

A LOCAL SENSITIVITY ANALYSIS FOR THE KINETIC KURAMOTO EQUATION WITH RANDOM INPUTS

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ABSTRACT. We present a local sensitivity analysis for the kinetic Kuramoto equation with random inputs in a large coupling regime. In our proposed random kinetic Kuramoto equation (in short, RKKE), the random inputs are encoded in the coupling strength. For the deterministic case, it is well known that the kinetic Kuramoto equation exhibits asymptotic phase concentration for well-prepared initial data in the large coupling regime. To see a response of the system to the random inputs, we provide propagation of regularity, local-in-time stability estimates for the variations of the random kinetic density function in random parameter space. For identical oscillators with the same natural frequencies, we introduce a Lyapunov functional measuring the phase concentration, and provide a local sensitivity analysis for the functional.

1. Introduction. Collective behaviors of oscillatory complex systems are ubiquitous in our nature, e.g., flashing of fireflies, chorusing of crickets, synchronous firing of cardiac pacemaker and metabolic synchrony in yeast cell suspension [1, 7, 15, 32, 33] etc. Aforementioned collective patterns come down to synchronization phenomena. The jargon “*synchronization*” represents the adjustment of rhythms in an ensemble of weakly coupled oscillators. Compared to long human history, a rigorous treatment for synchronization started only several decades ago in the pioneering works by Kuramoto and Winfree in [25, 26, 37]. They introduced simple, continuous dynamical systems for weakly coupled oscillators, and explained

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how collective behaviors in such simple models can emerge from initial configurations. Recently, emergent dynamics of coupled oscillators on networks has become an active, emerging research field in diverse disciplines such as biology, nonlinear dynamics, statistical physics and sociology. After Kuramoto and Winfree's seminal works, many phenomenological models have been used in the study of synchronization. Among them, we are mainly interested in the prototype model, namely the Kuramoto model. In order to fix the idea, let z be a random vector which is a measurable vector-valued function on a sample space Ω in \mathbb{R}^d and we denote its probability density function (pdf) by $\pi(z)$. For the simplicity of presentation, we assume that z is one-dimensional. Due to the randomness in system parameters such as the natural frequency and coupling strength, the phases of oscillators are random processes as well. More precisely, let $\theta_i = \theta_i(t, z)$ be the phase of the i -th Kuramoto oscillator whose dynamics is governed by the random Kuramoto model (in short, RKM):

$$\partial_t \theta_i(t, z) = \nu_i + \frac{1}{N} \sum_{j=1}^N \kappa_{ij}(z) \sin(\theta_j(t, z) - \theta_i(t, z)), \quad t > 0, \quad (1)$$

where ν_i is an intrinsic natural frequency of the i -th oscillator whose pdf is $g = g(\nu)$, and nonnegative random field $\kappa_{ij} = \kappa_{ij}(z)$ measures the random coupling strength between i and j -th oscillators. Throughout the paper, we assume that κ_{ij} is symmetric in i and j :

$$\kappa_{ij} = \kappa_{ji}, \quad 1 \leq i, j \leq N.$$

For the deterministic case where all randomness were quenched, i.e.,

$$\nu_i = \text{constant}, \quad \kappa_{ij}(z) = \kappa_{ij},$$

emergent dynamics of (1) has been extensively studied in [2, 6, 10, 11, 12, 13, 14, 20, 23, 28, 29, 30, 34, 35, 36] where the complete synchronization and stability conditions were proposed. The pathwise well-posedness of (1) can be done using the standard Cauchy-Lipschitz theory. In authors' recent work [18], a local sensitivity analysis for (1) has been addressed. In this paper, we are interested in the corresponding mean-field equation which can effectively describe the dynamics of system (1) with $N \gg 1$. Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and $f = f(t, \theta, \nu, z)$ be an one-oscillator probability density function on the extended phase space $\mathbb{T} \times \mathbb{R} \times \Omega$ at time t . Then, the density function f satisfies the RKKE [1, 27]:

$$\begin{aligned} \partial_t f + \partial_\theta(\omega[f]f) &= 0, \quad (\theta, \nu, z) \in \mathbb{T} \times \mathbb{R} \times \Omega, \quad t > 0, \\ \omega[f](t, \theta, \nu, z) &= \nu - \kappa(z) \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \theta_*) f(t, \theta_*, \nu_*, z) d\nu_* d\theta_*. \end{aligned} \quad (2)$$

When the extrinsic randomness in $\kappa(z)$ and initial data $f^0(z)$ are quenched, well-posedness and dynamic features of the kinetic Kuramoto equation have been studied in [4, 5, 8, 27].

In this paper, we address a local sensitivity analysis for (2) to see the effect of random parameter in $f = f(t, \theta, \nu, z)$ which is one of topics in uncertainty quantification (UQ). More precisely, we study the propagation of regularity of f in the random space, and provide concentration and stability estimates of z -variations $\{\partial_z^\alpha f\}$. The UQ for mean-field flocking models were first addressed in [3, 9] where particle based gPC methods were discussed. On the other hand, systematic local sensitivity analysis for the Cucker-smale and Kuramoto models have been addressed

in authors' series of recent works [16, 17, 18]. Thus, the current work is a continuation of this systematic research on the local sensitivity analysis for flocking and synchronization models.

The main results of this paper are two-fold. First, we present pathwise well-posedness and stability estimate of the RKKE by establishing a priori estimates (see Theorem 3.2, Theorem 3.3 and Theorem 3.4): for $T \in (0, \infty)$, $z \in \Omega$, $l \leq k$,

$$\begin{aligned} \sup_{0 \leq t < T} \|\partial_z^l f(t, z)\|_{W_{\theta, \nu}^{k-l, \infty}} &\leq C(z, T), \quad \sup_{0 \leq t < T} \|f(t)\|_{H_\pi^l(L_{\theta, \nu}^\infty)} \leq C(T) \|f^0\|_{H_\pi^l(L_{\theta, \nu}^\infty)}, \\ \sup_{0 \leq t < T} \sum_{l=0}^k \|\partial_z^l (f - \tilde{f})(t, z)\|_{W^{k-l, \infty}} &\leq C(z, T) \sum_{l=0}^k \|\partial_z^l (f^0 - \tilde{f}^0)(z)\|_{W^{k-l, \infty}}, \end{aligned}$$

where f and \tilde{f} are solution processes to (2) corresponding to initial data f^0 and \tilde{f}^0 , respectively.

Second, we consider identical oscillator with $\varphi(\nu) = \delta_0$. In this case, we can write $f(t, \nu, \theta, z) = \rho(t, \theta, z)\varphi(\nu)$ and ρ satisfies

$$\begin{aligned} \partial_t \rho + \partial_\theta(\tilde{\omega}[\rho]\rho) &= 0, \quad (\theta, \nu, z) \in \mathbb{T} \times \mathbb{R} \times \Omega, \quad t > 0, \\ \tilde{\omega}[\rho](t, \theta, z) &= -\kappa(z) \int_0^{2\pi} \sin(\theta - \theta_*) \rho(t, \theta_*, z) d\theta_*. \end{aligned}$$

Now, we introduce a Lyapunov functional \mathcal{L} measuring the phase concentration:

$$\mathcal{L}[\rho](t, z) := \int_{\mathbb{T}} |\theta - \theta_{\rho, c}(t, z)|^2 \rho(t, \theta, z) d\theta, \quad \theta_{\rho, c}(t, z) := \int_{\mathbb{T}} \theta \rho(t, \theta, z) d\theta. \quad (3)$$

Note that the zero convergence of $\mathcal{L}[\rho]$ as $t \rightarrow \infty$ implies the asymptotic formation of phase concentration in probability sense. This can be seen easily from Chebyshev inequality as follows:

$$\mathcal{L}[\rho](t, z) \geq \varepsilon^2 \int_{|\theta - \theta_{\rho, c}(t, z)| > \varepsilon} \rho d\theta = \varepsilon^2 \mathbb{P}[|\theta - \theta_{\rho, c}(t, z)| > \varepsilon].$$

This implies

$$\lim_{t \rightarrow \infty} \mathbb{P}[|\theta - \theta_{\rho, c}(t, z)| > \varepsilon] \leq \frac{1}{\varepsilon^2} \lim_{t \rightarrow \infty} \mathcal{L}[\rho](t, z) = 0.$$

Under suitable conditions on initial data and system parameters, we will show that there exists random functions $C(z)$ and $\Lambda(z)$ such that

$$\mathcal{L}[|\partial_z \rho|](t, z) \leq C(z) e^{-\Lambda(z)t}, \quad t \geq 0.$$

(see Theorem 4.3 for details). However, higher-order sensitivity analysis for $\mathcal{L}[|\partial_z^\alpha \rho|](t, z)$ with $\alpha \gg 1$ might lead to the exponential growth (see Remark 4), a phenomenon not observed in earlier works in this direction.

The rest of this paper is organized as follows. In Section 2, we briefly introduce the random kinetic Kuramoto equation and discuss its basic properties. In Section 3, we study a pathwise well-posedness of the RKKE by providing a priori estimates such as boundedness of $H_\pi^l(L_{\theta, \nu}^\infty)$ in any finite time interval, and provide the stability estimates for the RKKE. In Section 4, we perform a local sensitivity analysis for a Lyapunov functional (3). Finally, Section 5 is devoted to a brief summary of our main results and some remaining issues to be explored in future. In Appendices A and B we provide the proof for Theorem 3.2 and Lemma 4.2, respectively.

Gallery of Notation: Throughout the paper, we use the following notation:

Let $\pi : \Omega \rightarrow \mathbb{R}_+ \cup \{0\}$ be a nonnegative p.d.f. function, and let $y = y(z)$ be a scalar-valued random function defined on Ω . Then, we define the expected value as

$$\mathbb{E}[\varphi] := \int_{\Omega} \varphi(z)\pi(z)dz,$$

and a weighted L^2 -space:

$$L_{\pi}^2(\Omega) := \{y : \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} |y(z)|^2 \pi(z) dz < \infty\},$$

with an inner product and norm:

$$\langle y_1, y_2 \rangle_{L_{\pi}^2(\Omega)} := \int_{\Omega} y_1(z)y_2(z)\pi(z)dz, \quad \|y\|_{L_{\pi}^2(\Omega)} := \left(\int_{\Omega} |y(z)|^2 \pi(z) dz \right)^{\frac{1}{2}}.$$

For $k \in \mathbb{Z}_+ \cup \{0\}$, set

$$\|y\|_{H_{\pi}^l(\Omega)} := \left(\sum_{\ell=0}^k \|\partial_z^{\ell} y\|_{L_{\pi}^2(\Omega)}^2 \right)^{\frac{1}{2}}, \quad k \geq 1, \quad \|y\|_{H_{\pi}^0(\Omega)} := \|y\|_{L_{\pi}^2(\Omega)}.$$

Let $h = h(\theta, \nu, z)$ be scalar-valued random function defined on the extended phase space $\mathbb{T} \times \mathbb{R} \times \Omega$. For such h , we define a Sobolev norm $W_{\theta, \nu}^{k, \infty}$ and a mixed norm $H_{\pi}^l(L_{\theta, \nu}^{\infty})$ as follows.

$$\begin{aligned} \|h(z)\|_{W_{\theta, \nu}^{k, \infty}} &:= \sum_{0 \leq \alpha + \beta \leq k} \|\partial_{\theta}^{\alpha} \partial_{\nu}^{\beta} h(z)\|_{L^{\infty}(\mathbb{T} \times \mathbb{R})}, \\ \|h\|_{H_{\pi}^l(L_{\theta, \nu}^{\infty})} &:= \sum_{|\alpha| \leq l} \|\partial_z^{\alpha} h\|_{L_{\pi}^2(\Omega; L^{\infty}(\mathbb{T} \times \mathbb{R}))}. \end{aligned}$$

Moreover, as long as there is no confusion, we suppress π and Ω dependence in $L_{\pi}^2(\Omega)$ -norm and $H_{\pi}^k(\Omega)$ -norm:

$$\|y\|_{L_{\pi}^2} := \|y\|_{L_{\pi}^2(\Omega)}, \quad \|y\|_{H_{\pi}^k} := \|y\|_{H_{\pi}^k(\Omega)}.$$

2. Preliminaries. In this section, we briefly introduce the RKKKE and study its basic properties. Let $\theta_i = \theta_i(t, z)$ be a random phase process of the i -th Kuramoto oscillator whose dynamics is governed by the following system of random ordinary differential equations (ODEs):

$$\partial_t \theta_i(t, z) = \nu_i + \frac{1}{N} \sum_{j=1}^N \kappa_{ij}(z) \sin(\theta_j(t, z) - \theta_i(t, z)), \quad t > 0, \quad 1 \leq i \leq N. \quad (4)$$

Here, the coupling matrix $(\kappa_{ij}(z))$ is assumed to be a symmetric random matrix. In literature [1, 13, 21] on the Kuramoto model, the randomness in natural frequencies is assumed to be time-independent and quenched so that ν_i is a constant parameter. It will be interesting to see how random natural frequency ν_i and random coupling strength $\kappa_{ij}(z)$ interplay in the synchronization process of (4). This uncertain quantification (UQ) question for (4) was addressed in authors' recent work [18].

