$ and $j \in \mathcal{N}_2$,

$$
\begin{aligned}
|\dot{x}_i| &= \left|\frac{1}{N_1}\sum_{k\in\mathcal{N}_1}\phi(x_k-x_i)(x_k-x_i) - \frac{1}{N_2}\sum_{m\in\mathcal{N}_2}\frac{(y_\ell - x_i)}{|y_\ell - x_i|^\beta}\right| \leq \phi(0)\mathcal{D}(X) + \delta^{1-\beta}, \\
|\dot{y}_j| &= \left|\frac{1}{N_2}\sum_{\ell\in\mathcal{N}_2}\phi(y_\ell-y_j)(y_\ell-y_j) - \frac{1}{N_1}\sum_{k\in\mathcal{N}_1}\frac{(x_k - y_j)}{|x_k - y_j|^\beta}\right| \leq \psi(0)\mathcal{D}(Y) + \delta^{1-\beta},
\end{aligned}
$$

and we use the uniform upper and lower bounds guaranteed in Lemma 3.2 to get

(3.6) C-1-3
$$
\sup_{t\geq 0}\left\|\frac{dX}{dt}\right\| + \sup_{t\geq 0}\left\|\frac{dY}{dt}\right\| < \infty.
$$

Next, we also have

$$
|\ddot{x}_i| = \left|\frac{1}{N_1}\sum_{k\in\mathcal{N}_1}\Big[\left(D\phi(x_k-x_i)\cdot(\dot{x}_k-\dot{x}_i)\right)(x_k-x_i) + \phi(x_k-x_i)(\dot{x}_k-\dot{x}_i)\Big]\right.
$$

$$+\frac{\beta}{N_2}\sum_{\ell\in\mathcal{N}_2}\frac{((y_\ell-x_i)\cdot(\dot{y}_\ell-\dot{x}_i))\,(y_\ell-x_i)}{|y_\ell-x_i|^{\beta+2}}-\frac{1}{N_2}\sum_{\ell\in\mathcal{N}_2}\frac{\dot{y}_\ell-\dot{x}_i}{|y_\ell-x_i|^{\beta}}\Bigg|$$

$$\le 2\phi_{Lip}\left\|\frac{dX}{dt}\right\|\mathcal{D}(X)+2\phi(0)\left\|\frac{dX}{dt}\right\|+(\beta+1)\delta^{-\beta}\left(\left\|\frac{dY}{dt}\right\|+\left\|\frac{dX}{dt}\right\|\right),$$

where $D\phi$ denotes the gradient of ϕ and ϕ_{Lip} denotes the Lipschitz constant of ϕ. Similarly,

$$|\ddot{y}_j|\le 2\psi_{Lip}\left\|\frac{dY}{dt}\right\|\mathcal{D}(Y)+2\psi(0)\left\|\frac{dY}{dt}\right\|+(\beta+1)\delta^{-\beta}\left(\left\|\frac{dX}{dt}\right\|+\left\|\frac{dY}{dt}\right\|\right),$$

where ψ_{Lip} denotes the Lipschitz constant of ψ. Thus, one may use (3.6) to get

(3.7) C-1-4
$$\sup_{t\ge0}\left\|\frac{d^2X}{dt^2}\right\|+\sup_{t\ge0}\left\|\frac{d^2Y}{dt^2}\right\|<\infty.$$

Thus, one combines (3.6) with (3.7) to get

$$\begin{aligned}\sup_{t\ge0}\left|\frac{d}{dt}\|\nabla W\|^2\right|&=\sup_{t\ge0}\left|\frac{d}{dt}\left(\|\nabla_X W\|^2+\|\nabla_Y W\|^2\right)\right|\\&=\sup_{t\ge0}\left|\frac{d}{dt}\left(\left\|\frac{1}{N_1}\frac{dX}{dt}\right\|^2+\left\|\frac{1}{N_2}\frac{dY}{dt}\right\|^2\right)\right|\\&=\sup_{t\ge0}\left|\frac{2}{N_1^2}\frac{dX}{dt}\cdot\frac{d^2X}{dt^2}+\frac{2}{N_2^2}\frac{dY}{dt}\cdot\frac{d^2Y}{dt^2}\right|\\&\le\frac{2}{N_1^2}\sup_{t\ge0}\left\|\frac{dX}{dt}\right\|\cdot\sup_{t\ge0}\left\|\frac{d^2X}{dt^2}\right\|+\frac{2}{N_2^2}\sup_{t\ge0}\left\|\frac{dY}{dt}\right\|\cdot\sup_{t\ge0}\left\|\frac{d^2Y}{dt^2}\right\|<\infty.\end{aligned}$$

Thus, by the integrability (3.5) of $\|\nabla W\|^2$ and the uniform upper bound for $\frac{d}{dt}(\|\nabla W\|^2)$, we may use Barbalat's lemma to yield

$$\|\nabla W\|^2=\|\nabla_X W\|^2+\|\nabla_Y W\|^2=\left\|\frac{1}{N_1}\frac{d}{dt}X\right\|^2+\left\|\frac{1}{N_2}\frac{d}{dt}Y\right\|^2\to0,\quad\text{as } t\to\infty,$$

which implies our desired result.

□

3.2.1. *Proof of Theorem 2.1.* Now, we proceed to the proof of Theorem 2.1. First, we set

$$\begin{aligned}\mathcal{C}(t)=\mathcal{C}(X,Y)(t)&:=\frac{\kappa_1}{2N_1}\sum_{i\in\mathcal{N}_1}|x_i-x_c|^2+\frac{\kappa_2}{2N_2}\sum_{j\in\mathcal{N}_2}|y_j-y_c|^2\\&=\frac{\kappa_1}{2N_1}|X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c|^2+\frac{\kappa_2}{2N_2}|Y-(\mathbb{1}_{N_2}\otimes\mathbb{I}_d)y_c|^2.\end{aligned}$$

• (Step A: Asymptotic consensus): We differentiate the functional $\mathcal{C}$ with respect to t to get

$$
\begin{aligned}
\frac{d}{dt}\mathcal{C} &= \frac{\kappa_1}{N_1}(X - (\mathbb{1}_{N_1}\otimes \mathbb{I}_d)x_c)\cdot\left(\frac{dX}{dt} - (\mathbb{1}_{N_1}\otimes \mathbb{I}_d)\frac{dx_c}{dt}\right) \\
&\quad + \frac{\kappa_2}{N_2}(Y - (\mathbb{1}_{N_2}\otimes \mathbb{I}_d)y_c)\cdot\left(\frac{dY}{dt} - (\mathbb{1}_{N_2}\otimes \mathbb{I}_d)\frac{dy_c}{dt}\right),
\end{aligned}
$$

and note that

$$
\begin{aligned}
&(X - (\mathbb{1}_{N_1}\otimes \mathbb{I}_d)x_c)\cdot(\mathbb{1}_{N_1}\otimes \mathbb{I}_d)\frac{dx_c}{dt} = \sum_{i\in\mathcal{N}_1}(x_i - x_c)\cdot\frac{dx_c}{dt} = 0, \\
&(Y - (\mathbb{1}_{N_2}\otimes \mathbb{I}_d)y_c)\cdot(\mathbb{1}_{N_2}\otimes \mathbb{I}_d)\frac{dy_c}{dt} = \sum_{j\in\mathcal{N}_2}(y_j - y_c)\cdot\frac{dy_c}{dt} = 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{d}{dt}\mathcal{C} &= \frac{\kappa_1}{N_1}\frac{dX}{dt}\cdot(X - (\mathbb{1}_{N_1}\otimes \mathbb{I}_d)x_c) + \frac{\kappa_2}{N_2}\frac{dY}{dt}\cdot(Y - (\mathbb{1}_{N_2}\otimes \mathbb{I}_d)y_c) \\
&= -\kappa_1(\nabla_X W_X + \nabla_X\mathcal{D}^\beta)\cdot(X - (\mathbb{1}_{N_1}\otimes \mathbb{I}_d)x_c) \\
&\quad - \kappa_2(\nabla_Y W_Y + \nabla_Y\mathcal{D}^\beta)\cdot(Y - (\mathbb{1}_{N_2}\otimes \mathbb{I}_d)y_c).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{D}^\beta &= \nabla_X\mathcal{D}^\beta\cdot\frac{dX}{dt} + \nabla_Y\mathcal{D}^\beta\cdot\frac{dY}{dt} \\
&= -N_1\nabla_X\mathcal{D}^\beta\cdot\nabla_X W_X - N_1\|\nabla_X\mathcal{D}^\beta\|^2 \\
&\quad - N_2\nabla_Y\mathcal{D}^\beta\cdot\nabla_Y W_Y - N_2\|\nabla_Y\mathcal{D}^\beta\|^2.
\end{aligned}
$$

Then, one obtains

$$
\begin{aligned}
\frac{d}{dt}\left(\mathcal{C} + \mathcal{D}^\beta\right) &= -\kappa_1\nabla_X W_X\cdot(X - (\mathbb{1}_{N_1}\otimes \mathbb{I}_d)x_c) \\
&\quad - N_1\Big(\nabla_X\mathcal{D}^\beta + \frac{\kappa_1}{N_1}(X - (\mathbb{1}_{N_1}\otimes \mathbb{I}_d)x_c)\Big) - N_1\|\nabla_X\mathcal{D}^\beta\|^2 \\
&\quad - \kappa_2\nabla_Y W_Y\cdot(Y - (\mathbb{1}_{N_2}\otimes \mathbb{I}_d)y_c) \\
&\quad - N_2\Big(\nabla_Y\mathcal{D}^\beta + \frac{\kappa_2}{N_2}(Y - (\mathbb{1}_{N_2}\otimes \mathbb{I}_d)y_c)\Big) - N_2\|\nabla_Y\mathcal{D}^\beta\|^2 \\
&=: \sum_{r=1}^{4}\mathcal{I}_{3r}.
\end{aligned}
$$

We separately estimate $\mathcal{I}_{3r}$'s as follows.

