Local Sensitivity Analysis and Spectral Convergence of the Stochastic Galerkin Method for Discrete-Velocity Boltzmann Equations with Multi-scales and Random Inputs

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Abstract

In this paper we study the general discrete-velocity models of Boltzmann equation with uncertainties from collision kernel and random inputs. We follow the framework of Kawashima and extend it to the case of diffusive scaling in a random setting. First, we provide a uniform regularity analysis in the random space with the help of a Lyapunov-type functional, and prove a uniformly (in the Knudsen number) exponential decay towards the global equilibrium, under certain smallness assumption on the random perturbation of the collision kernel, for suitably small initial data. Then we consider the generalized polynomial chaos based stochastic Galerkin approximation (gPC-SG) of the model, and prove the spectral convergence and the exponential time decay of the gPC-SG error uniformly in the Knudsen number.

1 Introduction

In this paper, we are interested in the *discrete-velocity models* (DVMs) of the Boltzmann equations with multi-scales and random inputs. The study of DVMs of the Boltzmann equations is of considerable interest in the kinetic theory of gases, which describes the time evolution of particles of gases in the case where the particles are allowed to move in the space with finitely many velocities.

Starting from 1970s, there have been plenty of works that studied the discrete-velocity Boltzmann kinetic models. The diffusive limit for the Carleman-type kinetic models were investigated in [36, 33, 38, 37], while the L^1-stability was established in [9]. The decay of solutions of the Carleman model was given in [13, 14] without any scaling. For another example of DVMs, the Broadwell model, Inoue and Nishida showed the decay of solution in one dimension and the hydrodynamical limit in the compressible Euler scaling, as the mean free path goes to zero [15]. For general DVMs, we would like to mention the framework that Kawashima constructed [28, 24, 29, 25, 42, 39, 26]. In particular, he proved the global existence and long-time behavior for general type of DVMs and also applied this framework to systems of hyperbolic-parabolic-type equations [39]. Later on, the analysis for the diffusive limit of general DVMs was showed in [33, 2]. Interested readers could also consult an early review [35] for references. However, there has been no work studying the long-time behavior of solutions in diffusive scaling.

Deterministic models are ideal to mathematicians. In reality, there are many aspects of uncertainties contributed to the models, due to, for example, the lack of knowledge of the interaction mechanism between particles, and inaccurate measurements of initial or/and boundary data. Therefore, understanding and analyzlying the impact of uncertainties, often entering into the problem via random inputs, are crucial to the accessment, validation and calibration of kinetic modeling. The main goal

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of this paper is to study the general DVMs of the Boltzmann equations under the influence of random uncertainties from the collision kernel and initial data.

In the last two decades, *uncertainty quantification* (UQ) became a hot topic in many areas of science and engineering. However, the UQ for kinetic equations remains untouched until recent years [21]. One of the typical numerical methods for UQ is the generalized polynomial chaos approximation based on stochastic Galerkin (gPC-SG) methods [44, 43]. Compared to direct simulation methods such as Monte Carlo [1, 5], gPC-SG is more efficient and accurate if the solution is smooth enough in the random space. Another typical difficulty in kinetic modeling is due to small scales determined by the mean free path, relaxation time, etc. To cope with this difficulty, an efficient computational framework called *asymptotic-preserving* (AP) scheme was usually adopted [16]. Such framework is able to mimic the asymptotic transitions from kinetic equations to their diffusive or hydrodynamic limit in the numerically discrete space (see for examples [8, 17, 20, 30, 31]).

Some recent works attempt to address the two aforementioned difficulties in kinetic equations. The first such work is introduced by Jin, Xiu, and Zhu [19], in which the notion of *stochastic asymptotic-preserving* (s-AP) was introduced. Clearly, the convergence of s-AP methods requires the regularity in the random space. Subsequently, a series of regularity and/or local sensitivity analysis for various types of kinetic equations were conducted, including the linear semiconductor Boltzmann equation [19], linear transport equation [18], and Vlasov-Poisson-Fokker-Planck System [22]. In addition, the uniform regularity for the general linear transport equations conserving mass based on hypocoercivity is established in [32]. Uniform regularity is also obtained for nonlinear kinetic equations, such as the Vlasov-Poisson-Fokker-Planck system [22], the Fokker-Planck-incompressible Navier-Stokes system [40], and a general framework for nonlinear collisional kinetic equations was provided in [34]. See a recent review article for uncertainty quantification on kinetic equations [12]. As typical in hypocoercivity theory [10, 7, 34], the Lyapunov functional includes mixed space-velocity derivative, which is not available in the discrete-velocity setting.

In this paper, by extending the framework of Kawashima for deterministic models to the case of random DVMs, we carry out the regularity and local sensitivity analysis for DVMs with random inputs in the initial data and collision coefficients, in diffusive scaling under periodic boundary condition. Specifically, we establish a uniform regularity in the random space with the help of a Lyapunov-type functional, and a uniformly exponential decay towards the global equilibrium, under certain smallness assumption on the random perturbation of the collision kernel, for suitably small initial data. We use a weighted norm that is first introduced in [18]. This is the so-called sensitivity analysis [41], since it shows the insensitivity of the solution to the random perturbation under the assumed conditions. Then we consider the gPC-SG approximation for the same model, and prove the spectral convergence of the method and the exponential time decay of the gPC-SG error uniformly in the Knudsen number.

The study of discrete velocity kinetic models is not just of theoretical interest. It will also have an impact for numerical computations, since any numerical kinetic model needs to discretize velocity, thus becomes *de facto* discrete-velocity models. Moreover, the lattice Boltzmann methods [6], popular in numerical simulations of incompressible flows, are also discrete-velocity models.

This paper is organized as follows. In Section 2, we introduce the generalized form of DVMs of the Boltzmann equations with randomness and describe the notations used in this paper. In Section 3, we show the regularity of the solutions of DVMs, which results in the decay toward global equilibrium. Section 4 proves the spectral convergence and error estimates of the gPC-SG method.

2 DVM of Boltzmann Equations with Random Inputs

2.1 The Basic Setup

In this article, we consider the initial value problem for discrete-velocity Boltzmann equation in dimensionless form as following

$$\begin{cases} \frac{\partial f_i}{\partial t} + \frac{1}{\varepsilon} v_i \cdot \nabla_x f_i = \frac{1}{\varepsilon^2} \mathcal{B}_i(f,f), \qquad i = 1, 2, \dots, m, \\ f(0, \mathbf{x}, \mathbf{z}) = f_0(\mathbf{x}, \mathbf{z}), \quad x \in \omega \subset \mathbb{T}^d, z \in I_z \subset \mathbb{R}, \end{cases} \tag{2.1}$$

where $f_i = f_i(t, \mathbf{x}, \mathbf{z})$ represents the mass density of particles with velocity $\mathbf{v}_i \in \mathbb{R}^d$ at time t and position $\mathbf{x}$, depending on a random variable $\mathbf{z}$ with $\pi(\mathbf{z})$ as its probability density function. $d \geq 1$ is the dimension of space and velocity. The random variable z lies in I_z. f is a vector function with component f_i. ε is the Knudsen number, the ratio of the mean free path over a typical length scale of the problem. Each $\mathcal{B}_i$ is a binary collision operator given by

$$\mathcal{B}_i(f, g) = \sigma(\mathbf{z}) B_i(f, g), \tag{2.2}$$

where

$$B_i(f, g) = \frac{1}{2\alpha_i} \sum_{j,k,l} \{A_{kl}^{ij}(f_k g_l + f_l g_k) - A_{ij}^{kl}(f_i g_j + f_j g_i)\},$$

α_i are positive constants, and A_{kl}^{ij} are non-negative constants. A_{ij}^{kl} are so-called transition rates related to the collisions

$$(\mathbf{v}_i, \mathbf{v}_j) \leftrightarrow (\mathbf{v}_k, \mathbf{v}_l). \tag{2.3}$$

The transition rates are positive constants which, according to the indistinguishability property of the gas particles and the reversibility of the collision, satisfy

$$A_{lk}^{ij} = A_{kl}^{ij} = A_{kl}^{ji} \text{ and } A_{kl}^{ij} = A_{ij}^{kl} \text{ for all } i, j, k, l = 1, 2, \cdots, m. \tag{2.4}$$

Remark. *For discrete velocity Boltzmann equations, it is easy to deduce the high dimensional problems to one dimension [26].*

Here we consider periodic boundary condition, i.e. $\mathbb{T} = [-\pi, \pi]$.

2.2 Examples of DVMs

One famous example of discrete-velocity model of the Boltzmann equation is the Carleman Model [4],

$$\begin{aligned} \frac{\partial}{\partial t} f_1 + \frac{1}{\varepsilon} v \frac{\partial}{\partial x} f_1 &= \frac{1}{\varepsilon^2}(f_2^2 - f_1^2), \\ \frac{\partial}{\partial t} f_2 - \frac{1}{\varepsilon} v \frac{\partial}{\partial x} f_2 &= \frac{1}{\varepsilon^2}(f_1^2 - f_2^2), \end{aligned}$$

where $f = (f_1, f_2)^T$, $V = \text{diag}(v, -v)$.