Next, we consider a situation where the number of oscillators tend to infinity and the coupling strengths $\kappa_{ij}(z)$ are uniform and identical:

$$N \rightarrow \infty, \quad \kappa_{ij}(z) = \kappa(z), \quad 1 \leq i, j \leq N. \quad (5)$$

For the derivation of the mean-field model associated with (1) and (5), we refer to [27, 31] for details. It is more convenient to rewrite system (4) with (5) as a dynamical system on the extended phase space $\mathbb{T} \times \mathbb{R}$:

$$\begin{cases} \partial_t \theta_i(t, z) = \nu_i + \frac{\kappa(z)}{N} \sum_{j=1}^N \sin(\theta_j(t, z) - \theta_i(t, z)), & t > 0, \\ \partial_t \nu_i = 0. \end{cases} \quad (6)$$

Next, we return to the pathwise mean-field limit of (6) as $N \rightarrow \infty$. The formal mean-field limit equation can be easily identified using the standard BBGKY hierarchy based on the formal weak limit of marginal distribution functions and molecular chaos assumption (we refer to [22, 24] for a brief introduction of BBGKY hierarchy). Note that for a frozen $z \in \Omega$, the vector field generated by system (6) is bounded and Lipschitz continuous, so it does satisfy Neunzert’s framework in [31] based on particle-in-cell method and measure-valued solutions. In fact, this has been rigorously done in [27] in any finite-time interval for any initial data. Recently, for an augmented Kuramoto model uniform-in-time mean-field limit is also derived by combining uniform stability analysis and finite-in-time mean-field limit in [21]. Thus, for each $z \in \Omega$, we can perform the same argument as the deterministic case to derive the kinetic equation with random inputs:

$$\begin{aligned} \partial_t f + \partial_\theta(\omega[f]f) &= 0, & (\theta, \nu) \in \mathbb{T} \times \mathbb{R}, \quad t > 0, \\ \omega[f](t, \theta, \nu, z) &= \nu - \kappa(z) \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \theta_*) f(t, \theta_*, \nu_*, z) d\nu_* d\theta_* \end{aligned} \quad (7)$$

subject to initial data:

$$f(0, \theta, \nu, z) = f^0(\theta, \nu, z). \quad (8)$$

Here the initial datum f^0 is assumed to satisfy the following constraints:

$$\begin{aligned} f^0(\theta, \nu, z) &= f^0(\theta + 2\pi, \nu, z), & (\nu, z) \in \mathbb{R} \times \Omega, \\ \int_{\mathbb{T}} f^0(\theta, \nu, z) d\theta &= g(\nu, z), & \int_{\mathbb{T} \times \mathbb{R}} f^0(\theta, \nu, z) d\nu d\theta = 1. \end{aligned} \quad (9)$$

Lemma 2.1. *Let f be a smooth solution to (7), (8) and (9) satisfying additional periodic boundary condition:*

$$f(t, 0, \nu, z) = f(t, 2\pi, \nu, z), \quad (\nu, z) \in \mathbb{R} \times \Omega, \quad t > 0.$$

Then, for each $z \in \Omega$ and $t > 0$, we have

$$\begin{aligned} (i) \quad & \int_{\mathbb{T} \times \mathbb{R}} f(t, \theta, \nu, z) d\nu d\theta = \int_{\mathbb{T} \times \mathbb{R}} f^0(\theta, \nu, z) d\nu d\theta. \\ (ii) \quad & \int_{\mathbb{T} \times \mathbb{R}} \nu f(t, \theta, \nu, z) d\nu d\theta = \int_{\mathbb{T} \times \mathbb{R}} \nu f^0(\theta, \nu, z) d\nu d\theta. \end{aligned}$$

Proof. We multiply 1 and ν to (7)₁ and integrate the resulting relation over $\mathbb{T} \times \mathbb{R}$ to obtain the desired estimates:

$$\begin{aligned} 0 &= \partial_t \int_{\mathbb{T} \times \mathbb{R}} f d\nu d\theta + \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta(\omega[f]f) d\nu d\theta = \partial_t \int_{\mathbb{T} \times \mathbb{R}} f d\nu d\theta, \\ 0 &= \partial_t \int_{\mathbb{T} \times \mathbb{R}} \nu f d\nu d\theta + \int_{\mathbb{T} \times \mathbb{R}} \nu \partial_\theta(\omega[f]f) d\nu d\theta = \partial_t \int_{\mathbb{T} \times \mathbb{R}} \nu f d\nu d\theta. \end{aligned}$$

□

Remark 1. Note that f satisfies

$$f(t, \theta, \nu, z) = f(t, \theta + 2\pi, \nu, z), \quad \int_{\mathbb{T}} f(t, \theta, \nu, z) d\theta = g(\nu), \quad \int_{\mathbb{T} \times \mathbb{R}} f d\nu d\theta = 1.$$

Since the ν -variable is not a dynamic variable as can be seen from (6)₂, thus the ν -support of f will not be changed along the random Kuramoto flow. For a later use, we present the above argument in the following lemma.

Lemma 2.2. *Let f be a C^1 -regular process to (7), and suppose that the initial process $f^0(z)$ has a compact support in ν for each $z \in \Omega$: there exists a positive random function $M := M(z) > 0$ such that*

$$\sup_{\theta \in \mathbb{T}} |f^0(\theta, \nu, z)| = 0, \quad \text{for all } |\nu| \geq M(z), \quad \text{for each } z \in \Omega.$$

Then for each $z \in \Omega$ and $t > 0$, we have

$$\sup_{\theta \in \mathbb{T}} |f(t, \theta, \nu, z)| = 0, \quad \text{for all } |\nu| \geq M(z), \quad \text{for each } z \in \Omega.$$

Before we leave this section, we state a Grönwall-type lemma to be used in later sections.

Lemma 2.3. *Let $y : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+ \cup \{0\}$ be a differentiable function satisfying*

$$y' \leq -\alpha y + C e^{-\beta t}, \quad t > 0, \quad y(0) = y^0, \tag{10}$$

where α, β and C are non-negative constants $\alpha \neq \beta$. Then y satisfies

$$y(t) \leq y^0 e^{-\alpha t} + \frac{C}{\alpha - \beta} (e^{-\beta t} - e^{-\alpha t})$$

Proof. We multiply (10) by $e^{\alpha t}$ and integrate it over $(0, t]$ to give

$$y(t) e^{\alpha t} \leq y^0 + \frac{C}{\alpha - \beta} (e^{(\alpha - \beta)t} - 1).$$

This yields the desired estimate. □

3. Propagation of Sobolev regularity and stability estimate. In this section, we present a pathwise well-posedness of (2) and propagation of $H_\pi^l(L_{\theta, \nu}^\infty)$ -norm of f . First, we study a priori bound of $W_{\theta, \nu}^{k, \infty}$ -norm.

For each $l \in \mathbb{N} \cup \{0\}$, it is easy to check that the z -variations $\{\partial_z^l f\}$ satisfy a hierarchical system:

$$\begin{cases} \partial_t f + \omega[f] \partial_\theta f = -(\partial_\theta \omega[f]) f, & l = 0, \\ \partial_t (\partial_z^l f) + \omega[f] \partial_\theta (\partial_z^l f) = -(\partial_\theta \omega[f]) (\partial_z^l f) - \underbrace{\sum_{r=1}^l \binom{l}{r} \partial_\theta [(\partial_z^r \omega[f]) (\partial_z^{l-r} f)]}_{\text{L.O.T.}}, & l \geq 1. \end{cases} \tag{11}$$

Note that the L.H.S. for f and its z -variations $\partial_z^l f$ have the same transport structure, while the R.H.S. for (11)₂ has a lower-order z -variation terms. Thus, the characteristics for $\partial_z^l f$ will be the same as that of f and hence, independent of l . More precisely, for a given $(\theta, \nu) \in \mathbb{T} \times \mathbb{R}$ and $z \in \Omega$, we define a random forward characteristics

$$(\theta(t, z), \nu(t, z)) := (\theta(t; 0, \theta, \nu, z), \nu(t; 0, \theta, \nu, z))$$

as a solution to (11):

$$\begin{cases} \frac{\partial \theta(s, z)}{\partial s} = \omega[f](\theta(s, z), \nu(s, z)), & \frac{\partial \nu(s, z)}{\partial s} = 0, \quad s > 0, \\ (\theta(0, z), \nu(0, z)) = (\theta, \nu). \end{cases} \quad (12)$$

Now, we define the ν -support of the process $\partial_z^l f$ and its diameter as follows:

$$\begin{aligned} \mathcal{V}^l(t, z) &:= \overline{\{\nu \in \mathbb{R} \mid \sup_{\theta \in \mathbb{T}} |\partial_z^l f(\theta, \nu, t, z)| \neq 0\}}, \\ D(\mathcal{V}^l)(t, z) &:= \sup\{|\nu_1 - \nu_2| \mid \nu_1, \nu_2 \in \mathcal{V}^l(t, z)\}. \end{aligned} \quad (13)$$

For $l = 0$, we set

$$\mathcal{V}^0(t, z) =: \mathcal{V}(t, z) \quad \text{and} \quad D(\mathcal{V}^0)(t, z) =: D(\mathcal{V})(t, z).$$

Note that the ν -support of the process $\partial_z^l f$ is a subset of the ν -support of f . Then, since the ν -support of f does not change along the dynamics of (11)₁ by Lemma 2.2, we have

$$\mathcal{V}^l(t, z) \subseteq \mathcal{V}(t, z) = \mathcal{V}(z) \quad \text{and} \quad D(\mathcal{V}^l)(t, z) \leq D(\mathcal{V})(z) \quad l \geq 0, t \geq 0.$$

3.1. Propagation of $W_{\theta, \nu}^{k, \infty}$ -regularity. In this subsection, we study the propagation of $W_{\theta, \nu}^{k, \infty}$ -regularity of z -variations $\partial_z^l f := \partial_z^l f(t, z)$ whose dynamics are given by the hierarchical system (11). For notational simplicity, we set

$$\|h\|_{L^\infty} := \|h\|_{L_{\theta, \nu}^\infty}, \quad \|h\|_{W^{k, \infty}} := \|h\|_{W_{\theta, \nu}^{k, \infty}},$$

and we denote a generic non-negative random function by $C(z, T)$ which depends on T and z and it may differ from line to line.