$\diamond$ (Step A-1: Estimates for $\mathcal{I}_{31}$): We use the change of indices $i \leftrightarrow k$ to get

$$
\mathcal{I}_{31} = -\kappa_1\nabla_X W_X\cdot(X - (\mathbb{1}_{N_1}\otimes \mathbb{I}_d)x_c)
$$

$$
\begin{aligned}
&= \frac{\kappa_1}{N_1^2} \sum_{i,k \in \mathcal{N}_1} \phi(x_k - x_i)(x_i - x_k)(x_i - x_c) \\
&= -\frac{\kappa_1}{2N_1^2} \sum_{i,k \in \mathcal{N}_1} \phi(x_k - x_i)|x_k - x_i|^2 \\
&\le -\frac{\kappa_1 \phi_m}{2N_1^2} \sum_{i,k \in \mathcal{N}_1} |x_k - x_i|^2 = -\frac{\kappa_1 \phi_m}{N_1} |X - (\mathbb{1}_{N_1} \otimes \mathbb{I}_d) x_c|^2.
\end{aligned}
$$

$\diamond$ (Step A-2: Estimates for $\mathcal{I}_{32}$): First, we use Young's inequality to yield

$$
\begin{aligned}
\mathcal{I}_{32} &= -N_1 \nabla_X \mathcal{D}^\beta \cdot \Big(\frac{\kappa_1}{N_1} (X - (\mathbb{1}_{N_1} \otimes \mathbb{I}_d) x_c) + \nabla_X W_X \Big) - N_1 \|\nabla_X \mathcal{D}^\beta\|_2 \\
&\le \frac{N_1}{4} \Big\| \Big(\frac{\kappa_1}{N_1} (X - (\mathbb{1}_{N_1} \otimes \mathbb{I}_d) x_c) + \nabla_X W_X \Big) \Big\|^2.
\end{aligned} \tag{3.8}
$$

C-2

Here, we get

$$
\begin{aligned}
&\Big\| \Big(\frac{\kappa_1}{N_1} (X - (\mathbb{1}_{N_1} \otimes \mathbb{I}_d) x_c) + \nabla_X W_X \Big) \Big\|^2 \\
&= \frac{1}{N_1^4} \sum_{i \in \mathcal{N}_1} \Big| \sum_{k \in \mathcal{N}_1} (\phi(x_k - x_i) - \kappa_1)(x_k - x_i) \Big|^2 \\
&= \frac{1}{N_1^4} \sum_{n=1}^{d} \sum_{i \in \mathcal{N}_1} \Big| \sum_{k \in \mathcal{N}_1} (\phi(x_k - x_i) - \kappa_1)(x_{kn} - x_{in}) \Big|^2 \\
&= \frac{1}{N_1^2} \sum_{n=1}^{d} \sum_{i \in \mathcal{N}_1} \Big| \sum_{k \in \mathcal{N}_1} \mathcal{P}_{ik} (x_{kn} - x_{in}) \Big|^2,
\end{aligned}
$$

where x_{kn} denotes the n-th component of x_k and $\mathcal{P} = \mathcal{P}(X) \in M_{N_1 \times N_1}(\mathbb{R})$ is a matrix whose components are given as

$$
\mathcal{P}(X)_{ik} := \begin{cases} -\frac{1}{N_1} \sum\limits_{i' \in \mathcal{N}_1, i' \neq i} (\phi(x_{i'} - x_i) - \kappa_1), & \text{if } i = j, \\ \frac{1}{N_1} (\phi(x_k - x_i) - \kappa_1), & \text{if } i \neq j. \end{cases}
$$

Note that if $\{e_n\}_{n=1}^d$ is a standard orthonormal basis in $\mathbb{R}^d$, then for each $n = 1, \cdots, d$, $(e_n^T \otimes \mathbb{I}_{N_1})X$ is a vector in $\mathbb{R}^{N_1}$ whose k-th component is x_{kn}. Here, we recall that $(e_n^T \otimes \mathbb{I}_{N_1}) \in M_{N_1 \times N_1 d}(\mathbb{R})$ is written as

$$
(e_n^T \otimes \mathbb{I}_{N_1}) = \begin{pmatrix} e_n^T & & 0 \\ & \ddots & \\ 0 & & e_n^T \end{pmatrix} \in M_{N_1 \times N_1 d}(\mathbb{R}).
$$

Now, since the sum of each row in $\mathcal{P}$ is zero, one gets

$$\begin{aligned}
&\left\|\Big(\frac{1}{N_1}(X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c)+\nabla_X W_X\Big)\right\|^2 \\
&=\frac{1}{N_1^2}\sum_{n=1}^{d}\sum_{i\in\mathcal{N}_1}\left|\sum_{k\in\mathcal{N}_1}\mathcal{P}_{ik}(x_{kn}-x_{in})\right|^2=\frac{1}{N_1^2}\sum_{n=1}^{d}\sum_{i\in\mathcal{N}_1}\left|\sum_{k\in\mathcal{N}_1}\mathcal{P}_{ik}x_{kn}\right|^2 \\
&=\frac{1}{N_1^2}\sum_{n=1}^{d}\sum_{i\in\mathcal{N}_1}\left|\sum_{k\in\mathcal{N}_1}\mathcal{P}_{ik}(x_{kn}-x_{cn})\right|^2 \\
&=\frac{1}{N_1^2}\sum_{n=1}^{d}\left|\mathcal{P}(e_n^T\otimes\mathbb{I}_{N_1})(X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c)\right|^2 \\
&\le\frac{\lambda(\mathcal{P})^2}{N_1^2}\sum_{n=1}^{d}\left|(e_n^T\otimes\mathbb{I}_{N_1})(X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c)\right|^2 \\
&=\frac{\lambda(\mathcal{P})^2}{N_1^2}|X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c|^2,
\end{aligned}$$

where $\lambda(\mathcal{P})$ is the largest eigenvalue of $\mathcal{P}$ in $(\mathbb{1}_{N_1})^\perp$. Note that

$$(e_n^T\otimes\mathbb{I}_{N_1})(X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c)\cdot\mathbb{1}_{N_1}=\sum_{k\in\mathcal{N}_1}(x_{kn}-x_{cn})=0,$$

i.e. $(e_n^T\otimes\mathbb{I}_{N_1})(X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c)\in(\mathbb{1}_{N_1})^\perp$. To estimate $\lambda(\mathcal{P})$, consider any $U\in(\mathbb{1}_{N_1})^\perp$. Then, once we notice that $\mathcal{P}$ is symmetric and the sum of each row in $\mathcal{P}$ is zero, we have

$$\begin{aligned}
\langle PU,U\rangle&=\sum_{i,k=1}^{N_1}\mathcal{P}_{ik}u_iu_k=\sum_{i,k=1}^{N_1}\mathcal{P}_{ik}(u_i-u_k)u_k+\underbrace{\sum_{i,k=1}^{N_1}\mathcal{P}_{ik}u_k^2}_{=0} \\
&=-\frac{1}{2}\sum_{i\ne k}\mathcal{P}_{ik}|u_i-u_k|^2 \\
&\le\frac{1}{2N_1}\max\{|\phi(0)-\kappa_1|,|\kappa_1-\phi_m|\}\sum_{i\ne k}|u_i-u_k|^2 \\
&=\max\{|\phi(0)-\kappa_1|,|\kappa_1-\phi_m|\}|U|^2,
\end{aligned}$$

which implies

$$\lambda(\mathcal{P})\le\max\{|\phi(0)-1|,|1-\phi_m|\}.$$

Thus, we apply this result to (3.8) and get

$$\begin{aligned}
\mathcal{I}_{32}&\le\frac{N_1}{4}\left\|\Big(\frac{\kappa_1}{N_1}(X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c)+\nabla_X W_X\Big)\right\|^2 \\
&\le\frac{\lambda(\mathcal{P})^2}{4N_1}|X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c|^2
\end{aligned}$$

$$\leq \frac{\max\{|\phi(0)-\kappa_1|^2, |\kappa_1-\phi_m|^2\}}{4N_1}\big|X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c\big|^2.$$