The other example is the Broadwell model [3],

$$\begin{aligned} \frac{\partial}{\partial t} f_1 + \frac{1}{\varepsilon} v \frac{\partial}{\partial x} f_1 \quad &= \frac{1}{\varepsilon^2}(f_2^2 - f_1 f_3), \\ \frac{\partial}{\partial t} f_2 \quad &= \frac{1}{2\varepsilon^2}(f_1 f_3 - f_2^2), \\ \frac{\partial}{\partial t} f_3 - \frac{1}{\varepsilon} v \frac{\partial}{\partial x} f_3 \quad &= \frac{1}{\varepsilon^2}(f_2^2 - f_1 f_3), \end{aligned}$$

where $f = (f_1, f_2, f_3)^T$, $V = \text{diag}(v, 0, -v)$. Here v is a positive constant and $\sigma(z) = 1$.

2.3 Notations

In this paper, we will work on vector functions $f = (f_1, f_2, \ldots, f_m)^T \in \mathbb{R}^m$. If f and g are two complex-valued vectors, then the standard dot product (multiplication) in $\mathbb{C}$ is defined as

$$(f, g) = \sum_{j=1}^{m} f_j \overline{g_j}.$$

Denote the inner product as

$$\langle f,g\rangle_x=\int_{\mathbb{T}}(f,g)dx=\int_{\mathbb{T}}\sum_{j=1}^m f_j\overline{g_j}dx, \qquad \text{with norm } \|f\|_{L_x}^2=\langle f,f\rangle_x,$$

$$\langle f,g\rangle_\mu=\int_{I_z}(f,g)d\mu=\int_{I_z}\sum_{j=1}^m f_j\overline{g_j}d\mu, \qquad \text{with norm } \|f\|_{L_\mu}^2=\langle f,f\rangle_\mu,$$

$$\langle f,g\rangle=\int_{I_z}\int_{\mathbb{T}}(f,g)\,dx\,d\mu=\int_{I_z}\int_{\mathbb{T}}\sum_{j=1}^m f_j\overline{g_j}\,dx\,d\mu, \qquad \text{with norm } \|f\|^2=\langle f,f\rangle,$$

where $d\mu=\pi(z)dz$. For functions $f=f(x)$, we define the Sobolev norm (with x derivatives):

$$\|f\|_{H_x^s}^2=\sum_{0\le\alpha\le s}\|\partial_x^\alpha f\|_{L_x}^2.$$

For functions $f=f(x,z)$, the above norm is actually a function of z, we define the expect value of sum of square of Sobolev norm (including both z and x derivatives):

$$\|f\|_{H_x^sH_z^r}^2=\sum_{0\le\gamma\le r}\int_{I_z}\|\partial_z^\gamma f\|_{H_x^s}^2d\mu.$$

In particular, $\|f\|_{H_x^sL_z^2}=\int_{I_z}\|f\|_{H_x^s}^2d\mu$. In addition, define

$$\|f\|_{L_z^\infty(H_x^s)}^2=\sup_{z\in I_z}\|f\|_{H_x^s}^2.$$

Besides, for functions $f=f(z)$, we define the Sobolev norm in the random space as

$$\|f\|_{H_z^r}^2=\sum_{0\le\gamma\le r}\|\partial_z^\gamma f\|_{L_\mu}^2.$$

3 Uniformly Exponential Decay to the Global Equilibrium

In this section, we extend the deterministic framework of Kawashima [23, 26, 27] about convergence toward the global equilibrium for the DVMs of Boltzmann equation to the case with uncertainty.

In particular, we will consider a solution which is a small perturbation of the global equilibrium. To this aim, we shall introduce basic concepts concerning (2.1) and summarize their properties [23, 26, 2] which will be used later.

3.1 Preliminaries

Definition 1. A vector $\phi=(\phi_1,\dots,\phi_m)^T$ is called a *summational invariant* if

$$A_{kl}^{ij}\left(\frac{\phi_i}{\alpha_i}+\frac{\phi_j}{\alpha_j}-\frac{\phi_k}{\alpha_k}-\frac{\phi_j}{\alpha_j}\right)=0, \quad \text{for all } i,j,k,l=1,\dots,m.$$

We denote by $\mathcal{M}$ the set of summational invariants. Then $0<\dim\,\mathcal{M}<m$ because $(\alpha_1,\dots,\alpha_m)^T\in\mathcal{M}$ and $\mathcal{M}\neq\mathbb{R}^m$.

Denote $f=(f_1,\dots,f_m)>0$ if $f_i>0$ for all $i=1,\dots,m$. Let $d=\dim\,\mathcal{M}$. and $\psi^{(j)}$, $j=1,\dots,d$ and $\phi^{(k)}$, $k=d+1,\dots,m$, be constant vectors such that

$$\{\psi^{(1)},\dots\psi^{(d)}\}\text{ is a basis of }\mathcal{M},\ \text{ and }\{\phi^{(d+1)},\dots\phi^{(m)}\}\text{ is a basis of }\mathcal{M}^\perp, \tag{3.1}$$

where $\mathcal{M}^\perp$ denotes the orthogonal complement of $\mathcal{M}$ in $\mathbb{R}^m$. For $f\in\mathbb{R}^m$, we define

$$w=(w_1,\dots,w_d), \quad w_j=(f,\psi^{(j)}), \quad j=1,\dots,d. \tag{3.2}$$

Each w_j is called the $j-$th moment of f.

Definition 2. A vector $f = (f_1, \ldots, f_m) > 0$ is called a local equilibrium if

$$A_{kl}^{ij}(f_i f_j - f_k f_l) = 0, \quad \text{for all } i, j, k, l = 1, \ldots, m.$$

In particular, $f > 0$ is called a *global equilibrium* if it is a locally equilibrium and is independent of t and x, which means it is a constant vector.

Let $B(f, g) = (B_1(f, g), \ldots, B_m(f, g))^T$ and $\mathcal{B}(f, g) = \sigma(\mathbf{z})B(f, g)$.

Lemma 3.1. *Let $f = (f_1, \ldots, f_m) > 0$. The following four statements are equivalent.*

1. *f is a local equilibrium.*
2. *$A_{kl}^{ij} \log(\frac{f_i f_j}{f_k f_l}) = 0$ for all $i, j, k, l = 1, \ldots, m$, that is, $(\alpha_1 \log f_1, \ldots, \alpha_m \log f_m) \in \mathcal{M}$.*
3. *$B(f,\ f) = 0$.*
4. *$\sum \alpha_i \log f_i B_i(f,\ f) = 0$.*

Definition 3. A vector $M > 0$ is called the local equilibrium state associated with $f > 0$ if M is a local equilibrium state and satisfies $M = f$ on $\mathcal{M}$.

Lemma 3.2. *Let $f > 0$ be a given vector. Then there exists uniquely a local equilibrium state M associated with f. Moreover, M can be completely determined by its moments $w = (w_1, \ldots, w_d)$, where $w_i = (M, \psi^j)$.*

All the definitions and the lemmas above can be found in [23, 26].

Let M be the global equilibrium, which can be uniquely determined by the initial data. We shall seek the solution of (2.1)-(2.2) in the form

$$f = M + \varepsilon^2 \Lambda^{1/2} g, \tag{3.3}$$

where $M = (M_1, \ldots, M_m)^T > 0$ and

$$\Lambda = \text{diag}\{M_1/\alpha_1, \ldots M_m/\alpha_m\}.$$

The fluctuation g satisfies

$$\begin{cases} g_t + \frac{1}{\varepsilon} V g_x + \frac{1}{\varepsilon^2} \mathcal{L} g = \mathcal{B}(g, g), \\ g_0 = \frac{1}{\varepsilon^2} \Lambda^{-1/2}(f_0 - M), \end{cases} \tag{3.4}$$

where $V = \text{diag}\{v_1, v_2, \cdots, v_m\}$ and

$$\mathcal{L} g = \sigma(\mathbf{z}) L g = \sigma(\mathbf{z})(-2\Lambda^{-1/2} B(M, \Lambda^{1/2} g)). \tag{3.5}$$

The operators L and B have the following properties [23].

Lemma 3.3.

1. *L is real symmetric and positive semi-definite; its null space is given by*

$$\text{Null(L)} = \text{span}\{\Lambda^{1/2} \mathcal{M}\}.$$

2. *B is bi-linear and satisfies $B(f, g) \in \text{Null}(L)^{\perp}$ for any $f, g \in \mathbb{R}^m$, where $\text{Null(L)}^{\perp}$ is the orthogonal complement of Null(L) in $\mathbb{R}^m$.*
3. *There exist λ_0 and λ_1 such that*

$$\lambda_0 |P^{\perp} f|^2 \leq (Lf,\ f), \tag{3.6}$$

and

$$|Lf|^2 \leq \lambda_1 |P^{\perp} f|^2, \tag{3.7}$$

where $P^{\perp}$ denotes the orthogonal projection onto Null(L)$^{\perp}$.