Proposition 1. *For $k \in \mathbb{N}$ and $z \in \Omega$, let $f^0 := f^0(z)$ be the initial process satisfying*

$$\|f^0(z)\|_{W^{k, \infty}} < \infty, \quad D(\mathcal{V}(z)) < \infty, \quad \text{for each } z \in \Omega.$$

Then, for $T \in (0, \infty)$, there exists a unique $W^{k, \infty}$ -regular solution process $f := f(t, z)$ such that

$$\sup_{t \in [0, T]} \|f(t, z)\|_{W^{k, \infty}} \leq C(z, T) \|f^0(z)\|_{W^{k, \infty}}, \quad (14)$$

where $C(z, T)$ is a nonnegative random function $C(z, T)$.

Proof. The existence and uniqueness of the solution can be found in [27]. So we only provide a priori estimate for the solution process f . We apply $\partial_\theta^\alpha \partial_\nu^\beta$ to (2)₁ with $0 \leq \alpha + \beta \leq k$ to get

$$\partial_t (\partial_\theta^\alpha \partial_\nu^\beta f) + \partial_\theta^{\alpha+1} \partial_\nu^\beta (\omega[f]f) = 0. \quad (15)$$

Next, we split our estimate into two parts

$$\alpha + \beta = 0, \quad 1 \leq \alpha + \beta \leq k.$$

Let $z \in \Omega$ be fixed.

• **Case A** ($\alpha + \beta = 0$): Note that

$$\partial_t f + \omega[f] \partial_\theta f = -\partial_\theta \omega[f]f. \quad (16)$$

We use the method of characteristics to obtain

$$f(t, \theta(t, z), \nu(t, z), z) = f^0(\theta, \nu, z) - \int_0^t (\partial_\theta \omega[f]f)(s, \theta(s, z), \nu(s, z), z) ds. \quad (17)$$

Since

$$|\partial_\theta \omega[f](s, \theta(s, z), \nu(s, z), z)| \leq \kappa(z) \left(\int_{\mathbb{T} \times \mathbb{R}} |\cos(\theta_* - \theta)| f(s, \theta_*, \nu_*, z) d\nu_* d\theta_* \right) \leq \kappa(z),$$

it follows from (17) that

$$\|f(t, z)\|_{L^\infty} \leq \|f^0(z)\|_{L^\infty} + \kappa(z) \int_0^t \|f(\tau, z)\|_{L^\infty} d\tau. \quad (18)$$

• **Case B** ($1 \leq \alpha + \beta \leq k$): Next, we consider the cases:

$$\beta = 0 \quad \text{and} \quad \beta \geq 1.$$

◊ **Case B-1** ($\beta = 0$ case): In this case, we use the same argument as in Case A to see that $|\partial_\theta^\alpha f|$ satisfies

$$\partial_t(\partial_\theta^\alpha f) + \omega[f] \partial_\theta(\partial_\theta^\alpha f) = - \sum_{\mu=1}^{\alpha+1} \binom{\alpha+1}{\mu} (\partial_\theta^\mu \omega[f]) (\partial_\theta^{\alpha+1-\mu} f) =: \mathcal{R}_1. \quad (19)$$

On the other hand, since

$$|\partial_\theta^\mu \omega[f]| \leq \kappa(z) \int_{\mathbb{T} \times \mathbb{R}} f(\theta_*, \nu_*, z) d\nu_* d\theta_* = \kappa(z),$$

The R.H.S. of (19) can be estimated as

$$|\mathcal{R}_1| \leq C(\alpha, z) \sum_{\mu=1}^{\alpha+1} \|\partial_\theta^{\alpha+1-\mu} f\|_{L^\infty} \leq C(\alpha, z) \|f\|_{W^{\alpha, \infty}}. \quad (20)$$

Now, we integrate relation (19) along the characteristics and use (20) to get

$$\|\partial_\theta^\alpha f(t, z)\|_{L^\infty} \leq \|\partial_\theta^\alpha f^0(z)\|_{L^\infty} + C(\alpha, z) \int_0^t \|f(\tau, z)\|_{W^{\alpha, \infty}} d\tau. \quad (21)$$

◊ **Case B-2** ($\beta \geq 1$ case): It follows from (15) that

$$\partial_t(\partial_\theta^\alpha \partial_\nu^\beta f) + \omega[f] \partial_\theta(\partial_\theta^\alpha \partial_\nu^\beta f) + \sum_{\mu+\lambda \neq 0} \binom{\alpha+1}{\mu} \binom{\beta}{\lambda} (\partial_\theta^\mu \partial_\nu^\lambda \omega[f]) (\partial_\theta^{\alpha+1-\mu} \partial_\nu^{\beta-\lambda} f) = 0. \quad (22)$$

Since

$$\partial_\nu \omega[f] = 1, \quad \partial_\nu^\lambda \omega[f] = 0, \quad \lambda \geq 2,$$

relation (22) can be simplified as

$$\begin{aligned} & \partial_t(\partial_\theta^\alpha \partial_\nu^\beta f) + \omega[f] \partial_\theta^{\alpha+1} \partial_\nu^\beta f \\ & = -\beta \partial_\theta^{\alpha+1} \partial_\nu^{\beta-1} f - \sum_{\mu=1}^{\alpha+1} \binom{\alpha+1}{\mu} (\partial_\theta^\mu \omega[f]) (\partial_\theta^{\alpha+1-\mu} \partial_\nu^\beta f) =: \mathcal{R}_2. \end{aligned} \quad (23)$$

We integrate the above relation along the characteristics and use the estimate

$$|\mathcal{R}_2| \leq C(\alpha, \beta, z) \left(\|\partial_\theta^{\alpha+1} \partial_\nu^{\beta-1} f\|_{L^\infty} + \sum_{\mu=0}^{\alpha} \|\partial_\theta^\mu \partial_\nu^\beta f\|_{L^\infty} \right) \leq C(\alpha, \beta, z) \|f\|_{W^{\alpha+\beta, \infty}}$$

to obtain

$$\|\partial_\theta^\alpha \partial_\nu^\beta f(t, z)\|_{L^\infty} \leq \|\partial_\theta^\alpha \partial_\nu^\beta f^0(z)\|_{L^\infty} + C(\alpha, \beta, z) \int_0^t \|f(\tau, z)\|_{W^{\alpha+\beta, \infty}} d\tau. \quad (24)$$

Therefore, we obtain relations (21) and (24), sum the resulting relation over all $1 \leq \alpha + \beta \leq k$ and add (18) to obtain a Gronwall's inequality:

$$\|f(t, z)\|_{W^{k, \infty}} \leq \|f^0(z)\|_{W^{k, \infty}} + C(\alpha, \beta, z) \int_0^t \|f(\tau, z)\|_{W^{k, \infty}} d\tau.$$

Then Grönwall's lemma yields the desired estimate (14). \square

Remark 2. Note that the Sobolev embedding theorem yields that for $k > 2$, $W_{\theta, \nu}^{k, \infty}$ -solution is $C^1(\mathbb{T} \times \mathbb{R})$ along the path. Thus, it is a classical solution to (2) along the path.

Now, we provide a lemma regarding the estimates of the frequency $\omega[f]$.

Lemma 3.1. For $l \leq k \in \mathbb{N}$ and $T \in (0, \infty)$, suppose that the two solution processes $\partial_z^l f := \partial_z^l f(t, z)$ and $\partial_z^l \tilde{f} := \partial_z^l \tilde{f}(t, z)$ to (11) satisfy the following conditions: for each $z \in \Omega$ and $t \in [0, T]$,

$$\sum_{l=0}^k \left(\|\partial_z^l f(t, z)\|_{L^\infty} + \|\partial_z^l \tilde{f}(t, z)\|_{L^\infty} \right) < \infty, \quad D(\mathcal{V})(z) + D(\tilde{\mathcal{V}})(z) < \infty.$$

Then, for $t \in [0, T]$ and $z \in \Omega$ there exists a nonnegative random function $C(z)$ such that

- (i) $|\partial_\theta^\alpha \partial_z^k (\omega[f])(t, z)| \leq C(z) \sum_{l=0}^k \|\partial_z^l f(t, z)\|_{L^\infty}$,
- (ii) $|\partial_\theta^\alpha \partial_z^k (\omega[f] - \omega[\tilde{f}])(t, z)| \leq C(z) \sum_{l=0}^k \|\partial_z^l (f - \tilde{f})(t, z)\|_{L^\infty}$,
- (iii) $\partial_\nu \omega[f] = 1, \quad \partial_\nu^\alpha \omega[f] = 0, \quad \alpha \geq 2.$

Proof. We first recall the relation:

$$\omega[f](t, \theta, \nu, z) = \nu - \kappa(z) \int_{\mathbb{T} \times \mathbb{R}} \sin(\theta - \theta_*) f(t, \theta_*, \nu_*, z) d\nu_* d\theta_*.$$

(i) It follows from (2) that

$$\begin{aligned} |\partial_\theta^\alpha \partial_z^k \omega[f]| &= \left| \sum_{l=0}^k \binom{k}{l} \partial_z^{k-l} \kappa(z) \int_{\mathbb{T} \times \mathbb{R}} \partial_\theta^\alpha \{\sin(\theta - \theta_*)\} \partial_z^l f(\theta_*, \nu_*, z) d\theta_* d\nu_* \right| \\ &\leq 2\pi \sum_{l=0}^k \binom{k}{l} D(\mathcal{V})(z) |\partial_z^{k-l} \kappa(z)| \|\partial_z^l f\|_{L^\infty} \leq C(z) \sum_{l=0}^k \|\partial_z^l f\|_{L^\infty}, \end{aligned}$$

where we used $\binom{k}{l} \leq 2^k$ and $C(z)$ is given by

$$C(z) := 2^{k+1} \pi D(\mathcal{V})(z) \max_{0 \leq m \leq k} |\partial_z^m \kappa(z)|.$$

(ii) Similar to (i), we have

$$|\partial_\theta^\alpha \partial_z^k (\omega[f] - \omega[\tilde{f}])| \leq C(z) \sum_{l=0}^k \|\partial_z^l (f - \tilde{f})\|_{L^\infty},$$

where $C(z)$ is given by

$$C(z) := 2^{k+1} \pi \max\{D(\mathcal{V})(z), D(\tilde{\mathcal{V}})(z)\} \max_{0 \leq m \leq k} |\partial_z^m \kappa(z)|.$$

(iii) The third estimate follows from the defining relation of $\omega[f]$ in (2). \square

Now, we are ready to provide well-posedness of the process $\partial_z^l f$ for every $l \in \mathbb{N}$.