$\diamond$ (Step A-3: Estimates for $\mathcal{I}_{33}$): Similar to the estimates for $\mathcal{I}_{33}$, one can obtain

$$\mathcal{I}_{33} \leq -\frac{\kappa_2\psi_m}{N_2}|Y-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)y_c|^2.$$

$\diamond$ (Step A-4: Estimates for $\mathcal{I}_{34}$): We employ the same analysis for $\mathcal{I}_{32}$ to get

$$\mathcal{I}_{34} \leq \frac{\max\{|\psi(0)-\kappa_2|^2, |\kappa_2-\psi_m|^2\}}{4N_2}\big|Y-(\mathbb{1}_{N_2}\otimes\mathbb{I}_d)y_c\big|^2.$$

Hence, we combine all the estimates for $\mathcal{I}_{3r}$'s to yield

$$\begin{aligned}
&\frac{d}{dt}(\mathcal{C}+\mathcal{D}^\beta)\\
&\quad\leq -\frac{1}{4N_1}\left(4\kappa_1\phi_m - \max\{|\phi(0)-\kappa_1|^2, |\kappa_1-\phi_m|^2\}\right)\big|X-(\mathbb{1}_{N_1}\otimes\mathbb{I}_d)x_c\big|^2\\
&\qquad -\frac{1}{4N_2}\left(4\kappa_2\psi_m - \max\{|\psi(0)-\kappa_2|^2, |\kappa_2-\psi_m|^2\}\right)\big|Y-(\mathbb{1}_{N_2}\otimes\mathbb{I}_d)y_c\big|^2\\
&\quad\leq -\varepsilon\mathcal{C},
\end{aligned}$$

where $\varepsilon > 0$ is defined as

$$\varepsilon := \min\left(\frac{4\kappa_1\phi_m - \max\{|\phi(0)-1|^2, |1-\phi_m|^2\}}{2\kappa_1}, \frac{4\kappa_2\psi_m - \max\{|\psi(0)-1|^2, |1-\psi_m|^2\}}{2\kappa_2}\right).$$

Thus,

$$\mathcal{C}+\mathcal{D}^\beta+\varepsilon\int_0^t \mathcal{C}(s)ds \leq \mathcal{C}(0)+\mathcal{D}^\beta(0),$$

which gives

$$\int_0^\infty \mathcal{C}(s)ds < \infty,$$

and since the uniform bounds in Lemma 3.2 imply the uniform boundedness of $\left|\frac{d}{dt}\mathcal{C}\right|$, we can use Barbalat's lemma to conclude $\mathcal{C}(t)\to 0$ as $t\to\infty$.

$\bullet$ (Step B: Separation results): First, we let

$$\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(X,Y) := \frac{1}{2N_1}\sum_{i\in\mathcal{N}_1}|x_i-x_c|^2 + \frac{1}{2N_2}\sum_{j\in\mathcal{N}_2}|y_j-y_c|^2.$$

Then, one has

$$
\begin{aligned}
\frac{d}{dt}\tilde{\mathcal{C}} &= -\frac{1}{2N_1^2}\sum_{i,k\in\mathcal{N}_1}\phi(x_k-x_i)|x_k-x_i|^2 - \frac{1}{2N_2^2}\sum_{j,\ell\in\mathcal{N}_2}\psi(y_\ell-y_j)|y_\ell-y_j|^2 \\
&\quad - \frac{1}{N_1N_2}\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}\frac{(y_j-x_i)}{|y_j-x_i|^\beta}(x_i-x_c) - \frac{1}{N_1N_2}\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}\frac{(x_i-y_j)}{|x_i-y_j|^\beta}(y_j-y_c) \\
&= -\frac{1}{2N_1^2}\sum_{i,k\in\mathcal{N}_1}\phi(x_k-x_i)|x_k-x_i|^2 - \frac{1}{2N_2^2}\sum_{j,\ell\in\mathcal{N}_2}\psi(y_\ell-y_j)|y_\ell-y_j|^2 \\
&\quad - \frac{1}{4}\frac{d}{dt}\|x_c-y_c\|^2 + (\beta-2)\mathcal{D}^\beta.
\end{aligned} \tag*{(3.9) C-6}
$$

Here, we assume for a contradiction that $\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}|x_i-y_j|^2$ is bounded. Now, we note the following identity holds:

$$
\frac{1}{N_1N_2}\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}|x_i-y_j|^2 = \frac{1}{N_1}\sum_{i\in\mathcal{N}_1}|x_i-x_c|^2 + \|x_c-y_c\|^2 + \frac{1}{N_2}\sum_{j\in\mathcal{N}_2}|y_j-y_c|^2. \tag*{(3.10) C-7}
$$

Then, one uses (3.9) and (3.10) to get

$$
\begin{aligned}
&\frac{1}{4N_1N_2}\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}|x_i-y_j|^2 \\
&\geq \frac{1}{2N_1N_2}\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}|x_i-y_j|^2 - \frac{1}{4}\|x_c-y_c\|^2 \\
&= \tilde{\mathcal{C}}(t) + \frac{1}{4}\|x_c-y_c\|^2 \\
&\geq \tilde{\mathcal{C}}(0) + \frac{1}{4}\|(x_c-y_c)(0)\|^2 - 2\max\{\phi(0),\psi(0)\}\int_0^t \tilde{\mathcal{C}}(s)ds + (\beta-2)\int_0^t \mathcal{D}^\beta(s)ds.
\end{aligned} \tag*{(3.11) C-8}
$$

Since we know $\int_0^\infty \mathcal{C}(s)ds < \infty$ from Step A, we also have

$$
\int_0^\infty \tilde{\mathcal{C}}(s)ds < \infty,
$$

and hence $\int_0^\infty \mathcal{D}^\beta(s)ds < \infty$. However, if we use the uniform bounds in Lemma 3.2, we easily obtain that $\left|\frac{d}{dt}\mathcal{D}^\beta\right|$ is uniformly bounded and hence, by Barbalat's lemma, $\mathcal{D}^\beta \to 0$ as $t\to\infty$, which is a contradiction to the boundedness of $\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}|x_i-y_j|^2$. Thus, we find out that $\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}|x_i-y_j|^2$ is unbounded.

It remains to show that $\sum_{i\in\mathcal{N}_1,j\in\mathcal{N}_2}|x_i-y_j|^2$ grows to infinity as $t\to\infty$. Now, since $\tilde{\mathcal{C}}$ is uniformly bounded, we deduce from (3.10) that $\|x_c-y_c\|$ is unbounded. Then, we use relation (3.9) again to obtain

$$\tilde{\mathcal{C}}(t) + \frac{1}{4}\|x_c - y_c\|^2 + 2\min\{\phi_m, \psi_m\}\int_0^t \tilde{\mathcal{C}}(s)ds$$
$$\leq \tilde{\mathcal{C}}(0) + \frac{1}{4}\|(x_c - y_c)(0)\|^2 + (\beta - 2)\int_0^t \mathcal{D}^\beta(s)ds.$$

and we can deduce that

$$\int_0^t \mathcal{D}^\beta(s)ds \to \infty, \quad \text{as} \quad t \to \infty.$$

Thus, we use relation (3.11) again to conclude that $\sum_{i\in\mathcal{N}_1, j\in\mathcal{N}_2} |x_i - y_j|^2 \to \infty$, and hence from (3.10), $\|x_c - y_c\| \to \infty$ as $t \to \infty$. Since we already have the asymptotic consensus result in Step A, we find out that $\delta(X,Y)(t) \to \infty$ as $t \to \infty$. This completes the proof.

4. Dynamics of the opposing groups: the one-dimensional case

?⟨sec:4⟩? In this section, we focus on the dynamics of system (1.1) for one-dimensional case. First, we prove the collision avoidance result when $\beta > 1$.