Proof. The proof of 1 and 2 can be found in [23]. (3.6) and (3.7) can also be found in [2, 23, 25]. □

Remark. *(3.6) is also called "hypocoercivity".*

Denote P as the projection operator onto Null(L), then it is not hard to find:

Lemma 3.4.

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} (Pg(x,t))_j \, dx = 0. \tag{3.8}$$

Proof. Note that g is the perturbation around the global equilibrium M. Since f and M share the same moments, this directly yields this lemma. □

3.2 The Estimate for the Linearized Equation

Let's first consider linearized equation of (3.4) with the same initial data,

$$\begin{aligned} g_t + \frac{1}{\varepsilon} V g_x + \frac{1}{\varepsilon^2} \mathcal{L} g &= 0, \\ g_0 &= \frac{1}{\varepsilon^2} \Lambda^{-1/2}(f_0 - M), \end{aligned} \tag{3.9}$$

where $\mathcal{L} = \sigma(z)L$ is the bounded linear collision operator defined by (3.5).

We also assume that (3.9) is "dissipative" in the following sense (see [23, 24, 26]):

Assumption. *For any complex-valued vector function f, there exists a bounded real anti-symmetric matrix M such that the symmetric part of $MV + \mathcal{L}$ is positive definite. That is, there exist a constant $\lambda_2 > 0$ such that*

$$(([MV]' + \mathcal{L})f, f) \geq \lambda_2 |f|^2, \tag{3.10}$$

where $[MV]'$ denotes the symmetric part of MV.

Next we want to show the decay estimate following the framework of Kawashima [29, 26] using the Fourier transform. Suppose g can be written as

$$g(t,x,z) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \widehat{g}_k(t,z) e^{ikx}, \tag{3.11}$$

where the Fourier coefficient $\widehat{g}_k = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} g(x) e^{-ikx} dx$. We first state a technical lemma with more assumptions for the collision kernel.

Lemma 3.5. *Assume*

$$\sigma(z) = \sigma_0 + \varepsilon \sigma_1(z), \tag{3.12}$$

and for $n = 1, \ldots, r$, $0 < \sigma_{\min} \leq \sigma(z) \leq \sigma_{\max}$, and $|\partial_z^n \sigma| \leq \sigma_{\max}$ for some constant $\sigma_{\min}$ and $\sigma_{\max}$. Besides, assume that the Fourier coefficients $\widehat{g}_k$ of the initial data is in H_z^r. Then for all integer $0 \leq n \leq r$ and for all $k \in \mathbb{Z}/\{0\}$, there exist positive constants c_{rj}, a_{rj} and c_n such that

$$\begin{aligned} &\frac{1}{2}\partial_t \left\{ |\partial_z^n \widehat{g}_k|^2 + \sum_{j=0}^{n-1} c_{nj} |\partial_z^j \widehat{g}_k|^2 - \frac{\varepsilon \alpha_n k}{1+k^2} (iM\partial_z^n \widehat{g}_k, \partial_z^n \widehat{g}_k) - \sum_{j=0}^{n-1} c_{nj} \frac{\varepsilon \alpha_j k}{1+k^2} (iM\partial_z^j \widehat{g}_k, \partial_z^j \widehat{g}_k) \right\} \\ &+ \frac{\lambda_2 \alpha_n}{4} \frac{k^2}{1+k^2} |\partial_z^n \widehat{g}_k|^2 + \sum_{j=0}^{n-1} c_{nj} \frac{\lambda_2 \alpha_j}{4} \frac{k^2}{1+k^2} |\partial_z^j \widehat{g}_k|^2 \leq -\frac{c_n}{\varepsilon^2} \left(|P^\perp \partial_z^n \widehat{g}_k|^2 + \sum_{j=0}^{n-1} a_{nj} |P^\perp \partial_z^j \widehat{g}_k|^2 \right). \end{aligned} \tag{3.13}$$

Remark.

- *The Lyapunov functional*

$$E^{\alpha,n} = |\partial_z^n \widehat{g}_k|^2 + \sum_{j=0}^{n-1} c_{nj} |\partial_z^j \widehat{g}_k|^2 - \frac{\varepsilon \alpha_n k}{1+k^2} (iM\partial_z^n \widehat{g}_k, \partial_z^n \widehat{g}_k) - \sum_{j=0}^{n-1} c_{nj} \frac{\varepsilon \alpha_j k}{1+k^2} (iM\partial_z^j \widehat{g}_k, \partial_z^j \widehat{g}_k), \quad (3.14)$$

is positive and equivalent to the Sobolev norm for α_j small.

Proof. We will use mathematical induction in this proof. For $n = 0$, taking the Fourier transform in x, for $k \neq 0$ one gets

$$\partial_t(\widehat{g}_k) + (\frac{i}{\varepsilon} kV + \frac{1}{\varepsilon^2}\mathcal{L})\widehat{g}_k = 0. \quad (3.15)$$

Taking inner product with $\widehat{g}_k$ (in $\mathbb{C}^m$), and since V and $\mathcal{L}$ are real symmetric, then the real part of (3.15) reads

$$\partial_t(\frac{1}{2}|\widehat{g}_k|^2) + \frac{1}{\varepsilon^2} Re(\mathcal{L}\widehat{g}_k,\ \widehat{g}_k) = 0. \quad (3.16)$$

Multiply $(-\varepsilon ikM)$ and take inner product with $\widehat{g}_k$. Since iM is Hermitian, then the real part is

$$\partial_t \left\{ -\frac{1}{2}\varepsilon k(iM\widehat{g}_k,\ \widehat{g}_k) \right\} + k^2 \mathrm{Re}\left(([MV]' + \mathcal{L})\widehat{g}_k,\ \widehat{g}_k \right) - k^2 \mathrm{Re}(\mathcal{L}\widehat{g}_k, \widehat{g}_k) = \frac{1}{\varepsilon}\mathrm{Re}\left\{ ik(M\mathcal{L}\widehat{g}_k,\ \widehat{g}_k) \right\}. \quad (3.17)$$

Thus we have

$$\begin{aligned} &\partial_t\left(\frac{1}{2}|\widehat{g}_k|^2\right) + \frac{\lambda_0 \sigma_{\min}}{\varepsilon^2}|P^\perp \widehat{g}_k|^2 \le 0, \\ &\partial_t\left\{-\frac{1}{2}\varepsilon k(iM\widehat{g}_k,\ \widehat{g}_k)\right\} + \lambda_2 k^2 |\widehat{g}_k|^2 - \lambda_1 \sigma_{\max}^2 k^2 |P^\perp \widehat{g}_k|^2 \le \frac{1}{\varepsilon}\left(\frac{\sigma_{\max}^2 C_M}{4\delta_0}|L\widehat{g}_k|^2 + \delta_0 k^2 |\widehat{g}_k|^2\right). \end{aligned} \quad (3.18)$$

Here we use the boundedness of the matrix M. If one choose $\delta_0 = \frac{\varepsilon\lambda_2}{2}$, then it follows

$$\begin{aligned} &\partial_t\left(\frac{1}{2}|\widehat{g}_k|^2\right) + \frac{\lambda_0 \sigma_{\min}}{\varepsilon^2}|P^\perp \widehat{g}_k|^2 \le 0, \\ &\partial_t\left\{-\frac{1}{2}\varepsilon k(iM\widehat{g}_k,\ \widehat{g}_k)\right\} + \frac{\lambda_2 k^2}{2}|\widehat{g}_k|^2 - \lambda_1 \sigma_{\max}^2 k^2 |P^\perp \widehat{g}_k|^2 \le \frac{1}{2\varepsilon^2}\frac{\sigma_{\max}^2 C_M}{\lambda_2}|L\widehat{g}_k|^2 \le \frac{1}{2\varepsilon^2}\frac{\lambda_1 \sigma_{\max}^2 C_M}{\lambda_2}|P^\perp \widehat{g}_k|^2. \end{aligned} \quad (3.19)$$

One multiplies the first and the second inequalities by $(1+k^2)$ and α_0, respectively, and then adds them up. It follows

$$\begin{aligned} &\frac{1}{2}\partial_t\left\{(1+k^2)|\widehat{g}_k|^2 - \alpha_0 \varepsilon k(iM\widehat{g}_k,\ \widehat{g}_k)\right\} + \frac{\alpha_0\lambda_2}{2}k^2|\widehat{g}_k|^2 \\ &+ \left(\frac{\lambda_0\sigma_{\min}}{\varepsilon^2}(1+k^2) - \lambda_1\sigma_{\max}^2\alpha_0 k^2 - \frac{1}{2\varepsilon^2}\frac{\alpha_0\lambda_1\sigma_{\max}^2 C_M}{\lambda_2}\right)|P^\perp\widehat{g}_k|^2 \le 0. \end{aligned} \quad (3.20)$$