Theorem 3.2. *For $k \in \mathbb{N}$ and $T \in (0, \infty)$, suppose that the initial process f^0 satisfies the following conditions: for each $z \in \Omega$ and $l = 0, \dots, k$*

$$\|\partial_z^l f^0(z)\|_{W^{k-l, \infty}} < \infty, \quad D(\mathcal{V})(z) < \infty.$$

Then, there exists a unique W^{k-l} -regular process $\partial_z^l f$ to (11) satisfying

$$\sup_{t \in [0, T]} \|\partial_z^l f(t, z)\|_{W^{k-l, \infty}} \leq C(z, T), \quad \text{for each } z \in \Omega.$$

Proof. Since the proof is lengthy and tedious, we postpone its detailed proof in Appendix A. Here we briefly explain why one has a lower $W^{k-l, \infty}$ -regularity for higher z -variation $\partial_z^l f$. In Proposition 1, we have $\|f(t, z)\|_{W^{k, \infty}} \leq C(z, T)$. Then, it follows from (11) that

$$\partial_t(\partial_z f) + \omega[f] \partial_\theta(\partial_z f) = -(\partial_z \omega[f])(\partial_\theta f) + \dots.$$

Since R.H.S. of the above relation has a term $(\partial_\theta f)$, the above relation yields $W^{k-1, \infty}$ estimate for $\partial_z f$. Similarly, $\partial_z^2 f$ satisfies

$$\partial_t(\partial_z^2 f) + \omega[f] \partial_\theta(\partial_z^2 f) = -2(\partial_z \omega[f]) \partial_\theta(\partial_z f) - (\partial_z^2 \omega[f])(\partial_\theta f) + \dots.$$

Hence, the term $\partial_\theta(\partial_z^2 f)$ has $W^{k-2, \infty}$ -estimate. Thus, we can get at most $W^{k-2, \infty}$ estimate for $\partial_z^2 f$. Inductively, we can get $W^{k-l, \infty}$ -estimate for $\partial_z^l f$. This is why we have lower-order regularity for $\partial_z^l f$. \square

Next, we provide the boundedness of the solution process in $H_\pi^l(L_{\theta, \nu}^\infty)$ -norm.

Theorem 3.3. *For $k \in \mathbb{N}$ and $T \in (0, \infty)$, suppose that the initial process f^0 and coupling strength satisfy the following conditions:*

$$\sum_{l=0}^k \sup_{z \in \Omega} \|\partial_z^l f^0(z)\|_{W^{k-l, \infty}} < \infty, \quad \sup_{z \in \Omega} D(\mathcal{V})(z) < \infty, \quad \sum_{l=0}^k \sup_{z \in \Omega} |\partial_z^l \kappa(z)| < \infty.$$

Then, for $T \in (0, \infty)$ we have

$$\|f(t)\|_{H_\pi^k(L_{\theta, \nu}^\infty)} \leq C(T) \|f^0\|_{H_\pi^k(L_{\theta, \nu}^\infty)}, \quad t \in (0, T).$$

Proof. The proof is almost similar to that of Proposition 1. Thus, we briefly outline the proof here. By the same argument as in the proof of Proposition 1, we have

$$\sum_{l=0}^k \|\partial_z^l f(t, z)\|_{L^\infty} \leq C(T) \sum_{l=0}^k \left(\|\partial_z^l f^0(z)\|_{L^\infty} + \int_0^t \|\partial_z^l f(\tau, z)\|_{L^\infty} d\tau \right).$$

We use Grönwall's lemma to obtain

$$\sum_{l=0}^k \|\partial_z^l f(t, z)\|_{L^\infty} \leq C(T) \sum_{l=0}^k \|\partial_z^l f^0(z)\|_{L^\infty}. \quad (25)$$

Finally, we square both sides in (25), multiply by $\pi(z)$ and integrate over Ω to obtain the desired estimate:

$$\|f(t)\|_{H_\pi^k(L_{\theta, \nu}^\infty)}^2 \leq C(T) \|f^0\|_{H_\pi^k(L_{\theta, \nu}^\infty)}^2.$$

□

3.2. Local-in-time stability estimate. In this subsection, we provide a local-in-time $W^{k,\infty}$ -stability estimate for the RKKE. More precisely, we derive pathwise stability estimate of (2) with respect to initial data.

Proposition 2. *For $k \in \mathbb{N}$, $z \in \Omega$ and $T \in (0, \infty)$, let f and \tilde{f} be two $W^{k+1,\infty}$ -processes to (2) with the initial process f^0 and \tilde{f}^0 satisfying the following conditions: for each $t \in (0, T)$ and $z \in \Omega$,*

$$\left(\|f^0(z)\|_{W^{k+1,\infty}} + \|\tilde{f}^0(z)\|_{W^{k+1,\infty}} \right) < \infty, \quad \left(D(\mathcal{V})(z) + D(\tilde{\mathcal{V}})(z) \right) < \infty.$$

Then, there exists a positive random function $C(z, T)$ such that for each $z \in \Omega$,

$$\sup_{0 \leq t < T} \|(f - \tilde{f})(t, z)\|_{W^{k,\infty}} \leq C(z, T) \|(f^0 - \tilde{f}^0)(z)\|_{W^{k,\infty}}.$$

Proof. We use a similar argument as in Proposition 1 to derive the estimate

$$\|\partial_\theta^\alpha \partial_\nu^\beta (f - \tilde{f})\|_{L^\infty} \leq \|\partial_\theta^\alpha \partial_\nu^\beta (f^0 - \tilde{f}^0)\|_{L^\infty} + C(z, T) \int_0^t \|(f - \tilde{f})(\tau, z)\|_{W^{\alpha+\beta,\infty}} d\tau, \quad (26)$$

where $0 \leq \alpha + \beta \leq k$. Finally, we sum the relation (26) over all $0 \leq \alpha + \beta \leq k$ to derive

$$\|(f - \tilde{f})(t, z)\|_{W^{k,\infty}} \leq \|(f^0 - \tilde{f}^0)(z)\|_{W^{k,\infty}} + C(z, T) \int_0^t \|(f - \tilde{f})(\tau, z)\|_{W^{k,\infty}} d\tau. \quad (27)$$

Therefore, we use Grönwall's inequality on (27) to obtain the desired estimate. □

As an application of the arguments in Theorem 3.2 and Proposition 2, we get the local-in-time stability estimate of variations $\partial_z^l f$ in $W^{k-l,\infty}$ -norm.

Theorem 3.4. *For $k \in \mathbb{N}$ and $T \in (0, \infty)$, suppose that two initial data f^0 and \tilde{f}^0 satisfy the following conditions: for each $z \in \Omega$,*

$$\sum_{l=0}^k \left(\|\partial_z^l f^0(z)\|_{W^{k-l+1,\infty}} + \|\partial_z^l \tilde{f}^0(z)\|_{W^{k-l+1,\infty}} \right) < \infty, \quad D(\mathcal{V})(z) + D(\tilde{\mathcal{V}})(z) < \infty,$$

and let $f := f(t, z)$ and $\tilde{f} := \tilde{f}(t, z)$ be two $W^{k+1,\infty}$ -regular solution processes to (11) with initial data $f^0(z)$ and $\tilde{f}^0(z)$, respectively. Then, there exists a positive random function $C(T, z)$ such that

$$\sup_{0 \leq t < T} \sum_{l=0}^k \|\partial_z^l (f - \tilde{f})(t, z)\|_{W^{k-l,\infty}} \leq C(z, T) \sum_{l=0}^k \|\partial_z^l (f^0 - \tilde{f}^0)(z)\|_{W^{k-l,\infty}}.$$

Proof. We basically follow the arguments in Theorem 3.2 and Proposition 2. Thus, we omit the details. □

4. A local sensitivity analysis for phase concentration. In this section, we provide a local sensitivity analysis for the phase concentration that emerges in (2). Since the kinetic equation (2) has been derived from the first-order model, it is not easy to see how frequency synchronization emerges from (2). However, for the kinetic Kuramoto equation with $g(\nu) = \delta_0(\nu)$, we can study the phase synchronization using a Lyapunov functional approach. First, we set:

$$f(\theta, \nu, t, z) := \rho(\theta, t, z) \delta_0(\nu).$$

We substitute this ansatz into (2) to obtain an equation for ρ :

$$\begin{aligned} \partial_t \rho + \partial_\theta(\tilde{\omega}[\rho]\rho) &= 0, \quad \theta \in \mathbb{T}, t > 0, \\ \tilde{\omega}[\rho](\theta, t, z) &= \kappa(z) \int_0^{2\pi} \sin(\theta_* - \theta) \rho(\theta_*, t, z) d\theta_*, \end{aligned} \quad (28)$$

Recall a Lyapunov functional \mathcal{L} measuring the concentration of phases defined in (3):

$$\mathcal{L}[\rho] := \int_{\mathbb{T}} |\theta - \theta_{\rho,c}|^2 \rho(\theta) d\theta, \quad \theta_{\rho,c} := \int_{\mathbb{T}} \theta \rho(\theta) d\theta.$$

As discussed in Introduction, if $\mathcal{L}[\rho]$ goes to zero, then ρ tends to $\delta_{\theta_{\rho,c}}$ in probability. Next, we define several notation regarding the θ -support of ρ :

$$\begin{aligned} (\text{supp}_\theta \rho)(t, z) &:= \overline{\{\theta \in \mathbb{T} \mid \rho(\theta, t, z) \neq 0\}}, \\ D_\theta(\rho)(t, z) &:= \sup\{|\theta - \theta_*| \mid \theta, \theta_* \in \text{supp}_\theta \rho(t, z)\}. \end{aligned}$$

If there is no confusion, we set $\theta_c := \theta_{\rho,c}$, where ρ is the solution process to (28). At this point, we would like to see the basic properties of the solution process (28).

Proposition 3. *Let $\rho := \rho(t, z)$ be a \mathcal{C}^1 -regular process to (28). Then for each $z \in \Omega$, we have*

$$\begin{aligned} (i) \quad \int_0^{2\pi} \rho(t, \theta, z) d\theta &= \int_0^{2\pi} \rho^0(\theta, z) d\theta, \quad \int_0^{2\pi} \theta \rho(t, \theta, z) d\theta = \int_0^{2\pi} \theta \rho^0(\theta, z) d\theta. \\ (ii) \quad \inf_{\theta \in \mathbb{T}} \rho(t, \theta, z) &\geq 0, \quad \text{if} \quad \inf_{\theta \in \mathbb{T}} \rho^0(\theta, z) \geq 0. \end{aligned}$$

Proof. (i) The conservation of total phase can be followed by the direct integration of (28) using the periodic boundary condition in θ -variable. For the second relation, we multiply θ to (28)₁ to get

$$\partial_t \int_0^{2\pi} \theta \rho d\theta = \int_0^{2\pi} \tilde{w}[\rho] \rho d\theta = \kappa(z) \int_{\mathbb{T}^2} \sin(\theta_* - \theta) \rho(\theta_*, z) \rho(\theta, z) d\theta_* d\theta = 0,$$

where the last equality follows from the antisymmetry of the integrand.

(ii) For this, we consider the following characteristic:

$$\frac{\partial}{\partial s} \tilde{\theta}(s; 0, \theta, z) = \tilde{\omega}[\rho](\tilde{\theta}(s; 0, \theta, z), s, z), \quad \tilde{\theta}(0; 0, \theta, z) = \theta.$$

Now, we integrate (28) along the characteristic curve to yield

$$\rho(\theta, t, z) = \rho^0(\tilde{\theta}(0; 0, \theta, z), z) \exp\left(\int_0^t -\tilde{\omega}[\rho](\tilde{\theta}(s; 0, \theta, z), s, z) ds\right).$$

From this, we can deduce the non-negativity of the process ρ . \square

Remark 3. Proposition 3 yields that $\rho_c := \int_{\mathbb{T}} \rho d\theta$ and $\theta_{\rho,c}$ are constants. Moreover, as long as the initial process is nonnegative, ρ is also nonnegative. Hence, without loss of generality, we may assume

$$\rho_c(z) \equiv 1, \quad \theta_{\rho,c}(z) \equiv 0, \quad \inf_{\theta \in \mathbb{T}} \rho^0(\theta, z) \geq 0, \quad \text{for any } z \in \Omega. \quad (29)$$

Under the above setting, we have

$$\partial_z^l \mathcal{L}[\rho] = \partial_z^l \int_{\mathbb{T}} |\theta|^2 \rho(\theta, z) d\theta = \mathcal{L}[\partial_z^l \rho(z)].$$

Thus, for the local sensitivity analysis of $\mathcal{L}[\rho]$, we just need to consider $\mathcal{L}[\partial_z^l \rho]$, which is also bounded by $\mathcal{L}[|\partial_z^l \rho|]$.