⟨L4.1⟩ **Lemma 4.1.** *Let* (X,Y) *be a solution to system* (1.1) *with* $d = 1$ *satisfying*

$$\delta(X,Y)(0) > 0, \quad \beta > 1.$$

Then for any $t > 0$*, there is no collision between two groups* X *and* Y*:*

$$\delta(X,Y)(t) > 0, \quad \forall t > 0.$$

Proof. As we did in Lemma 3.1, we use a contradiction argument. Assume that there exists $T^* > 0$ such that

$$x_i(T^*) = y_j(T^*), \quad \text{for some} \quad (i,j) \in \mathcal{N}_1 \times \mathcal{N}_2.$$

Without loss of generality, we may let T^* be the first collision time. Then, we fix a particle x_{i^*} that collides with others at $t = T^*$ and consider particles in X and Y that collide with x_{i^*} at position $x_{i^*}(T^*)$ and time $t = T^*$. Then, we can find the clusters $\{\tilde{X}_k\}_{k=1}^r$, $\{\tilde{Y}_\ell\}_{\ell=1}^s$ of X-particles and Y-particles, respectively, such that particles in each cluster collide with x_{i^*} at $t = T^*$ and for each k and ℓ,

$$\tilde{X}_k = \{x_{a_{k-1}+1}, \cdots, x_{a_k}\}, \quad \tilde{Y}_\ell = \{y_{b_{\ell-1}+1}, \cdots, y_{b_\ell}\}, \quad a_0, b_0 \in \mathbb{N} \cup \{0\},$$

$$\tilde{\mathcal{X}}_k = \{a_{k-1}+1, \cdots, a_k\}, \quad \tilde{\mathcal{Y}}_\ell = \{b_{\ell-1}+1, \cdots, b_\ell\}, \quad \mathcal{I} := \bigcup_{k=1}^r \tilde{\mathcal{X}}_k, \quad \mathcal{J} := \bigcup_{\ell=1}^s \tilde{\mathcal{Y}}_\ell.$$

and the initial ordering among clusters does not change up to $t = T^*$, i.e. either one of the following four settings holds on $t \in [0, T^*)$:

(1) $\tilde{X}_1 < \tilde{Y}_1 < \cdots < \tilde{X}_p < \tilde{Y}_p$,
(2) $\tilde{X}_1 < \tilde{Y}_1 < \cdots < \tilde{X}_p < \tilde{Y}_p < \tilde{X}_{p+1}$,
(3) $\tilde{Y}_1 < \tilde{X}_1 < \cdots < \tilde{Y}_p < \tilde{X}_p$,
(4) $\tilde{Y}_1 < \tilde{X}_1 < \cdots < \tilde{Y}_p < \tilde{X}_p < \tilde{Y}_{p+1}$.

Here, the relation $\tilde{X}_r < \tilde{Y}_r$ is defined as

$$\tilde{X}_r < \tilde{Y}_r \quad \Longleftrightarrow \quad \max_{i \in \tilde{\mathcal{X}}_r} x_i < \min_{j \in \tilde{\mathcal{Y}}_r} y_j,$$

and $\tilde{Y}_r < \tilde{X}_{r+1}$ is defined in a similar way. Due to the symmetry in system (1.1), it suffices to consider the cases (1) and (2) without loss of generality. Before we move on, note that there exist positive constants $\eta > 0$ and $C = C(T^*) > 0$ such that

$$\eta := \min \left\{ \inf_{t<T^*} |x_i - y_j| \ : \ i \in \mathcal{I}, j \in \mathcal{N}_2 \setminus \mathcal{J} \ \text{ or } \ i \in \mathcal{N}_1 \setminus \mathcal{I}, j \in \mathcal{J} \right\},$$

$$\sup_{0 \le t < T^*} \max_{i \in \mathcal{N}_1} |x_i(t)| + \sup_{0 \le t < T^*} \max_{j \in \mathcal{N}_2} |y_j(t)| \le C(T^*),$$

and we set notation for maximal, minimal indices in each cluster: for each r,

$$x_{rm} := \min_{i \in \tilde{X}_r} x_i, \quad x_{rM} := \max_{i \in \tilde{X}_r} x_i,$$

and we set y_{rm} and y_{rM} similarly.

$\diamond$ Case (1): Here, for a.e. $t \in (0, T^*)$,

$$\begin{aligned} \dot{y}_{pM} - \dot{x}_{1m} &\ge -C(T^*) - \frac{1}{N_1} \sum_{k \in \mathcal{N}_1} \frac{x_k - y_{pM}}{|x_k - y_{pM}|^\beta} + \frac{1}{N_2} \sum_{\ell \in \mathcal{N}_2} \frac{y_\ell - x_{1m}}{|y_\ell - x_{1m}|^\beta} \\ &\ge -C(T^*) + \frac{1}{N_1} \sum_{k \in \mathcal{I}} \frac{y_{pM} - x_k}{|x_k - y_{pM}|^\beta} + \frac{1}{N_2} \sum_{\ell \in \mathcal{J}} \frac{y_\ell - x_{1m}}{|y_\ell - x_{1m}|^\beta} \\ &\ge -C(T^*) + \frac{2|\mathcal{I}|}{N_1} (y_{pM} - x_{1m})^{1-\beta}, \end{aligned}$$

where we used

$$\begin{aligned} &0 < y_{pM} - x_k \le y_{pM} - x_{1m}, \quad \forall k \in \mathcal{I}, \quad \forall t \in [0, T^*), \\ &0 < y_\ell - x_{1m} \le y_{pM} - x_{1m}, \quad \forall \ell \in \mathcal{J}, \quad \forall t \in [0, T^*). \end{aligned}$$

Since $y_{pM} - x_{1m} \to 0$ as $t \to T^*$, we can easily obtain a contradiction.

$\diamond$ Case (2): In this case, we have for each $r = 1, \cdots, p$ and a.e. $t > 0$,

$$\begin{aligned} &\dot{y}_{rm} - \dot{x}_{rM} \ge -C(T^*) - \frac{1}{N_1} \sum_{k \in \mathcal{I}} \frac{x_k - y_{rm}}{|x_k - y_{rm}|^\beta} + \frac{1}{N_2} \sum_{\ell \in \mathcal{J}} \frac{y_\ell - x_{rM}}{|y_\ell - x_{rM}|^\beta} \\ &\dot{x}_{(r+1)m} - \dot{y}_{rM} \ge -C(T^*) - \frac{1}{N_2} \sum_{\ell \in \mathcal{J}} \frac{y_\ell - x_{(r+1)m}}{|y_\ell - x_{(r+1)m}|^\beta} + \frac{1}{N_1} \sum_{k \in \mathcal{I}} \frac{x_k - y_{rM}}{|x_k - y_{rM}|^\beta}. \end{aligned} \tag{4.1}$$

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Then, we sum the relations in (4.1) for $r = 1, \cdots p$ to obtain

$$\sum_{r=1}^{p} (\dot{x}_{(r+1)m} - \dot{y}_{rM} + \dot{y}_{rm} - \dot{x}_{rM})$$

$$
\begin{aligned}
&\geq -C(T^*) + \frac{1}{N_1}\sum_{r=1}^{p}\sum_{k\in\mathcal{I}}\left(\frac{x_k - y_{rM}}{|x_k - y_{rM}|^\beta} - \frac{x_k - y_{rm}}{|x_k - y_{rm}|^\beta}\right)\\
&\quad - \frac{1}{N_2}\sum_{r=1}^{p}\sum_{\ell\in\mathcal{J}}\left(\frac{y_\ell - x_{(r+1)m}}{|y_\ell - x_{(r+1)m}|^\beta} - \frac{y_\ell - x_{rM}}{|y_\ell - x_{rM}|^\beta}\right)\\
&= -C(T^*) + \frac{1}{N_1}\sum_{r=1}^{p}\sum_{k\in\mathcal{I}}\left(\frac{x_k - y_{rM}}{|x_k - y_{rM}|^\beta} - \frac{x_k - y_{rm}}{|x_k - y_{rm}|^\beta}\right)\\
&\quad + \frac{1}{N_2}\sum_{r=2}^{p}\sum_{\ell\in\mathcal{J}}\left(\frac{y_\ell - x_{rM}}{|y_\ell - x_{rM}|^\beta} - \frac{y_\ell - x_{rm}}{|y_\ell - x_{rm}|^\beta}\right)\\
&\quad - \frac{1}{N_2}\sum_{\ell\in\mathcal{J}}\left(\frac{y_\ell - x_{(p+1)m}}{|y_\ell - x_{(p+1)m}|^\beta} - \frac{y_\ell - x_{1M}}{|y_\ell - x_{1M}|^\beta}\right).
\end{aligned}
$$

Here, if x_k $(k \in \mathcal{I})$ is ahead of the cluster $\tilde{Y}_r$, i.e. $x_k > y_{rM}$, then

$$
\begin{aligned}
\frac{x_k - y_{rM}}{|x_k - y_{rM}|^\beta} - \frac{x_k - y_{rm}}{|x_k - y_{rm}|^\beta} &= (x_k - y_{rM})^{1-\beta} - (x_k - y_{rm})^{1-\beta}\\
&= (1-\beta)(x_k^*)^{-\beta}(y_{rm} - y_{rM}) > 0,
\end{aligned}
$$

where we used mean-value theorem and $x_k^* \in (x_k - y_{rM}, x_k - y_{rm})$. Similarly, for the case $x_k < y_{rm}$,

$$
\frac{x_k - y_{rM}}{|x_k - y_{rM}|^\beta} - \frac{x_k - y_{rm}}{|x_k - y_{rm}|^\beta} > 0,
$$