Choosing α_0 such that

$$\frac{\lambda_0\sigma_{\min}}{2\varepsilon^2} \ge \frac{1}{2\varepsilon^2}\frac{\lambda_1\alpha_0\sigma_{\max}^2 C_M}{\lambda_2} \qquad \text{and} \qquad \frac{\lambda_0\sigma_{\min}}{2\varepsilon^2} \ge \lambda_1\alpha_0\sigma_{\max}^2,$$

(3.20) gives

$$\frac{1}{2}\partial_t\left\{|\widehat{g}_k|^2 - \frac{\alpha_0\varepsilon}{2}\frac{k}{1+k^2}(iM\widehat{g}_k,\ \widehat{g}_k)\right\} + \frac{\alpha_0\lambda_2}{2}\frac{k^2}{1+k^2}|\widehat{g}_k|^2 \le -\frac{\lambda_0\sigma_{\min}}{2\varepsilon^2}|P^\perp\widehat{g}_k|^2. \quad (3.21)$$

In this case, $c_{00} = a_{00} = 1$ and $c_0 = \frac{\lambda_0\sigma_{\min}}{2}$.

Assume that the inequality (3.13) holds true for all $n \leq r$. After adding all those inequalities, one arrives at

$$\begin{aligned}
&\frac{1}{2}\partial_t \left\{ \sum_{j=0}^{r} |\partial_z^j \widehat{g}_k|^2 + \sum_{l=1}^{r}\sum_{j=0}^{l-1} c_{lj} |\partial_z^j \widehat{g}_k|^2 - \frac{\varepsilon k}{1+k^2} \sum_{j=0}^{r} \alpha_j (iM\partial_z^j \widehat{g}_k,\ \partial_z^j \widehat{g}_k) - \frac{\varepsilon k}{1+k^2} \sum_{l=1}^{r}\sum_{j=0}^{l-1} \alpha_j c_{lj} (iM\partial_z^j \widehat{g}_k,\ \partial_z^j \widehat{g}_k) \right\} \\
&+ \frac{k^2}{1+k^2} \sum_{j=0}^{r} \frac{\alpha_j \lambda_2}{4} |\partial_z^j \widehat{g}_k|^2 + \frac{k^2}{1+k^2} \sum_{l=1}^{r}\sum_{j=0}^{l-1} \frac{\alpha_j \lambda_2}{4} |\partial_z^j \widehat{g}_k|^2 \\
&\leq -\frac{c_0}{\varepsilon^2} |P^\perp \widehat{g}_k|^2 - \sum_{j=1}^{r} \frac{c_j}{\varepsilon^2} |P^\perp \partial_z^j \widehat{g}_k|^2 - \sum_{l=1}^{r}\sum_{j=0}^{l-1} \frac{c_{l-1}}{\varepsilon^2} a_{lj} |P^\perp \partial_z^j \widehat{g}_k|^2 \\
&\leq -\frac{c}{\varepsilon^2} (|P^\perp \widehat{g}_k|^2 + \sum_{j=1}^{r} |P^\perp \partial_z^j \widehat{g}_k|^2 + \sum_{l=1}^{r}\sum_{j=0}^{l-1} a_{lj} |P^\perp \partial_z^j \widehat{g}_k|^2),
\end{aligned} \tag{3.22}$$

where $c = \min\{c_0, \ldots, c_n\}$. If we rewrite the equation, it follows

$$\begin{aligned}
&\frac{1}{2}\partial_t \left\{ \sum_{j=0}^{r} d_{rj} |\partial_z^j \widehat{g}_k|^2 - \frac{\varepsilon k}{1+k^2} \sum_{j=0}^{r} d_{rj} \alpha_j (iM\partial_z^j \widehat{g}_k, \partial_z^j \widehat{g}_k) \right\} \\
&+ \frac{\lambda_2 \tilde{\alpha}_r}{4} \frac{k^2}{1+k^2} \sum_{j=0}^{r} d_{rj} |\partial_z^j \widehat{g}_k|^2 \leq -\frac{c}{\varepsilon^2} \sum_{j=0}^{r} b_{rj} |P^\perp \partial_z^j \widehat{g}_k|^2,
\end{aligned} \tag{3.23}$$

where

$$d_{rj} = \begin{cases} 1 + \sum_{l=1}^{r} c_{l0} & \text{for } j = 0, \\ 1 + \sum_{l=j+1}^{r} c_{lj} & \text{for } 1 \leq j \leq r-1, \\ 1 & \text{for } j = r-1, \end{cases} \tag{3.24}$$

$$b_{rj} = \begin{cases} 1 + \sum_{l=1}^{r} a_{l0} & \text{for } j = 0, \\ 1 + \sum_{l=j+1}^{r} a_{lj} & \text{for } 1 \leq j \leq r-1, \\ 1 & \text{for } j = r-1, \end{cases} \tag{3.25}$$

and $\tilde{\alpha}_r = \min\{\alpha_0, \ldots, \alpha_r\}$.

For $n = r+1$, take derivative with respect to z variable to the equation.

$$\partial_t(\partial_z^{r+1} \widehat{g}_k) + \frac{i}{\varepsilon} k V (\partial_z^{r+1} \widehat{g}_k) + \frac{1}{\varepsilon^2} \mathcal{L}(\partial_z^{r+1} \widehat{g}_k) = -\frac{1}{\varepsilon^2} \sum_{l=0}^{r} \binom{r+1}{l} \partial_z^{r+1-l} \mathcal{L}(z) \partial_z^l \widehat{g}_k, \tag{3.26}$$

where $\partial_z^{r+1-l} \mathcal{L}(z) = \partial_z^{r+1-l} \sigma(z) L$.

After taking inner product with $\partial_z^{r+1} \widehat{g}_k$, the real part becomes

$$\frac{1}{2}\partial_t(|\partial_z^{r+1} \widehat{g}_k|^2) + \frac{1}{\varepsilon^2} \mathrm{Re}(\mathcal{L}\partial_z^{r+1} \widehat{g}_k, \partial_z^{r+1} \widehat{g}_k) = -\frac{1}{\varepsilon^2} \sum_{l=0}^{r} \binom{r+1}{l} \mathrm{Re}(\partial_z^{r+1-l} \mathcal{L} \partial_z^l \widehat{g}_k, \partial_z^{r+1} \widehat{g}_k), \tag{3.27}$$

which yields

$$\frac{1}{2}\partial_t(|\partial_z^{r+1} \widehat{g}_k|^2) + \frac{\lambda_0 \sigma_{\min}}{\varepsilon^2} |P^\perp \partial_z^{r+1} \widehat{g}_k|^2 \leq \frac{1}{\varepsilon^2} \frac{1+k^2}{4\delta k^2} \sum_{l=0}^{r} \binom{r+1}{l}^2 \sum_{l=0}^{r} |\partial_z^{r+1-l} \mathcal{L} \partial_z^l \widehat{g}_k|^2 + \frac{1}{\varepsilon^2} \frac{\delta k^2}{1+k^2} |\partial_z^{r+1} \widehat{g}_k|^2. \tag{3.28}$$

Multiplying (3.26) by $(-\varepsilon ikM)$ to both sides and taking inner product with $\partial_z^{r+1}\widehat{g}_k$, then the real part gives

$$-\frac{1}{2}\partial_t\left\{k\varepsilon(iM\partial_z^{r+1}\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)\right\}+k^2\mathrm{Re}(([MV]'+\mathcal{L})\partial_z^{r+1}\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)-k^2\mathrm{Re}(\mathcal{L}\partial_z^{r+1}\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)$$
$$=\frac{1}{\varepsilon}Re\left\{ik(M\mathcal{L}\partial_z^{r+1}\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)\right\}+\frac{1}{\varepsilon}Re\left\{ik(M\sum_{l=0}^{r}\binom{r+1}{l}\partial_z^{r+1-l}\mathcal{L}(z)\partial_z^l\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)\right\}. \tag{3.29}$$

Thus

$$\begin{aligned}&-\frac{1}{2}\partial_t\left\{k\varepsilon(iM\partial_z^{r+1}\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)\right\}+\lambda_2k^2|\partial_z^{r+1}\widehat{g}_k|^2-\lambda_1k^2\sigma_{\max}^2|P^\perp\partial_z^{r+1}\widehat{g}_k|^2\\ \le&\frac{1}{\varepsilon}\left(\frac{\sigma_{\max}^2}{4\delta_1}|L\partial_z^{r+1}\widehat{g}_k|^2+\delta_1k^2|\partial_z^{r+1}\widehat{g}_k|^2\right)\\ &+\frac{1}{\varepsilon}\left(\frac{\sigma_{\max}^2}{4\delta_1}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r+1-l}|\partial_z^{r+1-l}L(z)\partial_z^l\widehat{g}_k|^2+\delta_1k^2|\partial_z^{r+1}\widehat{g}_k|^2\right).\end{aligned} \tag{3.30}$$