Now we provide the contraction property of the θ -support of ρ in the following lemma.

Lemma 4.1. *Suppose that the θ -support of the initial process $\rho^0 := \rho^0(z)$ satisfies the following compactness condition:*

$$0 < D_\theta(\rho^0)(z) < \pi, \quad \text{for each } z \in \Omega.$$

Then, for \mathcal{C}^1 -regular solution process $\rho := \rho(t, z)$ to (28), we have

$$D_\theta(\rho)(t, z) \leq D_\theta(\rho^0)(z), \quad \text{for each } z \in \Omega.$$

Proof. Consider a forward characteristic $\tilde{\theta} := \tilde{\theta}(s; t, \theta, z)$ which is defined as a solution to the following equation:

$$\frac{\partial}{\partial s} \tilde{\theta}(s; t, \theta, z) = \tilde{\omega}[\rho](\tilde{\theta}(s; t, \theta, z), t, z), \quad \tilde{\theta}(t; t, \theta, z) = \theta.$$

First we consider characteristic curve starting from the maximal point $\theta_M(t, z) := \sup\{\theta \mid \theta \in \text{supp}_\theta \rho(t, z)\}$. Then, it is easy to see that it is nonincreasing:

$$\left. \frac{\partial}{\partial s} \tilde{\theta}(s; t, \theta_M, z) \right|_{s=t+} = \kappa(z) \int_{\mathbb{T}} \sin(\theta - \theta_M) \rho(\theta, z) d\theta \leq 0. \tag{30}$$

Similarly, the characteristics curve starting from the minimal point $\theta_m(t, z) := \inf\{\theta \mid \theta \in \text{supp}_\theta \rho(t, z)\}$ is nondecreasing:

$$\left. \frac{\partial}{\partial s} \tilde{\theta}(s; t, \theta_m, z) \right|_{s=t+} \geq 0. \tag{31}$$

Thus, we can deduce from (30) and (31) that $D_\theta(\rho)(t, z)$ does not increase at every time t . This implies our desired result. \square

Proposition 4. *Let $\rho := \rho(t, z)$ be a \mathcal{C}^1 -regular solution process satisfying the following condition:*

$$0 < D_\theta(\rho^0)(z) < \pi, \quad \text{for each } z \in \Omega$$

Then, $\mathcal{L}[\rho](t, z)$ decays exponentially fast along the sample path:

$$\mathcal{L}[\rho](t, z) \leq \mathcal{L}[\rho^0](z) e^{-2\kappa(z)R_0(z)t}, \quad t \geq 0,$$

where $R_0(z)$ is defined by the following relation:

$$R_0(z) := \frac{\sin D_\theta(\rho^0)(z)}{D_\theta(\rho^0)(z)}. \tag{32}$$

Proof. Under the setting (29), the functional $\mathcal{L}[\rho]$ becomes

$$\mathcal{L}[\rho] := \int_{\mathbb{T}} |\theta|^2 \rho(\theta) d\theta.$$

Then, it follows from (28) that

$$\begin{aligned} \partial_t \mathcal{L}[\rho](t, z) &= 2 \int_{\mathbb{T}} \theta \tilde{\omega}[\rho] \rho d\theta = 2\kappa(z) \int_{\mathbb{T}^2} \theta \sin(\theta_* - \theta) \rho(\theta_*, z) \rho(\theta, z) d\theta_* d\theta \\ &= -\kappa(z) \int_{\mathbb{T}^2} (\theta - \theta_*) \sin(\theta - \theta_*) \rho(\theta_*, z) \rho(\theta, z) d\theta_* d\theta, \end{aligned}$$

where we used the change of variable $\theta \leftrightarrow \theta_*$.

On the other hand, by assumption and Lemma 4.1,

$$|\theta - \theta_*| \leq D_\theta(\rho^0)(z) < \pi, \quad \forall \theta, \theta_* \in (\text{supp}_\theta \rho)(t, z). \tag{33}$$

Since $\sin x \geq R_0(z)x$, $x \in [0, D_\theta(\rho^0)]$, we have the following relation:

$$x \sin x \geq R_0(z)x^2, \quad \forall x \in [-D_\theta(\rho^0), D_\theta(\rho^0)], \quad D_\theta(\rho^0) \in (0, \pi). \quad (34)$$

We use (33) and (34) to yield

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{L}[\rho](t, z) &\leq -\kappa(z)R_0(z) \int_{\mathbb{T}^2} |\theta - \theta_*|^2 \rho(\theta_*, z) \rho(\theta, z) d\theta_* d\theta \\ &\leq -\kappa(z)R_0(z) \int_{\mathbb{T}^2} (\theta^2 + \theta_*^2) \rho(\theta_*, z) \rho(\theta, z) d\theta_* d\theta \\ &= -2\kappa(z)R_0(z) \mathcal{L}[\rho](t, z), \end{aligned} \quad (35)$$

where we used

$$|\theta - \theta_*|^2 = |\theta - \theta_c|^2 + |\theta_* - \theta_c|^2 + 2(\theta - \theta_c)(\theta_c - \theta_*)$$

and Proposition 4.1 (i).

Finally, we use Grönwall's lemma on (35) to obtain the desired result. \square

As a direct corollary of Proposition 4, we have the following exponential decay of $\mathbb{E}[\mathcal{L}[\rho]](t)$.

Corollary 1. *Suppose that the initial data and coupling strength satisfy the following conditions:*

$$0 \leq \sup_{z \in \Omega} D_\theta(\rho^0)(z) < \pi - \varepsilon \quad \text{and} \quad \inf_{z \in \Omega} \kappa(z) \geq \eta > 0$$

where ε is a positive constant. Then, for any \mathcal{C}^1 -regular solution process $\rho := \rho(t, z)$ to (28), there exists a positive constant $C = C(\varepsilon, \eta)$ such that

$$\mathbb{E}[\mathcal{L}[\rho]](t) \leq \mathbb{E}[\mathcal{L}[\rho^0]]e^{-2Ct}.$$

Before we consider the local sensitivity analysis about phase concentration, we first provide the following technical lemma.

Lemma 4.2. *For $k > 1$, let ρ be a \mathcal{C}^k -regular solution process to (28) satisfying*

$$D_\theta(\rho^0)(z) < \frac{\pi}{2}, \quad \text{for each } z \in \Omega.$$

Then for $\alpha < k$, we have the following upper and lower bound estimates:

$$(i) \quad \partial_t \mathcal{L}[\partial_\theta^\alpha \rho](t, z)$$

$$\begin{aligned} &\leq -2\kappa(z)R_0(z) \left(\mathcal{L}[\partial_\theta^\alpha \rho](t, z) + \|\partial_\theta^\alpha \rho(t, z)\|_{L_\theta^1} \mathcal{L}[\rho](t, z) \right) \\ &\quad + 2\kappa(z) \|\partial_\theta^\alpha \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]} + \alpha \kappa \mathcal{L}[\partial_\theta^\alpha \rho](t, z) + \kappa(z) \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \mathcal{L}[\partial_\theta^{\alpha+1-\mu} \rho](t, z), \end{aligned}$$

$$(ii) \quad \partial_t \mathcal{L}[\partial_\theta^\alpha \rho](t, z)$$

$$\begin{aligned} &\geq -2\kappa(z) \left(\mathcal{L}[\partial_\theta^\alpha \rho](t, z) + \|\partial_\theta^\alpha \rho(t, z)\|_{L_\theta^1} \mathcal{L}[\rho](t, z) \right) - 2\kappa(z) \|\partial_\theta^\alpha \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]} \\ &\quad + \alpha \kappa \cos D_\theta(\rho^0)(z) \mathcal{L}[\partial_\theta^\alpha \rho](t, z) - \kappa(z) \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \mathcal{L}[\partial_\theta^{\alpha+1-\mu} \rho](t, z), \end{aligned}$$

where $R_0(z)$ is defined in (32).

Proof. Since the proof is rather lengthy, we leave its proof to Appendix B. \square

Now, we are ready to provide the local sensitivity analysis for phase concentration. Note that for $l \in \mathbb{N}$, $\partial_z^l \rho$ satisfies

$$\partial_t(\partial_z^l \rho) + \tilde{\omega}[\rho] \partial_\theta(\partial_z^l \rho) = -(\partial_\theta \tilde{\omega}[\rho]) \partial_z^l \rho - \partial_\theta \left[\sum_{r=1}^l \binom{l}{r} \partial_z^r \tilde{\omega}[\rho] \partial_z^{l-r} \rho \right]. \quad (36)$$

Note that the θ -support of $\partial_z^l \rho$ is a subset of the θ -support of ρ , which is contractive. Now, we would like to analyze the first-order derivative of the functional $\Lambda[\rho]$ with respect to z -variable.

Theorem 4.3. *Suppose that initial density ρ^0 satisfies*

$$0 < D_\theta(\rho^0)(z) < \frac{\pi}{2}, \quad R_0(z) = \frac{\sin D_\theta(\rho^0)(z)}{D_\theta(\rho^0)(z)} > \frac{1}{2}, \quad \text{for each } z \in \Omega,$$

and the coupling strength κ is continuously differentiable and bounded. Let $\rho := \rho(t, z)$ be a C^2 -regular global solution to (28) satisfying a priori condition:

$$J[\rho](z) := \sup_{t \geq 0} (\|\partial_z \rho(z)\|_{L_\theta^1} + \|\partial_\theta \rho(z)\|_{L_\theta^1}) < \infty, \quad \text{for each } z \in \Omega.$$

Then we have

$$\mathcal{L}[|\partial_z \rho|](t, z) \leq \mathcal{L}[|\partial_z \rho^0|](z) e^{-2\kappa(z)R_0(z)t} + \frac{\mathcal{D}(z)}{\kappa(z)} e^{-\kappa(z)(2R_0(z)-1)t},$$

where the non-negative random function \mathcal{D} is given by

$$\begin{aligned} \mathcal{D}(z) &:= 2\kappa(z)J[\rho]\sqrt{\mathcal{L}[\rho^0]} \\ &+ (|\partial_z \kappa(z)| + \kappa(z)J[\rho]) \left(\mathcal{L}[\rho^0] + \mathcal{L}[|\partial_\theta \rho^0|] + \frac{2J[\rho]\sqrt{\mathcal{L}[\rho^0]} + \mathcal{L}[\rho^0]}{1 - R_0(z)} \right). \end{aligned}$$

Proof. We differentiate (28) with respect to z to yield

$$\partial_t(\partial_z \rho) + \partial_\theta(\partial_z \tilde{\omega}[\rho]\rho + \tilde{\omega}[\rho]\partial_z \rho) = 0. \quad (37)$$

We multiply (37) by $\text{sgn}(\partial_z \rho)$ to yield

$$\partial_t |\partial_z \rho| = -\partial_\theta(\tilde{\omega}[\rho]|\partial_z \rho|) - \partial_\theta(\partial_z \tilde{\omega}[\rho]\rho) \text{sgn}(\partial_z \rho). \quad (38)$$