Moreover, similar estimates give

$$
\frac{y_\ell - x_{rM}}{|y_\ell - x_{rM}|^\beta} - \frac{y_\ell - x_{rm}}{|y_\ell - x_{rm}|^\beta} > 0, \quad \ell \in \mathcal{J}.
$$

Thus,

$$
\begin{aligned}
&\sum_{r=1}^{p}(\dot{x}_{(r+1)m} - \dot{y}_{rM} + \dot{y}_{rm} - \dot{x}_{rM})\\
&\geq -C(T^*) + \frac{1}{N_2}\sum_{\ell\in\mathcal{J}}\left(\frac{x_{(p+1)m} - y_\ell}{|y_\ell - x_{(p+1)m}|^\beta} + \frac{y_\ell - x_{1M}}{|y_\ell - x_{1M}|^\beta}\right)\\
&\geq -C(T^*) + \frac{|\mathcal{J}|}{N_2}\left((x_{(p+1)m} - y_{1m})^{1-\beta} + (y_{pM} - x_{1M})^{1-\beta}\right),
\end{aligned}
$$

and this again leads to a contradiction, which implies our desired result. □

⟨R4.1⟩ **Remark 4.1.** *Consider the case when two groups are initially separated:*

$$
\text{(4.2) [D-2]} \quad x_1^0 < \cdots < x_{N_1}^0 < y_1^0 < \cdots < y_{N_2}^0, \quad \textit{or} \quad y_1^0 < \cdots < y_{N_2}^0 < x_1^0 < \cdots < x_{N_2}^0.
$$

Without loss of generality, we assume the first case in (4.3) *and let* $\delta(X,Y)(0) > 0$ *and* $\beta > 1$*. Then, due to the collision avoidance from Lemma 4.1, we have*

$$x_M(t) := \max_{1\le i\le N_1} x_i(t) < y_m(t) := \min_{1\le j\le N_2} y_j(t),$$

and moreover, for a.e. $t > 0$,

$$\begin{aligned}\dot{y}_m - \dot{x}_M &= \frac{1}{N_2}\sum_{\ell=1}^{N_2}\psi(y_\ell - y_m)(y_\ell - y_m) - \frac{1}{N_1}\sum_{k=1}^{N_1}\frac{(x_k - y_m)}{|x_k - y_m|^\beta}\\ &\quad - \frac{1}{N_1}\sum_{k=1}^{N_1}\phi(x_k - x_M)(x_k - x_M) + \frac{1}{N_2}\sum_{\ell=1}^{N_2}\frac{(y_\ell - x_M)}{|y_\ell - x_M|^\beta}\\ &\ge \left(\frac{1}{N_1} + \frac{1}{N_2}\right)\frac{1}{|y_m - x_M|^{\beta-1}}.\end{aligned}$$

Then, we multiply both sides by $|y_m - x_M|^{\beta-1}$ to obtain

$$\frac{d}{dt}|y_m - x_M|^\beta \ge \beta\left(\frac{1}{N_1} + \frac{1}{N_2}\right),$$

and hence, we obtain the asymptotic separation result:

$$(y_m - x_M) \ge \left((y_m^0 - x_M^0)^\beta + \beta\left(\frac{1}{N_1} + \frac{1}{N_2}\right)t\right)^{1/\beta}, \quad t \ge 0.$$

⟨R4.2⟩ **Remark 4.2.** *Thanks to Lemma 4.1, we can separate particles into several clusters $\{\mathcal{X}_k\}_{k=1}^r$, $\{\mathcal{Y}_\ell\}_{\ell=1}^s$ of X-particles and Y-particles, respectively, depending on the initial configuration: If we set*

$$X_k = \{x_{a_{k-1}+1}, \cdots, x_{a_k}\}, \quad Y_\ell = \{y_{b_{\ell-1}+1}, \cdots, y_{b_\ell}\}, \quad a_0, b_0 \in \mathbb{N}\cup\{0\},$$
$$\mathcal{X}_k = \{a_{k-1}+1, \cdots, a_k\}, \quad \mathcal{Y}_\ell = \{b_{\ell-1}+1, \cdots, b_\ell\},$$

and hence,

$$\mathcal{N}_1 = \bigcup_{k=1}^r \mathcal{X}_k, \quad \mathcal{N}_2 = \bigcup_{\ell=1}^s \mathcal{Y}_\ell.$$

Then Lemma 4.1 implies that the initial ordering among clusters does not change up to any finite time and those orderings are one of the following four settings:

(1) $X_1 < Y_1 < \cdots < X_p < Y_p$,
(2) $X_1 < Y_1 < \cdots < X_p < Y_p < X_{p+1}$,
(3) $Y_1 < X_1 < \cdots < Y_p < X_p$,
(4) $Y_1 < X_1 < \cdots < Y_p < X_p < Y_{p+1}$.

From now on, due to the symmetry in system (1.1)*, we only consider the cases (1) and (2) without loss of generality.*

We know from Lemma 4.1 that asymptotic consensus in each group can not be obtained in general. Instead, we can observe that asymptotic consensus emerges in each cluster.

⟨L4.2⟩ **Lemma 4.2.** *Let (X,Y) be a solution to system (1.1) satisfying $(\mathcal{A})$ and the following assumptions:*

$$\delta(X,Y)(0) > 0, \quad \beta > 1.$$

Then, we obtain

$$\mathcal{D}(X_r(t)) \le \mathcal{D}(X_r(0)) e^{-\frac{-\phi_r^0 |\mathcal{X}_r|}{N_1} t}, \quad \mathcal{D}(Y_r(t)) \le \mathcal{D}(Y_r(0)) e^{-\frac{\psi_r^0 |\mathcal{Y}_r|}{N_2} t}, \quad t > 0,$$

where ϕ_r^0 and ψ_r^0 are given by

$$\phi_r^0 := \phi(\mathcal{D}(X_r(0))), \quad \psi_r^0 := \psi(\mathcal{D}(Y_r(0))).$$

Proof. For the r-th cluster X_r in X-group, we set

$$x_{rM} := \max_{i \in \mathcal{X}_r} x_i, \quad x_{rm} := \min_{i \in \mathcal{X}_r} x_i.$$

Then, for a.e. $t > 0$,

$$\begin{aligned} \dot{x}_{rM} - \dot{x}_{rm} &= \frac{1}{N_1} \sum_{k=1}^{N_1} \left(\phi(x_k - x_{rM})(x_k - x_{rM}) - \phi(x_k - x_{rm})(x_k - x_{rm}) \right) \\ &\quad - \frac{1}{N_2} \sum_{\ell=1}^{N_2} \left(\frac{y_\ell - x_{rM}}{|y_\ell - x_{rM}|^\beta} - \frac{y_\ell - x_{rm}}{|y_\ell - x_{rm}|^\beta} \right) \\ &=: \mathcal{I}_{31} + \mathcal{I}_{32}. \end{aligned}$$

For $\mathcal{I}_{31}$, since $x_{rM} > x_{rm}$, we get $x_k - x_{rM} \le x_k - x_{rm}$ for each $1 \le k \le N_1$ and hence,

$$\phi(x_k - x_{rM})(x_k - x_{rM}) - \phi(x_k - x_{rm})(x_k - x_{rm}) \le 0,$$

due to the monotonicity of $\phi(x)x$. Thus, we obtain

$$\begin{aligned} \mathcal{I}_{31} &\le \frac{1}{N_1} \sum_{k \in X_r} \phi(x_k - x_{rM})(x_k - x_{rM}) - \phi(x_k - x_{rm})(x_k - x_{rm}) \\ &\le -\frac{|\mathcal{X}_r|}{N_1} \phi(\mathcal{D}(X_r(t)))(x_{rM} - x_{rm}). \end{aligned}$$

For $\mathcal{I}_{32}$, if $\ell \in \mathcal{Y}_s$ such that $X_r < Y_s$, i.e. if y_ℓ is in front of X-particles in the cluster X_r, then one gets

$$\begin{aligned} &\frac{y_\ell - x_{rM}}{|y_\ell - x_{rM}|^\beta} - \frac{y_\ell - x_{rm}}{|y_\ell - x_{rm}|^\beta} \\ &= \frac{1}{(y_\ell - x_{rM})^{\beta-1}} - \frac{1}{(y_\ell - x_{rm})^{\beta-1}} \\ &= (1-\beta)(x_r^*)^{-\beta}(x_{rm} - x_{rM}) > 0, \end{aligned}$$

where we used the mean-value theorem and $x_r^* \in (y_\ell - x_{rM}, y_\ell - x_{rm})$.