If one chooses $\delta_1=\frac{\lambda_2\varepsilon}{4}$, it follows

$$\begin{aligned}&-\frac{1}{2}\partial_t\left\{k\varepsilon(iM\partial_z^{r+1}\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)\right\}+\frac{\lambda_2}{2}k^2|\partial_z^{r+1}\widehat{g}_k|^2-\lambda_1k^2\sigma_{\max}^2|P^\perp\partial_z^{r+1}\widehat{g}_k|^2\\ &\le\frac{1}{\varepsilon^2}\frac{\sigma_{\max}^2C_M}{\lambda_2}|L\partial_z^{r+1}\widehat{g}_k|^2+\frac{1}{\varepsilon^2}\frac{\sigma_{\max}^2C_M}{\lambda_2}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r+1-l}|\partial_z^{r+1-l}L(z)\partial_z^l\widehat{g}_k|^2\\ &\le\frac{1}{\varepsilon^2}\frac{\lambda_1\sigma_{\max}^2C_M}{\lambda_2}|P^\perp\partial_z^{r+1}\widehat{g}_k|^2+\frac{1}{\varepsilon^2}\frac{\lambda_1\sigma_{\max}^2C_M}{\lambda_2}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r+1-l}|P^\perp\partial_z^l\widehat{g}_k|^2.\end{aligned} \tag{3.31}$$

Similar to the case of the zero-th derivative in z, one times inequality (3.31) by $\frac{\alpha_{r+1}}{1+k^2}$ and adds it with inequality (3.28) to get

$$\begin{aligned}&\frac{1}{2}\partial_t\left\{|\partial_z^{r+1}\widehat{g}_k|^2-\frac{\varepsilon\alpha_{r+1}k}{1+k^2}(iM\partial_z^{r+1}\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)\right\}+\frac{\alpha_{r+1}\lambda_2}{2}\frac{k^2}{1+k^2}|\partial_z^{r+1}\widehat{g}_k|^2\\ &+\frac{1}{1+k^2}\left\{(1+k^2)\frac{\lambda_0\sigma_{\min}}{\varepsilon^2}-\alpha_{r+1}\lambda_1k^2\sigma_{\max}^2-\frac{1}{\varepsilon^2}\alpha_{r+1}\frac{\lambda_1\sigma_{\max}^2C_M}{\lambda_2}\right\}|P^\perp\partial_z^{r+1}\widehat{g}_k|^2\\ &\le\frac{1}{1+k^2}\frac{1}{\varepsilon^2}\frac{\alpha_{r+1}\lambda_1\sigma_{\max}^2C_M}{\lambda_2}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2\\ &+\frac{1}{\varepsilon^2}\frac{1+k^2}{4\delta k^2}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|\partial_z^{r+1-l}\mathcal{L}(z)\partial_z^l\widehat{g}_k|^2+\frac{1}{\varepsilon^2}\frac{\delta k^2}{1+k^2}|\partial_z^{r+1}\widehat{g}_k|^2.\end{aligned} \tag{3.32}$$

One may choose $\delta=\frac{\varepsilon^2\lambda_2\alpha_{r+1}}{4}$ to get

$$\begin{aligned}&\frac{1}{2}\partial_t\left\{|\partial_z^{r+1}\widehat{g}_k|^2-\frac{\varepsilon\alpha_{r+1}k}{1+k^2}(iM\partial_z^{r+1}\widehat{g}_k,\partial_z^{r+1}\widehat{g}_k)\right\}+\frac{\alpha_{r+1}\lambda_2}{4}\frac{k^2}{1+k^2}|\partial_z^{r+1}\widehat{g}_k|^2\\ &+\frac{1}{1+k^2}\left\{(1+k^2)\frac{\lambda_0\sigma_{\min}}{\varepsilon^2}-\alpha_{r+1}\lambda_1k^2\sigma_{\max}^2-\frac{1}{\varepsilon^2}\alpha_{r+1}\frac{\lambda_1\sigma_{\max}^2C_M}{\lambda_2}\right\}|P^\perp\partial_z^{r+1}\widehat{g}_k|^2\\ \le&\frac{1}{1+k^2}\frac{1}{\varepsilon^2}\frac{\alpha_{r+1}\lambda_1\sigma_{\max}^2C_M}{\lambda_2}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2+\frac{1}{\varepsilon^2}\frac{1+k^2}{k^2}\frac{1}{\varepsilon^2\lambda_2\alpha_{r+1}}\varepsilon^2\sigma_{\max}^2C_M\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2\\ \le&\frac{1}{1+k^2}\frac{1}{\varepsilon^2}\frac{\alpha_{r+1}\lambda_1\sigma_{\max}^2C_M}{\lambda_2}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2+\frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2C_M}{\lambda_2\alpha_{r+1}}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2,\end{aligned} \tag{3.33}$$

where $|\partial_z^{r+1-l}L(z)\partial_z^l\widehat{g}_k|^2 \leq \varepsilon^2\sigma_{\max}^2|P^\perp\partial_z^l\widehat{g}_k|^2$ and $\frac{1+k^2}{k^2} \leq 2$ for $k \geq 1$.

As what was done before, choose α_{r+1} such that

$$\alpha_{r+1} \leq \frac{\lambda_0\lambda_2\sigma_{\min}}{2\lambda_1\sigma_{\max}^2 C_M} \qquad \text{and} \qquad \alpha_{r+1} \leq \frac{1}{\varepsilon^2}\frac{\lambda_0\sigma_{\min}}{2\lambda_1\sigma_{\max}^2}.$$

It follows

$$\begin{aligned}
&\frac{1}{2}\partial_t\left\{|\partial_z^{r+1}\widehat{g}_k|^2 - \frac{\varepsilon\alpha_{r+1}k}{1+k^2}(iM\partial_z^{r+1}\widehat{g}_k, \partial_z^{r+1}\widehat{g}_k)\right\} + \frac{\alpha_{r+1}\lambda_2}{4}\frac{k^2}{1+k^2}|\partial_z^{r+1}\widehat{g}_k|^2 \\
&+ \frac{\lambda_0\sigma_{\min}}{2\varepsilon^2}|P^\perp\partial_z^{r+1}\widehat{g}_k|^2 \\
\leq &\frac{1}{\varepsilon^2}\frac{\lambda_0\sigma_{\max}^2 C_M}{2}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2 + \frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2 C_M}{\lambda_2\alpha_{r+1}}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2.
\end{aligned} \tag{3.34}$$

Multiplying (3.23) with β, which will be determined later, and adding it to (3.34) yields

$$\begin{aligned}
&\frac{1}{2}\partial_t\left\{|\partial_z^{r+1}\widehat{g}_k|^2 + \sum_{j=0}^{r}\beta d_{rj}|\partial_z^j\widehat{g}_k|^2 - \frac{\varepsilon\alpha_{r+1}k}{1+k^2}(iM\partial_z^{r+1}\widehat{g}_k, \partial_z^{r+1}\widehat{g}_k) - \frac{\varepsilon k}{1+k^2}\sum_{j=0}^{r}\beta d_{rj}\alpha_j(iM\partial_z^j\widehat{g}_k, \partial_z^j\widehat{g}_k)\right\} \\
&+ \frac{\alpha_{r+1}\lambda_2}{4}\frac{k^2}{1+k^2}|\partial_z^{r+1}\widehat{g}_k|^2 + \frac{\lambda_2\tilde{\alpha}_r}{4}\frac{k^2}{1+k^2}\sum_{j=0}^{r}\beta d_{rj}|\partial_z^j\widehat{g}_k|^2 \\
\leq& -\frac{\lambda_0\sigma_{\min}}{2\varepsilon^2}|P^\perp\partial_z^{r+1}\widehat{g}_k|^2 - \sum_{j=0}^{r}\frac{c}{\varepsilon^2}\beta b_{rj}|P^\perp\partial_z^j\widehat{g}_k|^2 + \frac{1}{\varepsilon^2}\frac{\lambda_0\sigma_{\max}^2 C_M}{2}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2 \\
&+ \frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2 C_M}{\lambda_2\alpha_{r+1}}\sum_{l=0}^{r}\binom{r+1}{l}^2\sum_{l=0}^{r}|P^\perp\partial_z^l\widehat{g}_k|^2 \\
=& -\frac{\lambda_0\sigma_{\min}}{2\varepsilon^2}|P^\perp\partial_z^{r+1}\widehat{g}_k|^2 - \sum_{j=0}^{r}\left(\frac{c}{\varepsilon^2}\beta b_{rj} - \frac{1}{\varepsilon}\frac{\lambda_0\sigma_{\max}^2 C_M}{2}4^{r+1} - \frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2 C_M}{\lambda_2\alpha_{r+1}}4^{r+1}\right)|P^\perp\partial_z^j\widehat{g}_k|^2 \\
=& -\frac{\lambda_0\sigma_{\min}}{2\varepsilon^2}\left\{|P^\perp\partial_z^{r+1}\widehat{g}_k|^2 + \sum_{j=0}^{r}\frac{2\varepsilon^2}{\lambda_0\sigma_{\min}}\left(\frac{c}{\varepsilon^2}\beta b_{rj} - \frac{1}{\varepsilon}\frac{\lambda_0\sigma_{\max}^2 C_M}{2}4^{r+1} - \frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2 C_M}{\lambda_2\alpha_{r+1}}4^{r+1}\right)|P^\perp\partial_z^j\widehat{g}_k|^2\right\}.
\end{aligned} \tag{3.35}$$