Then, one can use (38) to obtain

$$\partial_t \mathcal{L}[|\partial_z \rho|] = - \int_{\mathbb{T}} \theta^2 \{ \partial_\theta(\tilde{\omega}[\rho]|\partial_z \rho|) + \partial_\theta(\partial_z \tilde{\omega}[\rho]\rho) \text{sgn}(\partial_z \rho) \} d\theta =: \mathcal{I}_{11} + \mathcal{I}_{12}. \quad (39)$$

Next, we estimate \mathcal{I}_{11} and \mathcal{I}_{12} as follows:

- **Step A (Estimates for \mathcal{I}_{11}):** By direct calculation, one has

$$\begin{aligned}
\mathcal{I}_{11} &= 2 \int_{\mathbb{T}} \theta \tilde{\omega}[\rho] |\partial_z \rho| d\theta \\
&= 2\kappa \int_{\mathbb{T}^2} \theta \sin(\theta_* - \theta) \rho(\theta_*) |\partial_z \rho(\theta)| d\theta_* d\theta \\
&= \kappa \int_{\mathbb{T}^2} \sin(\theta_* - \theta) (\theta \rho(\theta_*) |\partial_z \rho(\theta)| - \theta_* \rho(\theta) |\partial_z \rho(\theta_*)|) d\theta_* d\theta \\
&= -\kappa \int_{\mathbb{T}^2} (\theta - \theta_*) \sin(\theta - \theta_*) (\rho(\theta_*) |\partial_z \rho(\theta)| + \rho(\theta) |\partial_z \rho(\theta_*)|) d\theta_* d\theta \\
&\quad + \kappa \int_{\mathbb{T}^2} \sin(\theta_* - \theta) (\theta_* \rho(\theta_*) |\partial_z \rho(\theta)| - \theta \rho(\theta) |\partial_z \rho(\theta_*)|) d\theta_* d\theta \\
&=: \mathcal{I}_{111} + \mathcal{I}_{112}.
\end{aligned} \tag{40}$$

- ◇ (Estimate of \mathcal{I}_{111}): We use relation (34) and similar argument as in \mathcal{I}_{311} of Appendix B to yield

$$\mathcal{I}_{111} \leq -2\kappa R_0 \left(\mathcal{L}[|\partial_z \rho|] + \|\partial_z \rho\|_{L_\theta^1} \mathcal{L}[\rho] \right). \tag{41}$$

- ◇ (Estimate of \mathcal{I}_{112}): Again similar to \mathcal{I}_{312} in Appendix B, we use Proposition 4 to find

$$\mathcal{I}_{112} \leq 2\kappa \|\partial_z \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]} \leq 2\kappa \|\partial_z \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho^0]} e^{-\kappa R_0 t}. \tag{42}$$

- In (40), we combine (41) and (42) to obtain

$$\mathcal{I}_{11} \leq -2\kappa R_0 \left(\mathcal{L}[|\partial_z \rho|] + \|\partial_z \rho\|_{L_\theta^1} \mathcal{L}[\rho] \right) + 2\kappa \|\partial_z \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho^0]} e^{-\kappa R_0 t}. \tag{43}$$

- **Step B (Estimates for \mathcal{I}_{12}):** Note that

$$\mathcal{I}_{12} = - \int_{\mathbb{T}} \theta^2 (\partial_\theta \partial_z \tilde{\omega}[\rho] \rho + \partial_z \tilde{\omega}[\rho] \partial_\theta \rho) \operatorname{sgn}(\partial_z \rho) d\theta =: \mathcal{I}_{121} + \mathcal{I}_{122}. \tag{44}$$

- ◇ (Estimate of \mathcal{I}_{121}): In this case, one gets

$$\begin{aligned}
\mathcal{I}_{121} &= \int_{\mathbb{T}^2} \theta^2 \cos(\theta_* - \theta) (\partial_z \kappa \rho(\theta_*) + \kappa \partial_z \rho(\theta_*)) \rho(\theta) \operatorname{sgn}(\partial_z \rho(\theta)) d\theta_* d\theta \\
&\leq \left(|\partial_z \kappa| + \kappa \|\partial_z \rho\|_{L_\theta^1} \right) \mathcal{L}[\rho].
\end{aligned} \tag{45}$$

- ◇ (Estimate of \mathcal{I}_{122}): Note that

$$\begin{aligned}
\mathcal{I}_{122} &= \int_{\mathbb{T}^2} \theta^2 \cos(\theta_* - \theta) (\partial_z \kappa \rho(\theta_*) + \kappa(z) \partial_z \rho(\theta_*)) \partial_\theta \rho(\theta) \operatorname{sgn}(\partial_z \rho(\theta)) d\theta_* d\theta \\
&\leq \left(|\partial_z \kappa| + \kappa \|\partial_z \rho\|_{L_\theta^1} \right) \mathcal{L}[\partial_\theta \rho].
\end{aligned} \tag{46}$$

It follows from Proposition 4 and Lemma 4.2 that

$$\begin{aligned}
&\frac{\partial}{\partial t} \mathcal{L}[\partial_\theta \rho](t, z) \\
&\leq -2\kappa R_0 \left(\mathcal{L}[\partial_\theta \rho] + \|\partial_\theta \rho\|_{L_\theta^1} \mathcal{L}[\rho] \right) + 2\kappa \|\partial_\theta \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]} + \kappa \mathcal{L}[\partial_\theta \rho] + \kappa \mathcal{L}[\rho] \\
&\leq -\kappa(2R_0 - 1) \mathcal{L}[\partial_\theta \rho] + \kappa \left(2\|\partial_\theta \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]} + \mathcal{L}[\rho] \right) \\
&\leq -\kappa(2R_0 - 1) \mathcal{L}[\partial_\theta \rho] + \kappa \left(2J[\rho] \sqrt{\mathcal{L}[\rho^0]} + \mathcal{L}[\rho^0] \right) e^{-\kappa R_0 t},
\end{aligned} \tag{47}$$

where $J := J[\rho](z)$ denotes the following random functional:

$$J[\rho](z) := \sup_{t \geq 0} \left(\|\partial_z \rho\|_{L^1_\theta} + \|\partial_\theta \rho\|_{L^1_\theta} \right).$$

Thus we use the Grönwall-type inequality in Lemma 2.3 on (47) to yield

$$\mathcal{L}[|\partial_\theta \rho|] \leq \left(\mathcal{L}[|\partial_\theta \rho^0|] + \frac{2J[\rho]\sqrt{\mathcal{L}[\rho^0]} + \mathcal{L}[\rho^0]}{1 - R_0} \right) e^{-\kappa(2R_0-1)t}.$$

Therefore, we can obtain

$$\begin{aligned} \mathcal{I}_{12} &\leq \left(|\partial_z \kappa| + \kappa \|\partial_z \rho\|_{L^1_\theta} \right) \mathcal{L}[\rho] + \left(|\partial_z \kappa| + \kappa \|\partial_z \rho\|_{L^1_\theta} \right) \mathcal{L}[|\partial_\theta \rho|] \\ &\leq (|\partial_z \kappa| + \kappa J[\rho]) \mathcal{L}[\rho^0] e^{-2\kappa R_0 t} \\ &\quad + (|\partial_z \kappa| + \kappa J[\rho]) \left(\mathcal{L}[|\partial_\theta \rho^0|] + \frac{2J[\rho]\sqrt{\mathcal{L}[\rho^0]} + \mathcal{L}[\rho^0]}{1 - R_0} \right) e^{-\kappa(2R_0-1)t} \\ &\leq (|\partial_z \kappa| + \kappa J[\rho]) \left(\mathcal{L}[\rho^0] + \mathcal{L}[|\partial_\theta \rho^0|] + \frac{2J[\rho]\sqrt{\mathcal{L}[\rho^0]} + \mathcal{L}[\rho^0]}{1 - R_0} \right) e^{-\kappa(2R_0-1)t}. \end{aligned} \tag{48}$$

We combine (43) and (48) to yield

$$\begin{aligned} &\frac{\partial}{\partial t} \mathcal{L}[|\partial_z \rho|] \\ &\leq -2\kappa R_0 \left(\mathcal{L}[|\partial_z \rho|] + \|\partial_z \rho\|_{L^1_\theta} \mathcal{L}[\rho] \right) + 2\kappa \|\partial_z \rho\|_{L^1_\theta} \sqrt{\mathcal{L}[\rho^0]} e^{-\kappa R_0 t} \\ &\quad + (|\partial_z \kappa| + \kappa J[\rho]) \left(\mathcal{L}[\rho^0] + \mathcal{L}[|\partial_\theta \rho^0|] + \frac{2J[\rho]\sqrt{\mathcal{L}[\rho^0]} + \mathcal{L}[\rho^0]}{1 - R_0} \right) e^{-\kappa(2R_0-1)t} \\ &\leq -2\kappa R_0 \mathcal{L}[|\partial_z \rho|] + \mathcal{D} e^{-\kappa(2R_0-1)t}, \end{aligned}$$

where the random function $\mathcal{D} := \mathcal{D}(z)$ was given by

$$\begin{aligned} &\mathcal{D}(z) \\ &:= 2\kappa J[\rho] \sqrt{\mathcal{L}[\rho^0]} + (|\partial_z \kappa| + \kappa J[\rho]) \left(\mathcal{L}[\rho^0] + \mathcal{L}[|\partial_\theta \rho^0|] + \frac{2J[\rho]\sqrt{\mathcal{L}[\rho^0]} + \mathcal{L}[\rho^0]}{1 - R_0} \right). \end{aligned}$$

Hence, we use the Grönwall type inequality to obtain

$$\mathcal{L}[|\partial_z \rho|] \leq \mathcal{L}[|\partial_z \rho^0|] e^{-2\kappa R_0 t} + \frac{\mathcal{D}}{\kappa} e^{-\kappa(2R_0-1)t}.$$

This yields our desired result. \square

Remark 4. In this remark, we discuss our results about local sensitivity analysis for the functional $\mathcal{L}[\rho]$.

1. Note that in the proof of Theorem 4.3, we needed the temporal decay of $\mathcal{L}[|\partial_\theta \rho|](t, z)$ to derive the temporal decay of $\mathcal{L}[|\partial_z \rho|](t, z)$. Similarly, we needed the decay of $\mathcal{L}[|\partial_\theta^2 \rho|](t, z)$ to get the decay of $\mathcal{L}[|\partial_z^2 \rho|](t, z)$. Recall that the estimate (i) in Lemma 4.2 yields that for $\alpha = 2$,

$$\partial_t \mathcal{L}[|\partial_\theta^2 \rho|](t, z) \leq 2\kappa(z) (1 - R_0(z)) \mathcal{L}[|\partial_\theta^2 \rho|](t, z) + \text{L.O.T.}$$

Since the coefficient $2\kappa(z)(1 - R_0(z)) > 0$, the above differential inequality does not yield the time-decay of $\mathcal{L}[|\partial_\theta^2 \rho|]$. Thus, it prohibits our local sensitivity analysis for the phase concentration for higher order derivatives.

2. On the other hand, it follows from Lemma 4.2 that

$$\partial_t \mathcal{L}[|\partial_\theta^\alpha \rho|](t, z) \geq \kappa(z) \left(\alpha \kappa \cos D_\theta(\rho^0)(z) - 2 \right) \mathcal{L}[|\partial_\theta^\alpha \rho|](t, z) + \text{L.O.T.}$$

Thus, if α is large enough to satisfy

$$\alpha \cos D_\theta(\rho^0)(z) - 2 > 0 \quad \text{for some } z \in \Omega,$$

then we can deduce the exponential growth of $\mathcal{L}[|\partial_\theta^{\alpha'} \rho|](z)$ for $\alpha' \geq \alpha$ and such $z \in \Omega$. Thus, it might lead the exponential growth for $\mathcal{L}[|\partial_z^\alpha \rho|]$ for $\alpha \gg 1$.