On the other hand, if $\ell \in \mathcal{Y}_s$ such that $Y_s < X_r$, i.e. if y_ℓ is behind the X-particles in the cluster X_r,

$$\frac{y_\ell - x_{rM}}{|y_\ell - x_{rM}|^\beta} - \frac{y_\ell - x_{rm}}{|y_\ell - x_{rm}|^\beta}$$

$$= -\frac{1}{(x_{rM} - y_\ell)^{\beta-1}} + \frac{1}{(x_{rm} - y_\ell)^{\beta-1}}$$
$$= -(1-\beta)(x_r^{**})^{-\beta}(x_{rM} - x_{rm}) > 0,$$

where we again used the mean-value theorem and $x_r^{**} \in (x_{rm} - y_\ell, x_{rM} - y_\ell)$. Thus, we can conclude that

$$\mathcal{I}_{32} \le 0.$$

Therefore, we combine the estimates for $\mathcal{I}_{31}$ and $\mathcal{I}_{32}$ to get

(4.3) D-2
$$\dot{x}_{rM} - \dot{x}_{rm} \le -\frac{|X_r|}{N_1}\phi(\mathcal{D}(X_r(t)))(x_{rM} - x_{rm}).$$

Then, we have

$$\mathcal{D}(X_r(t)) \le \mathcal{D}(X_r(0)), \quad \forall t > 0,$$

and hence, we apply the monotonicity of ϕ and Grönwall's lemma to (4.3) to obtain the desired estimate. Since the estimates for $\mathcal{D}(Y_r)$ are similar, we omit the proof. □

Next, we show that if two groups are not initially separated (i.e. the case of Remark 4.1), then they are confined in a finite interval.

⟨L4.4⟩ **Lemma 4.3.** *Let* (X, Y) *be a solution to system* (1.1) *satisfying* $(\mathcal{A})$ *and the following assumptions:*

$$\delta(X,Y)(0) > 0, \quad \beta > 2.$$

Moreover, assume that the initial configuration (4.3) *is excluded. Then,*

$$\sup_{t\ge 0}\mathcal{D}(X) + \sup_{t\ge 0}\mathcal{D}(Y) < \infty.$$

Proof. Here, we may use Lemma 3.2 to obtain that

$$W(t) \le W(0),$$

which yields $\inf_{t\ge 0}\delta(X,Y)(t) > 0$ and

$$W(0) \ge \frac{1}{2N_1^2}\sum_{i,k\in\mathcal{N}_1}\Phi(|x_i - x_k|) + \frac{1}{2N_2^2}\sum_{j,\ell\in\mathcal{N}_2}\Psi(|y_j - y_\ell|)$$
$$\ge \frac{1}{2N_1^2}\Phi(\mathcal{D}(X)) + \frac{1}{2N_2^2}\Psi(\mathcal{D}(Y)).$$

Since we have assumed $\phi(x)x$ and $\psi(x)x$ are monotonically increasing in x, it is obvious that $\Phi(x)$ and $\Psi(x)$ monotonically increases to infinity. Thus, this implies the desired result. □

?⟨C4.1⟩? **Corollary 4.1.** *Let* (X, Y) *be a solution to system* (1.1) *satisfying* $(\mathcal{A}1)$-$(\mathcal{A}2)$ *and the following assumptions:*

$$\delta(X,Y)(0) > 0, \quad \beta > 2.$$

Moreover, assume that the initial configuration (4.3) *is excluded. Then, there exists a bounded interval* K *such that*

$$x_i(t), y_j(t) \in K, \quad \textit{for any} \quad (i,j) \in \mathcal{N}_1 \times \mathcal{N}_2 \quad \textit{and any} \quad t > 0.$$

Proof. We argue by contradiction. Since the other cases are similar, we only consider case (1) in Remark 4.2. In this case, the following inequality holds for all $t \geq 0$:

$$2x_{1m}(t) \leq x_c(t) + y_c(t) \leq 2y_{pM}(t). \tag{4.4}$$

C4-1.1

Together with the results in Lemma 4.3, the nonexistence of such an interval K implies either one of the followings hold:

$$\text{either} \quad \liminf_{t\to\infty} x_{pM}(t) = -\infty, \quad \text{or} \quad \limsup_{t\to\infty} x_{1m}(t) = \infty.$$

Here, due to the collision avoidance between the two groups X and Y, we have

$$\begin{cases} \liminf_{t\to\infty} x_{pM}(t) = -\infty \implies \liminf_{t\to\infty} y_j(t) = -\infty, \quad \forall j \in Y_r \quad (1 \leq r < p) \\ \limsup_{t\to\infty} x_{1m}(t) = \infty \quad \implies \limsup_{t\to\infty} y_j(t) = \infty, \quad \forall j \in Y_r \quad (1 \leq r \leq p). \end{cases}$$

Moreover, since $\mathcal{D}(Y)$ is uniformly bounded,

$$\liminf_{t\to\infty} x_{pM}(t) = -\infty \quad \implies \quad \liminf_{t\to\infty} y_{pM}(t) = -\infty.$$

Thus, the inequality (4.4) yields

$$\begin{cases} \liminf_{t\to\infty} x_{pM}(t) = -\infty \implies \liminf_{t\to\infty} \Big(x_c(t) + y_c(t)\Big) = -\infty, \\ \limsup_{t\to\infty} x_{1m}(t) = \infty \quad \implies \limsup_{t\to\infty} \Big(x_c(t) + y_c(t)\Big) = \infty. \end{cases}$$

However, this contradicts Lemma 2.1, since $x_c(t)+y_c(t)$ is a conserved quantity along time. Therefore, such a finite interval K exists.

□

4.0.1. *Proof of Theorem 2.2.* Thanks to Lemma 3.2 and Lemma 4.2, it suffices to show the convergence toward asymptotic limits. For simplicity, we set $W(t) := W(X(t), Y(t))$. First, we recall from Lemma 3.2 that

$$\frac{d}{dt}W(t) = -N_1\|\nabla_X W(t)\|^2 - N_2\|\nabla_Y W(t)\|^2 \leq -\|\nabla W(t)\|^2.$$

Since $W(X(t), Y(t))$ is nonnegative and monotonically decreasing, we can find $W^\infty \geq 0$ satisfying

$$\lim_{t\to\infty} W(t) = W^\infty.$$

Moreover, we have

$$\int_{t_0}^\infty \|\nabla W(t)\|^2 dt \leq -\int_{t_0}^\infty \frac{d}{dt}W(t)dt = W(t_0) - W^\infty.$$

Together with $\inf_{t\geq 0} \delta(X(t), Y(t)) > 0$ and Corollary 4.1, we have $\nabla W(t)$ is uniformly continuous. Thus,

$$\lim_{t\to\infty} \nabla W(X(t), Y(t)) = 0.$$

On the other hand, since $(X(t),Y(t))$ is uniformly bounded, we can find a sequence $t_n \nearrow \infty$ and $(X^\infty, Y^\infty) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ such that

$$\lim_{n\to\infty} (X(t_n), Y(t_n)) = (X^\infty, Y^\infty).$$

Hence, we have

$$0 = \lim_{n\to\infty} \|\nabla W(t_n)\| = \|\nabla W(X^\infty, Y^\infty)\|.$$

Here, note that the potential W is analytic on the following region:

$$\mathcal{O} := \{(x_1, \cdots, x_{N_1}, y_1, \cdots, y_{N_2}) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \mid x_i \neq y_j \quad \text{for all} \quad (i,j) \in \mathcal{N}_1 \times \mathcal{N}_2\}.$$

So, Lemma 2.2 guarantees the existence of positive constants $\eta \in [1/2, 1)$, $c, r > 0$ such that

$$\|\nabla W(X,Y)\| \geq c|W(X,Y) - W^\infty|^\eta, \quad \forall (X,Y) \in B((X^\infty, Y^\infty), r), \tag{4.5}$$

D-4

where $B((X,Y),r)$ denotes the ball of radius r in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ centered at $(X,Y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

Our claim is to show that, for any $0 < \varepsilon < r$, there exists $t^\varepsilon > 0$ such that

$$(X(t), Y(t)) \in B((X^\infty, Y^\infty), \varepsilon), \quad \forall t \geq t^\varepsilon,$$

which implies the desired result. Now, we fix $\varepsilon < r$ and let $g(t) := |W(t) - W^\infty|^{1-\eta}$. Since $g(t) \searrow 0$ as $t \to \infty$ and g is uniformly continuous, we can choose $t^\varepsilon > 0$ satisfying

$$\frac{N_1 N_2}{c(1-\eta)}(g(t) - g(t^\varepsilon)) < \frac{\varepsilon}{2}, \quad \forall t \geq t^\varepsilon,$$

where c is given in (4.5). Due to the sequence $\{t_n\}_{n\in\mathbb{N}}$, we may choose t^ε sufficiently large so that

$$|(X(t^\varepsilon), Y(t^\varepsilon)) - (X^\infty, Y^\infty)| < \frac{\varepsilon}{2}.$$