If we choose β such that

$$\frac{c}{\varepsilon^2}\beta b_{rj} - \frac{1}{\varepsilon}\frac{\lambda_0\sigma_{\max}^2 C_M}{2}4^{r+1} - \frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2 C_M}{\lambda_2\alpha_{r+1}}4^{r+1} > 0,$$

then we can set

$$\frac{2\varepsilon^2}{\lambda_0\sigma_{\min}}\left(\frac{c}{\varepsilon^2}\beta b_{rj} - \frac{1}{\varepsilon}\frac{\lambda_0\sigma_{\max}^2 C_M}{2}4^{r+1} - \frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2 C_M}{\lambda_2\alpha_{r+1}}4^{r+1}\right) = a_{r+1,j} \qquad \text{and} \qquad \beta d_{rj} = d_{r+1,j}.$$

Finally, it follows

$$\begin{aligned}
&\frac{1}{2}\partial_t\left\{|\partial_z^{r+1}\widehat{g}_k|^2 + \sum_{j=0}^{r}d_{r+1,j}|\partial_z^j\widehat{g}_k|^2 - \frac{\varepsilon\alpha_{r+1}k}{1+k^2}(iM\partial_z^{r+1}\widehat{g}_k, \partial_z^{r+1}\widehat{g}_k) - \frac{\varepsilon k}{1+k^2}\sum_{j=0}^{r}d_{r+1,j}\alpha_j(iM\partial_z^j\widehat{g}_k, \partial_z^j\widehat{g}_k)\right\} \\
&+ \frac{\alpha_{r+1}\lambda_2}{4}\frac{k^2}{1+k^2}|\partial_z^{r+1}\widehat{g}_k|^2 + \frac{\lambda_2\tilde{\alpha}_r}{4}\frac{k^2}{1+k^2}\sum_{j=0}^{r}d_{r+1,j}|\partial_z^j\widehat{g}_k|^2 \\
&\leq -\frac{\lambda_0}{2\varepsilon^2}\left(|P^\perp\partial_z^{r+1}\widehat{g}_k|^2 + \sum_{j=0}^{r}a_{r+1,j}|P^\perp\partial_z^j\widehat{g}_k|^2\right).
\end{aligned} \tag{3.36}$$

Then we finish proving our lemma. $\square$

Next we obtain the time decay of g for the linearized equation (3.9).

Theorem 3.1. *With the assumption in Lemma 3.5, let M be the global equilibrium with positive components. Suppose that the initial data $g_0 \in H_x^s H_z^r, r \geq 0, s \geq 0$, and $\|g_0\|^2_{H_x^s H_z^r}$ is small enough, then the solution of the linearized equation (3.9) satisfies*

$$\|g(t)\|^2_{H_x^s H_z^r} \leq e^{-\delta t}\|g_0\|^2_{H_x^s H_z^r} \tag{3.37}$$

for some constant $\delta > 0$ independent of ε.

Proof. We expand $g(t,x,z)$ by the Fourier transform in x direction

$$\begin{aligned} g(t,x,z) &= \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \widehat{g}_k(t,z)e^{ikx} \\ &= \frac{1}{\sqrt{2\pi}} P\widehat{g}_0(t,z) + \frac{1}{\sqrt{2\pi}} P^\perp \widehat{g}_0(t,z) + \frac{1}{\sqrt{2\pi}} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \widehat{g}_k(t,z)e^{ikx}. \end{aligned} \tag{3.38}$$

Then by Parseval's identity, one gets

$$\int_{\mathbb{T}} |g(t,x)|^2 dx = \sum_{k\in\mathbb{Z}} |\widehat{g}_k(0)|^2 = |P\widehat{g}_0(t)|^2 + |P^\perp \widehat{g}_0(t)|^2 + \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |\widehat{g}_k(t)|^2. \tag{3.39}$$

Take $k=0$ in (3.15), we have

$$(\widehat{g}_0)_t + \frac{1}{\varepsilon^2}\mathcal{L}\widehat{g}_0 = 0. \tag{3.40}$$

Applying $P^\perp$ to (3.40) and multiplying it by the complex conjugate of $P^\perp\widehat{g}_0$, it follows

$$\frac{1}{2}\frac{d}{dt}|P^\perp \widehat{g}_0|^2 \leq -\frac{\lambda_0}{\varepsilon^2}|P^\perp \widehat{g}_0|^2, \tag{3.41}$$

which implies

$$|P^\perp \widehat{g}_0(t)|^2 \leq e^{-\frac{2\lambda_0}{\varepsilon^2}t}|P^\perp \widehat{g}(0)|^2. \tag{3.42}$$

For each component $j = 1, \ldots, m$, using (3.8) one has

$$\left(P\widehat{g}_0(t)\right)_j = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} \left(Pg(x,t)\right)_j dx = 0. \tag{3.43}$$

Thus $|P\widehat{g}_0(t)|^2 = \sum_{j=1}^m \left(P\widehat{g}_0(t)\right)_j^2 = 0$.

For $k \neq 0$, from Lemma 3.5, one can have

$$\begin{aligned} &\frac{1}{2}\partial_t \left\{ |\partial_z^r \widehat{g}_k|^2 + \sum_{j=0}^{r-1} c_{rj}|\partial_z^j \widehat{g}_k|^2 - \frac{\varepsilon\alpha_r k}{1+k^2}(iM\partial_z^r\widehat{g}_k, \partial_z^r\widehat{g}_k) - \sum_{j=0}^{r-1} c_{rj}\frac{\varepsilon\alpha_j k}{1+k^2}(iM\partial_z^j\widehat{g}_k, \partial_z^j\widehat{g}_k) \right\} \\ &+ \frac{\lambda_2\alpha_r}{4}\frac{k^2}{1+k^2}|\partial_z^r\widehat{g}_k|^2 + \sum_{j=0}^{r-1} c_{rj}\frac{\lambda_2\alpha_j}{4}\frac{k^2}{1+k^2}|\partial_z^j\widehat{g}_k|^2 \leq -c_r\left(|P^\perp\partial_z^r\widehat{g}_k|^2 + \sum_{j=0}^{r-1} a_{rj}|P^\perp\partial_z^j\widehat{g}_k|^2\right) \leq 0. \end{aligned} \tag{3.44}$$

As long as $\alpha_j \quad j = 1, \ldots, r$ are small enough, we can construct

$$E^{\alpha_1,\ldots,\alpha_r,r} = |\partial_z^r\widehat{g}_k|^2 + \sum_{j=0}^{r-1} c_{rj}|\partial_z^j\widehat{g}_k|^2 - \frac{\varepsilon\alpha_r k}{1+k^2}(iM\partial_z^r\widehat{g}_k, \partial_z^r\widehat{g}_k) - \sum_{j=0}^{r-1} c_{rj}\frac{\varepsilon\alpha_j k}{1+k^2}(i\partial_z^j\widehat{g}_k, \partial_z^j\widehat{g}_k)$$

satisfying

$$\frac{1}{2}\left\{|\partial_z^r \widehat{g}_k|^2 + \sum_{j=0}^{r-1} c_{rj}|\partial_z^j \widehat{g}_k|^2\right\} \le E^{\alpha_1,\dots,\alpha_r,r} \le 2\left\{|\partial_z^r \widehat{g}_k|^2 + \sum_{j=0}^{r-1} c_{rj}|\partial_z^j \widehat{g}_k|^2\right\},$$

If one chooses $\alpha = \min\{\alpha_0, \dots \alpha_r\}$, then

$$(E^{\alpha_1,\dots,\alpha_r,r})_t + \frac{\lambda_2\alpha}{4}\frac{k^2}{1+k^2}E^{\alpha_1,\dots,\alpha_r,r} \le 0. \tag{3.45}$$