3. We may provide some heuristic reason for the above observation; By Proposition 4, we have

$$\rho(t, \theta, z) \rightarrow \delta_{\theta_c(z)}(\theta) \quad \text{in probability, as } t \rightarrow \infty.$$

Since ρ converges to a Dirac measure concentrated at a single phase $\theta_c(z)$, the graph of ρ is expected to show a steep slope near $\theta = \theta_c(z)$, which implies a rapid growth of $|\partial_\theta^\alpha \rho|$ near $\theta = \theta_c(z)$. However, since the θ -support of ρ also shrinks to the phase $\theta_c(z)$, the term $|\theta - \theta_c(z)|^2$ in the integrand of $\mathcal{L}[|\partial_\theta^\alpha \rho|]$ weakens the growth effect from $|\partial_\theta^\alpha \rho|$. Thus, the growth and decay of $\mathcal{L}[|\partial_\theta^\alpha \rho|]$ will be determined by the dominance of the decay term $|\theta - \theta_c(z)|^2$ over the growth term $|\partial_\theta^\alpha \rho|$. Hence, the possible exponential growth of $\mathcal{L}[|\partial_\theta^\alpha \rho|]$ can be due to the dominance of the rapid growth of $|\partial_\theta^\alpha \rho|$ over the decay from $|\theta - \theta_c(z)|^2$.

5. Conclusion. In this paper, we provided a local sensitivity analysis for the kinetic Kuramoto equation with random inputs in a large coupling regime. In the absence of random inputs, it is well known that the kinetic Kuramoto model exhibits a phase concentration phenomena in the large coupling regime. In authors' earlier series of works, we have begun a systematic local sensitivity analysis for the Kuramoto model with random inputs. For the Kuramoto model, we provided a sufficient framework for the local sensitivity analysis on the asymptotic dynamics of solution process. In this work, we have not only shown the well-posedness of the uncertain problem and stability under random perturbation, but also conducted local sensitivity analysis regarding the phase concentration that could be observed in the Kuramoto model for identical oscillators. For this, we have considered a Lyapunov functional measuring the phase concentration and performed a local sensitivity analysis on the functional. In summary, we found two interesting effects due to uncertainties for (2):

- (Decreasing (θ, ν) -regularity for higher-order z -regularity estimate for f): Propagation of high-order regularity in z -variable is measured in low-order Sobolev norm in the sense that for $T \in (0, \infty)$, $z \in \Omega \subset \mathbb{R}$, $l \leq k$,

$$\|\partial_z^l f^0(z)\|_{W_{\theta, \nu}^{k-l, \infty}} < \infty \quad \implies \quad \sup_{0 \leq t < T} \|\partial_z^l f(t, z)\|_{W_{\theta, \nu}^{k-l, \infty}} \leq C(z, T).$$

- (Formation of zero θ -variance and unbounded variance of $\partial_\theta^\alpha \rho$): The θ -variances for $\partial_\theta^\alpha \rho$ with $|\alpha| \leq 1$ tend to zero exponentially fast, whereas variances of higher order quantity $\partial_\theta^\alpha \rho$ with $\alpha \gg 1$ grows exponentially fast.

Of course, there are several interesting remaining issues on the role of random inputs. For example, in a small and critical coupling regimes, it is not known how the random inputs affects the overall collective dynamics or not. This will be addressed in future works.

Appendix A. Proof of Theorem 3.2. Since the local existence of regular solutions can be done using the standard argument based on contraction mapping theorem, we only provide a priori estimates to conclude the global-in-time existence of regular solutions in any finite-time interval. A priori estimates can be done inductively.

Recall that $\partial_z^l f$ satisfies

$$\begin{aligned} \partial_t f + \omega[f] \partial_\theta f + (\partial_\theta \omega[f]) f &= 0, \quad l = 0, \\ \partial_t (\partial_z^l f) + \omega[f] \partial_\theta (\partial_z^l f) + (\partial_\theta \omega[f]) (\partial_z^l f) &= - \sum_{r=1}^l \binom{l}{r} \partial_\theta \left[(\partial_z^r \omega[f]) (\partial_z^{l-r} f) \right], \quad l \geq 1. \end{aligned} \tag{49}$$

- (Initial step): For $l = 0$, Proposition 1 yields

$$\sup_{t \in [0, T]} \|f(t, z)\|_{W^{k, \infty}} \leq C(z, T) \|f^0(z)\|_{W^{k, \infty}}.$$

- (Inductive step): Suppose that $l \geq 1$ and for $0 \leq r < l$, there exists a unique $W^{k-r, \infty}$ -regular process $\partial_z^r f$ to (11) for each $z \in \Omega$ satisfying the following finiteness condition:

$$\sup_{t \in [0, T]} \|\partial_z^r f(t, z)\|_{W^{k-r, \infty}} \leq C(z, T), \quad 0 \leq r \leq l-1. \tag{50}$$

In order to get the desired estimate, we need to estimate $\|\partial_\theta^\alpha \partial_\nu^\beta f\|_{L^\infty}$. For this, we consider the following two cases:

$$(\alpha, \beta) = (0, 0), \quad 1 \leq \alpha + \beta \leq k - l.$$

- ◊ **Case A** $((\alpha, \beta) = (0, 0))$: It follows from (11) that

$$\begin{aligned} &\partial_t (\partial_z^l f) + \omega[f] \partial_\theta (\partial_z^l f) \\ &= -(\partial_\theta \omega[f]) (\partial_z^l f) - \sum_{r=1}^l \binom{l}{r} \partial_\theta \partial_z^r \omega[f] \partial_z^{l-r} f - \sum_{r=1}^l \binom{l}{r} \partial_z^r \omega[f] \partial_\theta \partial_z^{l-r} f \tag{51} \\ &=: \mathcal{R}_3. \end{aligned}$$

We use Lemma 3.1, (50), $|\partial_\theta \omega[f]| \leq \kappa(z)$ and $\binom{l}{r} \leq 2^l$ to estimate the R.H.S. of (51) as follows.

$$\begin{aligned} |(\partial_\theta \omega[f]) (\partial_z^l f)| &\leq \kappa(z) \|\partial_z^l f\|_{L^\infty}, \\ \left| \sum_{r=1}^l \binom{l}{r} \partial_\theta \partial_z^r \omega[f] \partial_z^{l-r} f \right| &\leq 2^l C(z) \sum_{r=1}^l \left(\sum_{p=0}^r \|\partial_z^p f\|_{L^\infty} \right) \|\partial_z^{l-r} f\|_{L^\infty} \leq C(z, T), \\ \left| \sum_{r=1}^l \binom{l}{r} \partial_z^r \omega[f] \partial_\theta \partial_z^{l-r} f \right| &\leq 2^l C(z) C(z, T) \sum_{r=1}^l \left(\sum_{p=0}^r \|\partial_z^p f\|_{L^\infty} \right) \leq C(z, T). \end{aligned}$$

Thus, we have

$$|\mathcal{R}_3| \leq C(z, T) (\|\partial_z^l f\|_{L^\infty} + 1). \tag{52}$$

Next, we integrate the above relation (51) along the characteristics using the relation (52) to get

$$\|\partial_z^l f(t, z)\|_{L^\infty} \leq \|\partial_z^l f^0(z)\|_{L^\infty} + C(z, T) \left(1 + \int_0^t \|\partial_z^l f(\tau, z)\|_{L^\infty} d\tau \right). \quad (53)$$

◇ **Case B** ($1 \leq \alpha + \beta \leq k$): For $1 \leq \alpha + \beta \leq k - l$, we differentiate (2) with respect to θ and ν for α - and β -times respectively, to get

$$\partial_t(\partial_\theta^\alpha \partial_\nu^\beta \partial_z^l f) + \sum_{\mu=0}^{\alpha+1} \sum_{\lambda=0}^{\beta} \sum_{r=0}^l \binom{\alpha+1}{\mu} \binom{\beta}{\lambda} \binom{l}{r} \partial_\theta^\mu \partial_\nu^\lambda \partial_z^r \omega[f] \partial_\theta^{\alpha+1-\mu} \partial_\nu^{\beta-\lambda} \partial_z^{l-r} f = 0. \quad (54)$$

◇ **Case B.1** ($\beta \geq 1$): In this case,

$$\begin{aligned} & \partial_t(\partial_\theta^\alpha \partial_\nu^\beta \partial_z^l f) + \omega[f] \partial_\theta(\partial_\theta^\alpha \partial_\nu^\beta \partial_z^l f) \\ &= -\beta \partial_\theta^{\alpha+1} \partial_\nu^{\beta-1} \partial_z^l f - \sum_{\mu=1}^{\alpha+1} \binom{\alpha+1}{\mu} \partial_\theta^\mu \omega[f] \partial_\theta^{\alpha+1-\mu} \partial_\nu^\beta \partial_z^l f \\ & \quad - \sum_{\mu=0}^{\alpha+1} \sum_{r=1}^l \binom{\alpha+1}{\mu} \binom{l}{r} \partial_\theta^\mu \partial_z^r \omega[f] \partial_\theta^{\alpha+1-\mu} \partial_\nu^\beta \partial_z^{l-r} f \\ &=: \mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23}. \end{aligned} \quad (55)$$

Below, we separately estimate \mathcal{I}_{2n} 's as follows.

◇ (Estimate of \mathcal{I}_{21}): It is easy to see that

$$|\mathcal{I}_{21}| \leq \beta \|\partial_z^l f\|_{W^{\alpha+\beta, \infty}}. \quad (56)$$

◇ (Estimate of \mathcal{I}_{22}): We use $|\partial_\theta^\mu \omega[f]| \leq \kappa(z)$ to get

$$\begin{aligned} |\mathcal{I}_{22}| &\leq \sum_{\mu=1}^{\alpha+1} \binom{\alpha+1}{\mu} \left| \partial_\theta^\mu \omega[f] \partial_\theta^{\alpha+1-\mu} \partial_\nu^\beta \partial_z^l f \right| \\ &\leq C(z) \sum_{\mu=1}^{\alpha+1} \|\partial_\theta^{\alpha+1-\mu} \partial_\nu^\beta \partial_z^l f\|_{L^\infty} \leq C(z) \|\partial_z^l f\|_{W^{\alpha+\beta, \infty}}. \end{aligned} \quad (57)$$

◇ (Estimate of \mathcal{I}_{23}): We use Lemma 3.1 to obtain

$$\begin{aligned} |\mathcal{I}_{23}| &\leq \sum_{\mu=0}^{\alpha+1} \sum_{r=1}^l \binom{\alpha+1}{\mu} \binom{l}{r} \left| \partial_\theta^\mu \partial_z^r \omega[f] \partial_\theta^{\alpha+1-\mu} \partial_\nu^\beta \partial_z^{l-r} f \right| \\ &\leq 2^{\alpha+1} 2^l C(z) \sum_{\mu=0}^{\alpha+1} \sum_{r=1}^l \left(\sum_{p=0}^r \|\partial_z^p f\|_{L^\infty} \right) \|\partial_\theta^{\alpha+1-\mu} \partial_\nu^\beta \partial_z^{l-r} f\|_{L^\infty}. \end{aligned} \quad (58)$$

Now, we combine all estimates (56), (57), (58) and use the induction assumption to get

$$\begin{aligned} |\mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23}| &\leq C(z, T) \left(\|\partial_z^l f\|_{W^{\alpha+\beta, \infty}} + \sum_{\mu=0}^{\alpha+1} \sum_{r=1}^l \|\partial_\theta^{\alpha+1-\mu} \partial_\nu^\beta \partial_z^{l-r} f\|_{L^\infty} \right) \\ &\leq C(z, T) (\|\partial_z^l f\|_{W^{\alpha+\beta, \infty}} + 1), \end{aligned} \quad (59)$$

Next, we integrate (55) along the characteristics and use (59) to yield

$$\|\partial_\theta^\alpha \partial_\nu^\beta \partial_z^l f(t, z)\|_{L^\infty} \leq \|\partial_\theta^\alpha \partial_\nu^\beta \partial_z^l f^0(z)\|_{L^\infty} + C(z, T) \left(1 + \int_0^t \|\partial_z^l f(\tau, z)\|_{W^{\alpha+\beta, \infty}} d\tau \right). \quad (60)$$

◇ **Case B.2** ($\beta = 0$): Similar to Case B.1, we get

$$\|\partial_\theta^\alpha \partial_z^l f(t, z)\|_{L^\infty} \leq \|\partial_\theta^\alpha \partial_z^l f^0(z)\|_{L^\infty} + C(z, T) \left(1 + \int_0^t \|\partial_z^l f(\tau, z)\|_{W^{\alpha, \infty}} d\tau\right). \quad (61)$$

Finally, we gather the results in (60) and (61), sum those over all $1 \leq \alpha + \beta \leq k - l$ and add the result in (53) to obtain

$$\|\partial_z^l f(t, z)\|_{W^{k-l, \infty}} \leq \|\partial_z^l f^0(z)\|_{W^{k-l, \infty}} + C(z, T) \left(1 + \int_0^t \|\partial_z^l f(\tau, z)\|_{W^{k-l, \infty}} d\tau\right).$$

Finally, we use Grönwall's lemma to derive the desired estimate.