Now, we set

$$T^\varepsilon := \sup\{t \geq t^\varepsilon \mid (X(t), Y(t)) \in B((X^\infty, Y^\infty), \varepsilon)\}.$$

We claim that $T^\varepsilon = \infty$. Assume for a contradiction that $T^\varepsilon < \infty$. Then, we note that

$$\frac{d}{dt} g(t) = (1-\eta)|W(t) - W^\infty|^{-\eta} \frac{d}{dt} W(t) \leq -c(1-\eta)|W(t) - W^\infty|^{-\eta} \|\nabla W(t)\|^2. \tag{4.6}$$

D-5

Then, for $t \in [t^\varepsilon, T^\varepsilon]$, we use (4.5) to get

$$\|\nabla W(t)\| \leq -\frac{dg}{dt}\left(\frac{1}{(1-\eta)|W(t) - W^\infty|^{-\eta}\|\nabla W\|}\right) \leq -\frac{1}{c(1-\eta)}\frac{dg}{dt}.$$

Thus, we obtain

$$\begin{aligned} \int_{t^\varepsilon}^{T^\varepsilon} \left| \frac{d}{dt} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \right| dt &= \int_{t^\varepsilon}^{T^\varepsilon} \left| \begin{pmatrix} N_1 \nabla_X W(t) \\ N_2 \nabla_Y W(t) \end{pmatrix} \right| dt \\ &\leq N_1 N_2 \int_{t^\varepsilon}^{T^\varepsilon} \|\nabla W(t)\| dt \leq \frac{N_1 N_2}{c(1-\eta)} (g(t^\varepsilon) - g(T^\varepsilon)) < \frac{\varepsilon}{2}. \end{aligned}$$

This implies

$$\begin{aligned}\varepsilon &= |(X(T^\varepsilon), Y(T^\varepsilon)) - (X^\infty, Y^\infty)| \\ &\le |(X(T^\varepsilon), Y(T^\varepsilon)) - (X(t^\varepsilon), Y(t^\varepsilon))| + |(X(t^\varepsilon), Y(t^\varepsilon)) - (X^\infty, Y^\infty)| \\ &\le \int_{t^\varepsilon}^{T^\varepsilon} \left| \frac{d}{dt} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \right| dt + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

which is a contradiction. Thus $T^\varepsilon = \infty$ and hence, we can obtain our desired convergence:

$$\lim_{t\to\infty} (X(t), Y(t)) = (X^\infty, Y^\infty).$$

Remark 4.3. *The same argument as the proof of Theorem 2.2 implies that there exists $t^* > 0$ such that*

$$\frac{N_1 N_2}{c(1-\eta)}(g(t) - g(t^*)) < \frac{r}{2}, \quad \forall t \ge t^*, \quad \text{and} \quad |(X(t^*), Y(t^*)) - (X^\infty, Y^\infty)| < \frac{r}{2}.$$

Then, the contradiction argument yields

$$T := \sup\{t \ge t^* \mid (X(t), Y(t)) \in B((X^\infty, Y^\infty), r)\} = \infty,$$

and hence, we can combine (4.5) *with* (4.6) *to get*

$$\begin{aligned}\frac{dg}{dt} &\le -(1-\eta)|W(t) - W^\infty|^{-\eta} \|\nabla W(t)\|^2 \\ &\le -c^2(1-\eta)|W(t) - W^\infty|^\eta = -c^2(1-\eta) g^{\frac{\eta}{1-\eta}}, \quad t \ge t^*,\end{aligned}$$

where c is given in (4.5).

If $\eta = 1/2$, then we may use Grönwall's lemma to obtain

$$g(t) \le g(t^*) e^{-c^2(1-\eta)(t-t^*)}, \quad t \ge t^*.$$

If $\eta \in (1/2, 1)$, we can get

$$g^{-\frac{\eta}{1-\eta}} \frac{dg}{dt} = \frac{1-\eta}{1-2\eta} \frac{d}{dt} g^{\frac{1-2\eta}{1-\eta}} \le -c^2(1-\eta), \quad t \ge t^*,$$

which implies

$$g(t) \le \left(g(t^*)^{-\frac{2\eta-1}{1-\eta}} + c^2(2\eta-1)(t-t^*) \right)^{-\frac{1-\eta}{2\eta-1}}, \quad t \ge t^*.$$

In either case, we have $g(t) \to 0$ as $t \to \infty$ at least at an algebraic rate. Thus,

$$\begin{aligned}|(X(t), Y(t)) - (X^\infty, Y^\infty)| &\le \int_t^\infty \left| \frac{d}{ds} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} \right| ds \\ &\le N_1 N_2 \int_t^\infty \|\nabla W(s)\| ds \le \frac{N_1 N_2}{c(1-\eta)} g(t), \quad t \ge t^*,\end{aligned}$$

which implies that after some time, the convergence rate toward the asymptotic limits is at least algebraic.

5. Numerical simulations

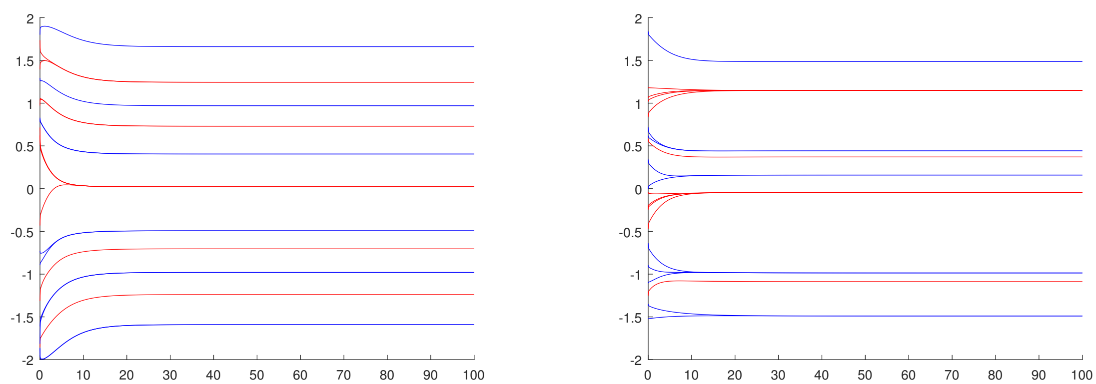
?⟨sec:5⟩? In this section, we present several results from numerical simulations concerning system (1.1). Since the dynamics observed in (1.1) appears different depending on the dimension, we provide the results for numerical simulations separately. For any dimension, we employed the fourth-order Runge-Kutta method.

5.1. **The one-dimensional case.** In this subsection, we provide the results of numerical simulations for the one-dimensional case. Notable situations in the one-dimensional case are the order preservation between clusters and the existence of asymptotic limits. We would like to observe such phenomena varying the choice of communication weight functions ϕ and ψ, and the singularity exponent β. For our numerical simulations, we choose the following type of communication weight functions:

$$\phi(x) = \frac{1}{(1+x^2)^{\alpha_1}}, \quad \psi(x) = \frac{1}{(1+x^2)^{\alpha_2}}, \quad \alpha_1, \alpha_2 > 0.$$

Note that if $\beta > 1$ and $\alpha_1, \alpha_2 \in (0, 1/2]$, they satisfy all the conditions in (1) of Theorem 2.2, while $\beta > 2$ and $\alpha_1, \alpha_2 \in (0, 1/2]$ coincide with the conditions in (2).

Now, we consider the case that satisfies our sufficient framework given in Theorem 2.2 and the cases that do not. Note that the red line denotes the position of particles in X-group, while the blue line denotes the position of particles in Y-group. In Figure 1(a), the coefficients satisfy the condition and we observe both the consensus in each cluster and convergence toward asymptotic limits. However, even for the case $\beta = 1.5$ (see Figure 1(b)), we can observe both phenomena, where only the consensus in each cluster is guaranteed.



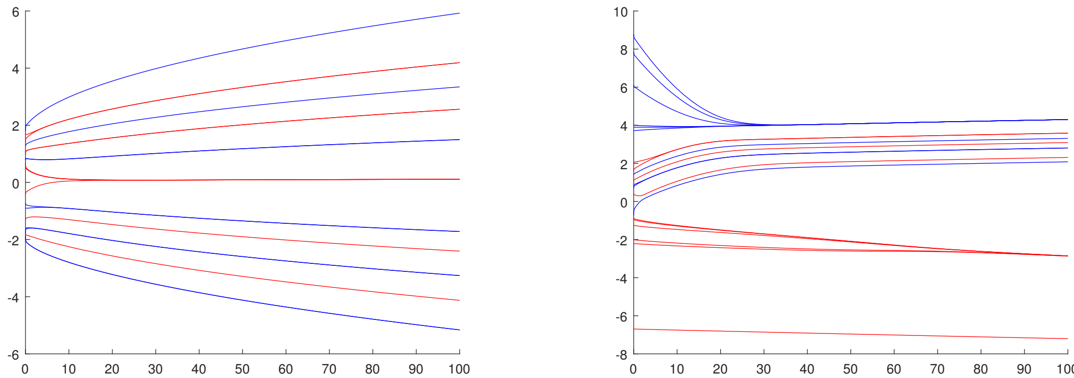
(a) $\alpha_1 = \alpha_2 = 0.25$, $\beta = 2.5$ (b) $\alpha_1 = \alpha_2 = 0.25$, $\beta = 1.5$

Figure 1. Consensus in each cluster and convergence toward the asymptotic limits under the sufficient and non-sufficient conditions

?⟨F1⟩?