This inequality implies

$$|\widehat{g}_k(t)|^2_{H^{r*}} \le e^{-\frac{\delta k^2}{(1+k^2)}t}|\widehat{g}_k(0)|^2_{H^{r*}} \le e^{-\frac{\delta}{2}t}|\widehat{g}_k(0)|^2_{H^{r*}}, \tag{3.46}$$

where $\delta = \frac{\lambda_2\alpha}{4}$ and

$$|\widehat{g}_k(t)|^2_{H^{r*}} := |\partial_z^r \widehat{g}_k|^2 + \sum_{j=0}^{r-1} c_{rj}|\partial_z^j \widehat{g}_k|^2. \tag{3.47}$$

Note we defined a weighted Sobolev norm

$$\|f\|^2_{H^{r*}} = \int_{I_z} |f(z)|^2_{H^{r*}}\pi(z)dz \tag{3.48}$$

that is equivalent to the standard Sobolev norm $\|f\|_{H^r}$in the random space. Then one can have $\|\widehat{g}_k(t)\|^2_{H^r} \le Ce^{-\delta t}\|\widehat{g}_k(0)\|^2_{H^r}$, which implies

$$\|g(t)\|^2_{H_x^0 H_z^r} \le Ce^{-\delta t}\|f(0)\|^2_{H_x^0 H_z^r}, \tag{3.49}$$

where C is a constant independent of ε. And

$$\delta = \frac{\lambda_2\alpha}{4},$$

where $\alpha = \min\{\alpha_0, \dots, \alpha_r\}$ for $\alpha_i \le \min\{\frac{\lambda_0\lambda_2\sigma_{\min}}{2\lambda_1\sigma_{\max}^2 C_M}, \frac{1}{\varepsilon^2}\frac{\lambda_0\sigma_{\min}}{2\lambda_1\sigma_{\max}^2}\}$. It is easy to generate to higher regularity in x space, since we can have

$$\|g(t)\|^2_{H_x^s H_z^r} = \sum_{0\le\alpha\le s}\sum_{k\in\mathbb{Z}} \|\widehat{g}_k(t)\|^2_{H^r}|ik|^{2\alpha} \le Ce^{-\delta t}\sum_{0\le\alpha\le s}\sum_{k\in\mathbb{Z}} \|\widehat{g}_k(0)\|^2_{H^r}|ik|^{2\alpha} = Ce^{-\delta t}\|g(0)\|^2_{H_x^s H_z^r}. \tag{3.50}$$

□

Remark. *Note that although δ depends on ε (through α), it depends on varepsilon in a good way. For example, without loss of generality one can assume $\varepsilon \le 1$, then (eq: g(t)-s-r) yields a uniform exponential decay.*

The next result states that one can still obtain exponential decay of the solution to (3.4) even with the bilinear operator $\mathcal{B}$, if the initial data is small enough.

Theorem 3.2. *With the assumption in Theorem 3.1, the solution of the generalized discrete-velocity model of Boltzmann equation* (3.4) *satisfies*

$$\|g(t)\|^2_{H_x^s H_z^r} \le e^{-\delta t}\|g_0\|^2_{H_x^s H_z^r}. \tag{3.51}$$

Proof. Consider a semi-group generated by

$$\mathcal{G} = -\frac{1}{\varepsilon}V\partial_x - \frac{1}{\varepsilon^2}\mathcal{L}.$$

Therefore, one gets the formula

$$g(t) = e^{t\mathcal{G}}g(0) + \int_0^t e^{(t-\tau)\mathcal{G}}\mathcal{B}(g,g)(\tau)d\tau, \tag{3.52}$$

which implies

$$\begin{aligned} \|g(t)\|^2_{H^s_x H^r_z} &\le \|e^{t\mathcal{G}}g(0)\|^2_{H^s_x H^r_z} + \int_0^t \|e^{(t-\tau)\mathcal{G}}\mathcal{B}(g,g)(\tau)\|^2_{H^s_x H^r_z}d\tau \\ &\le Ce^{-\delta t}\|g(0)\|^2_{H^s_x H^r_z} + C\int_0^t e^{-\delta(t-\tau)}\left(\|g(\tau)\|^2_{H^s_x H^r_z}\right)^2 d\tau. \end{aligned} \tag{3.53}$$

Set $G(t) = \sup_{0\le\tau\le t} e^{\delta\tau}\|g(\tau)\|^2_{H^s_x H^r_z}$. By the definition of $G(t)$, the last term on the right-hand side of (3.53) is dominated by $G(t)^2\int_0^t e^{-\delta t}e^{-\delta\tau}d\tau$. Therefore we arrive at the inequality

$$G(t) \le CE_0 + CG(t)^2, \tag{3.54}$$

where $E_0 = \|g(0)\|^2_{H^s_x H^r_z}$, from which follows the desired estimate $G(t) \le CE_0$ if E_0 is small enough.

□

Remark. *The results of Theorem 3.2 prove that the random fluctuation g decays exponentially, thus f of the solution to* (3.4) *converges to the deterministic global equilibrium M. In other words, the solution is insensitive to the random inputs in initial data and collision kernel $\sigma(z)$ under the assumption* (3.12).

4 The Spectral Convergence of the gPC-SG Method

In this section, we will first give a brief review of the generalized polynomial chaos approach in the stochastic Galerkin (SG) framework, state some properties of the SG solution and then prove the spectral convergence of the SG method and the exponential decay in time toward the global equilibrium of its solution.

4.1 The gPC-SG Approximation

Let $\{\phi_k(z)\}_{k=1}^\infty$ be the series of orthonormal polynomial basis in the Hilbert space $W^\infty_\mu(I_z)$ corresponding to a random measure $d\mu$, where

$$\langle\phi_i(z),\ \phi_j(z)\rangle_\mu = \delta_{ij} \quad \text{and} \quad W^\infty_\mu(I_z) = \{f: I_z \to \mathbb{R} : f \in \text{span}\{\phi_k(z)\}_{k=1}^\infty\}. \tag{4.1}$$

Here δ_{ij} is the Kronecker delta function. One can expand f as

$$f(t,x,z) = \sum_{k=1}^\infty \tilde{f}^k(t,x)\phi_k(z),$$

where

$$\tilde{f}^k(t,x) = \int_{I_z} f(t,x,z)\phi_k(z)d\mu$$

is the coefficient of the gPC expansion. For any fixed integer K, define the projection operator P^K : $W^\infty_\mu(I_z) \to W^K_\mu$ where W^K_μ is the subspace spanned by $\{\phi_k(z)\}_{k=1}^K$. Then

$$P^K f = \sum_{k=1}^K \tilde{f}^k(t,x)\phi_k(z).$$

We seek the solution in W_μ^K, that is in the form of

$$f^K = \sum_{k=1}^K f^k(t,x)\phi_k(z). \tag{4.2}$$

Correspondingly,

$$g^K = \sum_{k=1}^K g^k(t,x)\phi_k(z), \tag{4.3}$$

where $g^K = \frac{1}{\varepsilon^2}\Lambda^{-1/2}(f^K - M)$. Insert this ansatz into equation (3.4), one obtains the gPC-SG system for g^k :

$$\begin{aligned} &\partial_t g^k(t,x) + \frac{1}{\varepsilon} V g_x^k(t,x) + \frac{1}{\varepsilon^2}\mathcal{L}_k(g^K) = \mathcal{B}_k(g^K, g^K), \\ &g^k(0,x) = g_0^k(x), \end{aligned} \tag{4.4}$$

for each $1 \le k \le K$ and the initial condition is given by

$$g_0^k = \int_{I_z} g_0(x,z)\phi_k(z)\pi(z)dz.$$

The collision operators are given by

$$\mathcal{L}_k(g^K) = \sum_{i=1}^K \tilde{S}_{ik} L g^i, \qquad \mathcal{B}_k(g^K, g^K) = \sum_{i,j=1}^K S_{ijk} B(g^i, g^j), \tag{4.5}$$

where

$$\tilde{S}_{ik} = \int_{I_z} \sigma(z)\phi_i(z)\phi_k(z)\pi(z)dz, \qquad S_{ijk} = \int_{I_z} \sigma(z)\phi_i(z)\phi_j(z)\phi_k(z)\pi(z)dz.$$

4.2 Estimate for the gPC Coefficients

To get the spectral convergence of the gPC method, we follow the argument in [40]. We shall get an estimate on the solutions first. Assume that

$$\|\phi_k\|_\infty \le C k^p, \quad \forall k, \tag{4.6}$$

for some positive constant p. Then it follows that

$$|S_{ijk}| \le \sigma_{\max}\|\phi_i\|_\infty \langle |\phi_j|,\ |\phi_k|\rangle_z \le \sigma_{\max}\|\phi_i\|_\infty \le C i^p. \tag{4.7}$$