Appendix B. Proof of lemma 4.2. In this section, we present a proof of Lemma 4.2. It follows from (28) that

$$\partial_t(\partial_\theta^\alpha \rho) + \partial_\theta^{\alpha+1}(\tilde{\omega}[\rho]) = 0, \quad (62)$$

or equivalently,

$$\partial_t(\partial_\theta^\alpha \rho) + \partial_\theta(\tilde{\omega}[\rho] \partial_\theta^\alpha \rho) + \alpha \partial_\theta \tilde{\omega}[\rho] \partial_\theta^\alpha \rho + \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \partial_\theta^\mu \tilde{\omega}[\rho] \partial_\theta^{\alpha+1-\mu} \rho = 0. \quad (63)$$

We multiply (63) by $\text{sgn}(\partial_\theta^\alpha \rho)$ to yield

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{L}[|\partial_\theta^\alpha \rho|] &= \int_{\mathbb{T}} \theta^2 \partial_t |\partial_\theta^\alpha \rho| d\theta = 2 \int_{\mathbb{T}} \theta \tilde{\omega}[\rho] |\partial_\theta^\alpha \rho| d\theta - \alpha \int_{\mathbb{T}} \theta^2 \partial_\theta \tilde{\omega}[\rho] |\partial_\theta^\alpha \rho| d\theta \\ &\quad - \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \int_{\mathbb{T}} \text{sgn}(\partial_\theta^\alpha \rho) \theta^2 \partial_\theta^\mu \tilde{\omega}[\rho] \partial_\theta^{\alpha+1-\mu} \rho d\theta. \\ &=: \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33} \end{aligned} \quad (64)$$

(i) (An upper bound estimate): we separately estimate \mathcal{I}_{3n} 's as follows:

• **Case A (Estimates for \mathcal{I}_{31}):** In this case, one can yield

$$\begin{aligned} \mathcal{I}_{31} &= 2\kappa(z) \int_{\mathbb{T}^2} \theta \sin(\theta_* - \theta) \rho(\theta_*) |\partial_\theta^\alpha \rho(\theta)| d\theta_* d\theta \\ &= \kappa(z) \int_{\mathbb{T}^2} \sin(\theta_* - \theta) (\theta \rho(\theta_*) |\partial_\theta^\alpha(\theta)| - \theta_* \rho(\theta) |\partial_\theta^\alpha(\theta_*)|) d\theta_* d\theta \\ &= -\kappa(z) \int_{\mathbb{T}^2} (\theta - \theta_*) \sin(\theta - \theta_*) (\rho(\theta) |\partial_\theta^\alpha \rho(\theta_*)| + \rho(\theta_*) |\partial_\theta^\alpha \rho(\theta)|) d\theta_* d\theta \\ &\quad + \kappa(z) \int_{\mathbb{T}^2} \sin(\theta - \theta_*) (\theta \rho(\theta) |\partial_\theta^\alpha \rho(\theta_*)| - \theta_* \rho(\theta_*) |\partial_\theta^\alpha \rho(\theta)|) d\theta_* d\theta. \\ &=: \mathcal{I}_{311} + \mathcal{I}_{312}. \end{aligned} \quad (65)$$

◇ (Estimate of \mathcal{I}_{311}): Direct calculation leads to

$$\begin{aligned} \mathcal{I}_{311} &\leq -\tilde{\kappa}(z) \int_{\mathbb{T}^2} |\theta - \theta_*|^2 (\rho(\theta) |\partial_\theta^\alpha \rho(\theta_*)| + \rho(\theta_*) |\partial_\theta^\alpha \rho(\theta)|) d\theta_* d\theta \\ &= -\kappa(\tilde{z}) \int_{\mathbb{T}^2} (\theta^2 + \theta_*^2) (\rho(\theta) |\partial_\theta^\alpha \rho(\theta_*)| + \rho(\theta_*) |\partial_\theta^\alpha \rho(\theta)|) d\theta_* d\theta \\ &= -2\tilde{\kappa} \left(\mathcal{L}[|\partial_\theta^\alpha \rho|] + \|\partial_\theta^\alpha \rho\|_{L^1_\theta} \mathcal{L}[\rho] \right). \end{aligned} \quad (66)$$

◇ (Estimate of \mathcal{I}_{312}): Here, one obtains

$$\begin{aligned}\mathcal{I}_{312} &\leq \kappa(z) \int_{\mathbb{T}^2} |\theta_*|\rho(\theta_*)|\partial_\theta^\alpha \rho(\theta)| + |\theta|\rho(\theta)|\partial_\theta^\alpha \rho(\theta_*)|d\theta_*d\theta \\ &= 2\kappa(z)\|\partial_\theta^\alpha \rho\|_{L_\theta^1} \int_{\mathbb{T}} |\theta|\rho(\theta)d\theta \\ &\leq 2\kappa(z)\|\partial_\theta^\alpha \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]},\end{aligned}\tag{67}$$

where we used Cauchy-Schwarz inequality. In (65), we combine (66) and (67) to obtain

$$\mathcal{I}_{31} \leq -2\tilde{\kappa} \left(\mathcal{L}[\partial_\theta^\alpha \rho] + \|\partial_\theta^\alpha \rho\|_{L_\theta^1} \mathcal{L}[\rho] \right) + 2\kappa(z)\|\partial_\theta^\alpha \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]}.\tag{68}$$

• **Case B (Estimates for \mathcal{I}_{32}):** Note that

$$\begin{aligned}\mathcal{I}_{32} &= \alpha\kappa(z) \int_{\mathbb{T}^2} \theta^2 \cos(\theta_* - \theta)\rho(\theta_*)|\partial_\theta^\alpha \rho(\theta)|d\theta_*d\theta \\ &\leq \alpha\kappa(z) \int_{\mathbb{T}} \theta^2 |\partial_\theta^\alpha \rho(\theta)|d\theta = \alpha\kappa(z)\mathcal{L}[\partial_\theta^\alpha \rho].\end{aligned}\tag{69}$$

• **Case C (Estimates for \mathcal{I}_{33}):** One can get easily that

$$\mathcal{I}_{33} \leq \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \int_{\mathbb{T}} \theta^2 |\partial_\theta^{\alpha+1-\mu} \rho| |\partial_\theta^\mu \tilde{w}[\rho]|d\theta \leq \kappa(z) \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \mathcal{L}[\partial_\theta^{\alpha+1-\mu} \rho],\tag{70}$$

where we used

$$|\partial_\theta^\mu \tilde{w}[\rho]| \leq \kappa(z) \int_{\mathbb{T}} \rho(\theta_*)d\theta_* = \kappa(z).$$

Therefore, we combine all the estimates for \mathcal{I}_{3n} 's to obtain

$$\begin{aligned}\frac{\partial}{\partial t} \mathcal{L}[\partial_\theta^\alpha \rho] &\leq -2\tilde{\kappa} \left(\mathcal{L}[\partial_\theta^\alpha \rho] + \|\partial_\theta^\alpha \rho\|_{L_\theta^1} \mathcal{L}[\rho] \right) + 2\kappa(z)\|\partial_\theta^\alpha \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]} \\ &\quad + \alpha\kappa\mathcal{L}[\partial_\theta^\alpha \rho] + \kappa \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \mathcal{L}[\partial_\theta^{\alpha+1-\mu} \rho],\end{aligned}$$

which is our desired upper-bound estimate.

(ii) (A lower bound estimate): we will estimate separately \mathcal{I}_{3n} 's similarly to the upper bound case. Thus we estimate first \mathcal{I}_{31k} 's as follows:

$$\begin{aligned}\mathcal{I}_{311} &\geq -\kappa(z) \int_{\mathbb{T}^2} |\theta - \theta_*|^2 (\rho(\theta)|\partial_\theta^\alpha \rho(\theta_*)| + \rho(\theta_*)|\partial_\theta^\alpha \rho(\theta)|)d\theta_*d\theta \\ &\geq -\kappa(z) \left(\mathcal{L}[\partial_\theta^\alpha \rho] + \|\partial_\theta^\alpha \rho\|_{L_\theta^1} \mathcal{L}[\rho] \right), \\ \mathcal{I}_{312} &\geq -2\kappa(z)\|\partial_\theta^\alpha \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]}.\end{aligned}$$

For \mathcal{I}_{32} , one gets

$$\mathcal{I}_{32} \geq \alpha\kappa(z) \cos D_\theta(\rho^0)(z) \int_{\mathbb{T}} \theta^2 |\partial_\theta^\alpha \rho(\theta)|d\theta = \alpha\kappa(z) \cos D_\theta(\rho^0)(z) \mathcal{L}[\partial_\theta^\alpha \rho].$$

For \mathcal{I}_{33} , one can obtain

$$\mathcal{I}_{33} \geq -\sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \int_{\mathbb{T}} \theta^2 |\partial_\theta^{\alpha+1-\mu} \rho| |\partial_\theta^\mu \tilde{w}[\rho]|d\theta \geq -\kappa(z) \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \mathcal{L}[\partial_\theta^{\alpha+1-\mu} \rho].$$

We combine all estimates for \mathcal{I}_{3n} 's to yield

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{L}[|\partial_\theta^\alpha \rho|] &\geq -2\kappa \left(\mathcal{L}[|\partial_\theta^\alpha \rho|] + \|\partial_\theta^\alpha \rho\|_{L_\theta^1} \mathcal{L}[\rho] \right) - 2\kappa(z) \|\partial_\theta^\alpha \rho\|_{L_\theta^1} \sqrt{\mathcal{L}[\rho]} \\ &\quad + \alpha \kappa \cos D_\theta(\rho^0) \mathcal{L}[|\partial_\theta^\alpha \rho|] - \kappa \sum_{\mu=2}^{\alpha+1} \binom{\alpha+1}{\mu} \mathcal{L}[|\partial_\theta^{\alpha+1-\mu} \rho|], \end{aligned}$$

which gives our desired lower-bound estimate.

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