On the other hand, in the case of $\alpha_1 = \alpha_2 = 1.5$ with $\beta = 2.5$ (see Figure 2(a)), it does not satisfy any of our condition and the boundedness $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ and convergence toward asymptotic limits are not observed. However, we can observe the consensus in each cluster. When $\alpha_1 = 5$, $\alpha_2 = 0.25$ and $\beta = 2.5$, the consensus in the Y-cluster is observed.

As we may observe in Figure 2(b), the consensus in each cluster of Y emerges while the consensus in some clusters of X does not.



(a) $\alpha_1 = \alpha_2 = 1.5$, $\beta = 2.5$ (b) $\alpha_1 = 5$, $\alpha_2 = 0.25$, $\beta = 2.5$

FIGURE 2. Emergence and non-emergence of phenomena under non-sufficient conditions

?⟨F2⟩?

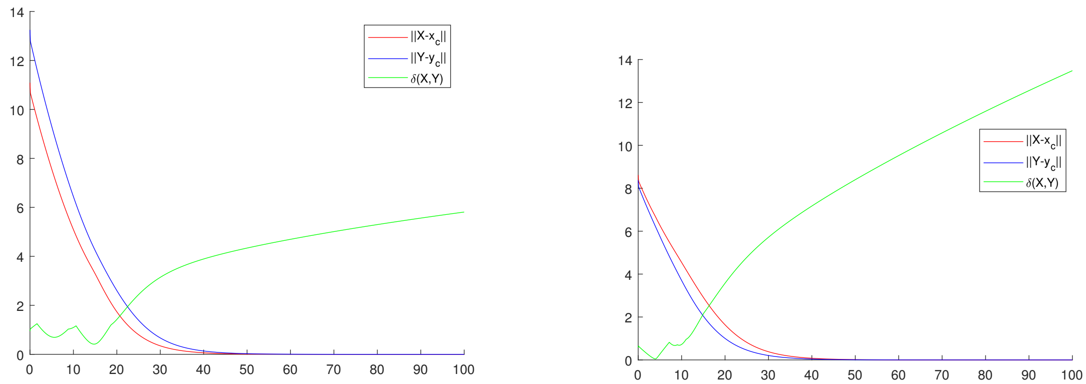
5.2. **The multi-dimensional case.** In this subsection, we consider the multi-dimensional case. Here, we only consider $d = 2$. As we did for the one-dimensional case, we would like to observe the phenomena varying communication weights and the singularity exponent. Here, we also use the same type of communication weights:

$$\phi(x) = \frac{1}{(1+|x|^2)^{\alpha_1}}, \quad \psi(x) = \frac{1}{(1+|x|^2)^{\alpha_2}}, \quad \alpha_1, \alpha_2 > 0.$$

Unlike the one-dimensional case, now we can expect the emergence of asymptotic consensus and separation. We will use the same parameters as the one-dimensional case and discuss what is different and what is similar. For the quantities in figures, we write

$$\|X - x_c\| := \left(\sum_{i=1}^{N_1} |x_i - x_c|^2 \right)^{1/2}, \quad \|Y - y_c\| := \left(\sum_{j=1}^{N_2} |y_j - y_c|^2 \right)^{1/2}.$$

First, when $\alpha_1 = \alpha_2 = 0.25$ and $\beta = 2.5$, we can observe the emergence of asymptotic consensus and separation, which is also observed in the case $\alpha_1 = \alpha_2 = 0.25$ and $\beta = 1.5$ (see Figure 3(a) and 3(b)), although our condition does not guarantee the collision avoidance when $\beta = 1.5$,

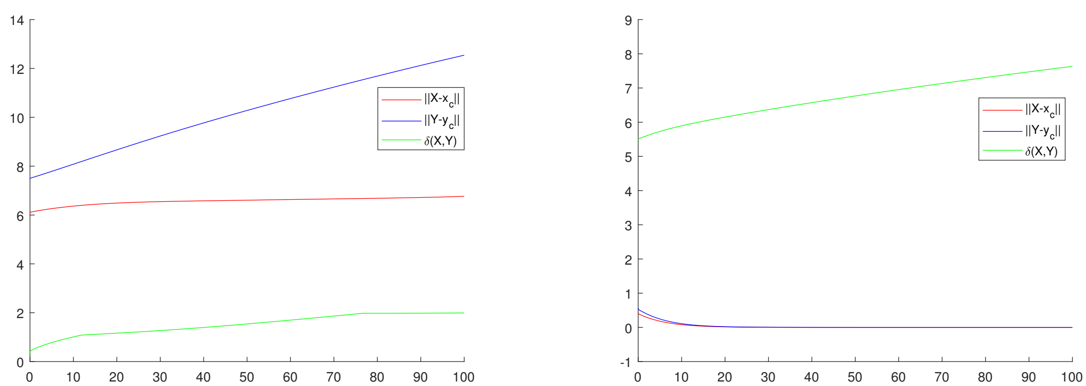


(a) $\alpha_1 = \alpha_2 = 0.25$, $\beta = 2.5$ (b) $\alpha_1 = \alpha_2 = 0.25$, $\beta = 1.5$

FIGURE 3. Emergence of asymptotic consensus and separation

?⟨F4⟩?

However, when $\alpha_1 = \alpha_2 = 1.5$ and $\beta = 2.5$, the behavior of system (1.1) becomes different. When the initial data is large, then only the collision avoidance is observed (see Figure 4(a)). On the other hand, when the initial data is small, asymptotic consensus and separation seem to emerge as depicted in Figure 4(b).

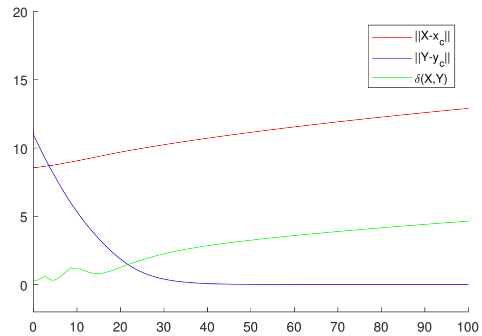


(a) $\alpha_1 = \alpha_2 = 1.5$, $\beta = 2.5$ with large initial data (b) $\alpha_1 = \alpha_2 = 1.5$, $\beta = 2.5$ with small initial data

FIGURE 4. Emergent behaviors depending on the choice of initial data

?⟨F5⟩?

Finally, we present a case where partial asymptotic consensus emerges, i.e. one group reaches an asymptotic consensus while the other does not. When $\alpha_1 = 5$, $\alpha_2 = 0.25$ and $\beta = 2.5$, we can see the same result without the smallness condition on initial data (see Figure 5(a)).



(a) $\alpha_1 = 5$, $\alpha = 0.25$, $\beta = 2.5$

?⟨F6⟩? FIGURE 5. Partial consensus

6. CONCLUSION

?⟨sec:5⟩? In this paper, we discussed the asymptotic dynamics of two opposing groups under attractive and repulsive forces. Due to the difference between the one-dimensional and multi-dimensional cases, we first considered the multi-dimensional case. Here, under a suitable assumption for the singularity exponent in the repulsion force, we proved the collision avoidance between two groups. Furthermore, with suitable conditions on system parameters, initial data and communication weights, we proved the emergence of both asymptotic consensus and separation. In the one-dimensional case, due to collision avoidance, particles are separated into several clusters by the particles from the other group. Moreover, the order between these clusters is preserved along time, which hinders the asymptotic consensus in general. Moreover, we showed that each cluster reaches a consensus and tends to an asymptotic limit.

Still, there are many interesting issues to be discussed concerning the dynamics of opposing groups. For example, the dynamics of multiple opposing groups, opposing groups governed by second-order systems communicating via singular kernel, etc. These will be treated in future works.

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(Jinwook Jung)
Department of Mathematical Sciences,
Seoul National University, Seoul, 08826, Republic of Korea
E-mail address: warp100@snu.ac.kr

(Shi Jin)
School of Mathematical Sciences, Institute of Natural Sciences and MOE-LSC,
Shanghai Jiao Tong University, Shanghai 200240, China
E-mail address: shijin-m@sjtu.edu.cn