Here are some examples satisfying (4.7). For the case $I_z = [-1,1]$ with uniform distribution, ϕ_k is the normalized Legendre polynomials, and (4.7) holds for $p = \frac{1}{2}$. For the case $I_z = [-1,1]$ with the distribution $\pi(z) = \frac{2}{\pi\sqrt{1-z^2}}$, ϕ_k are the normalized Chebyshev polynomials, and (4.7) holds with $p = 0$. Since ϕ_k is a $(k-1)^{th}$ degree polynomial, orthogonal to all lower order polynomials and if we are assuming that $\sigma(z)$ is linearly depending on z, $S_{ijk} = 0$ if $(i-1)+(j-1)+1 < k-1$. Thus S_{ijk} may be nonzero only when

$$i + j \ge k \tag{4.8}$$

holds. Note that there is symmetry for i,j,k in S_{ijk}, and S_{ijk} may be nonzero also when

$$j + k \ge i, \qquad k + i \ge j \tag{4.9}$$

hold. One can derive from (4.7) that

$$|S_{ijk}| \le C \cdot \min\{i,j,k\}^p. \tag{4.10}$$

Define the energy by

$$E^K(t) \triangleq \sum_{k=1}^{K} \|k^q g^k\|_{H_x^s}^2, \tag{4.11}$$

and we want to estimate this energy. To this aim, after multiplying k^q to system (4.4), one arrives

$$\partial_t(k^q g^k) + \frac{1}{\varepsilon} V \partial_x(k^q g^k) + \frac{1}{\varepsilon^2}\mathcal{L}_k(k^q g^K) = k^q \mathcal{B}_k(g^K, g^K). \tag{4.12}$$

Then we have the following lemma:

Lemma 4.1. *Assume condition* (4.6). *Let $q > p+2$ and suppose the collision kernel linearly depends on z, i.e. $\sigma(z) = \sigma_0 + \sigma_1 z$. Then*

$$\sum_{k=1}^{K} k^{2q} \|\mathcal{B}_k(g^K, g^K)\|_{H_x^s}^2 \le C(p,q) \sum_{i=1}^{K} \|i^p g^i\|_{H_x^s}^2 \sum_{j=1}^{K} \|j^p g^j\|_{H_x^s}^2. \tag{4.13}$$

Proof. We begin by rewriting the left side of (4.13) into

$$\sum_{k=1}^{K} k^{2q} \|\mathcal{B}_k(g^K, g^K)\|_{H_x^s}^2 = \sum_{k=1}^{K} \frac{k^{2q}}{i^{2q} j^{2q}} \left\| \sum_{i,j=1}^{K} S_{ijk} B(i^q g^i, j^q g^j) \right\|_{H_x^s}^2. \tag{4.14}$$

Consider the case of $i \ge j$. Since $i^q \ge \left(\frac{k}{2}\right)^q$ and (4.10), then

$$\frac{k^{2q}}{i^{2q} j^{2q}} |S_{ijk}|^2 \le C \frac{\left(\frac{k}{2}\right)^{2q} 2^{2q}}{i^{2q} j^{2q}} j^{2p} \le C 2^{2q} j^{2(p-q)}. \tag{4.15}$$

Thus the $i \ge j$ term in RHS of (4.14) can be estimated by

$$\begin{aligned} &\sum_{k=1}^{K} \frac{k^{2q}}{i^{2q} j^{2q}} \left\| \sum_{i,j=1; i\ge j}^{K} \chi_{ijk} S_{ijk} B(i^q g^i, j^q g^j) \right\|_{H_x^s}^2 \\ \le & C(q) \sum_{i,j,k=1; i\ge j}^{K} j^{2(p-q)} \chi_{ijk} \|i^q g^i\|_{H_x^s}^2 \|j^q g^j\|_{H_x^s}^2 \\ \le & C(q) \sum_{i,j,k=1}^{K} j^{2(p-q)} \chi_{ijk} \|i^q g^i\|_{H_x^s}^2 \|j^q g^j\|_{H_x^s}^2, \end{aligned} \tag{4.16}$$

where in the second inequality we use (4.15), and χ_{ijk} is the indicator function for index (i,j,k). If fixing i, one can rewrite the RHS of (4.16) as

$$\sum_{i=1}^{K} \|i^q g^i\|_{H_x^s}^2 \cdot I_i, \qquad I_i = \sum_{j,k=1}^{K} j^{2(p-q)} \|j^q g^j\|_{H_x^s}^2 \chi_{ijk}. \tag{4.17}$$

By (4.8) and (4.9), $\chi_{ijk} = 0$ only when $i - j \le k \le i + j$. It means that there are at most $2j$ terms in I_i above. With assumption $q > p+2$, it holds that

$$I_i \le 2 \sum_{j=1}^{K} j^{2(p-q)+1} \|j^q g^j\|_{H_x^s}^2 \le \sum_{j=1}^{K} j^{p-q+\frac{1}{2}} \sum_{j=1}^{K} \|j^q g^j\|_{H_x^s}^2 \le C \sum_{j=1}^{K} \|j^q g^j\|_{H_x^s}^2.$$

For the case of $i \le j$, one can exchange the indexes i and j to have the same estimate. Then we finish the proof.

□

Remark. *The assumption on the linearity in z is a common practice in UQ research. It is known that uncertainties are usually modelled by stochastic process, and according to the Karhunen-Loeve theory, any stochastic process can be approximated by a linear combination of uncorrelated random variables (z in this paper). Our analysis could be extended to more general function of z but the algebra will become messy and lose the clarity of the analysis, so we do not carry it out here.*

Next we obtain the exponential decay of $E^K(t)$ for $\sigma(z)$ with a smaller random perturbation.

Theorem 4.1. *Assume condition (4.6). Let $q > p+2$ and suppose the collision kernel linearly depends on z in the following way, $\sigma(z) = \sigma_0 + \varepsilon\sigma_1 z$ with $0 < \sigma_{\min} \le \sigma(z) \le \sigma_{\max}$. And if $\varepsilon \le \sqrt[2]{\frac{\lambda_2^2\lambda_0^2\sigma_{\min}^2}{\sigma_{\max}^4 C_M^2 \lambda_1 2^{2q+9}}}$, then the energy defined by (4.11) can be estimate as*

$$E^K(t) \le Ce^{-\delta t}E^K(0). \tag{4.18}$$

Proof. The proof is similar to Theorem 3.2, the interested reader will find them in Appendix A. ☐

Once we obtain the estimate of energy, we can get exponential decay of the gPC solutions.

Corollary 4.1. *With the assumption above, there exist constants C and C' which are independent of ε and K so that*

$$\|g^K\|^2_{L^\infty_z(H^s_x)} \le C_0 e^{-\delta t}, \tag{4.19}$$

and

$$\|g^K\|^2_{H^s_x L^2_z} \le C_0' e^{-\delta t}, \tag{4.20}$$

Proof.

$$\begin{aligned}\|g^K\|^2_{L^\infty_z(H^s_x)} &= \sup_{z\in I_z}\|\sum_{k=1}^K g^k\phi_k(z)\|^2_{H^s_x} \le C\sum_{k=1}^K \|g^k\|^2_{H^s_x}k^{2p}\\ &\le C\left(\sum_{k=1}^K \|k^q g^k\|^2_{H^s_x}\right)\left(\sum_{k=1}^K k^{2(p-q)}\right)\\ &\le C\left(\sum_{k=1}^K \|k^q g^k\|^2_{H^s_x}\right) \le C_0 e^{-\delta t},\end{aligned} \tag{4.21}$$

due to $q > p+2$. In addition,

$$\|g^K\|^2_{H^s_x L^2_z} = \int_{I_z}\|g^K\|\|^2_{H^s_x}d\mu \le C\int_{I_z}\|g^K\|^2_{L^\infty_z(H^s_x)}d\mu \le C_0' e^{-\delta t}. \tag{4.22}$$

☐

4.3 The gPC Error Estimate

In order to estimate the gPC error $g - g^K$, we denote

$$g^e = g - g^K = \underbrace{g - P^K g}_{\mathcal{R}^K} + \underbrace{P^K g - g^K}_{\mathcal{E}^K},$$

where $\mathcal{R}^K$ and $\mathcal{E}^K$ refer to truncation error and projection error, respectively. Then using the strategy of [34], we can have the following theorem.

Theorem 4.2. *Assume condition (4.6). Let $q > p+2$ and suppose the collision kernel linearly depending on z, i.e. $\sigma(z) = \sigma_0 + \varepsilon^2\sigma_1 z$ with σ_0, σ_1 independent of z (thus $0 < \sigma_{\min} \le \sigma(z) \le \sigma_{\max}$). If initially $\|g^e_{in}\|^2_{H^s_x H^r_z} \le C_I$, $\|g_0\|^2_{H^s_x H^r_z} \le C_0$, and if $\tilde{C}_0 = C_0\max\{C_\pi C_0, 1\}$ such that*

$$\frac{3\tilde{C}_0}{\delta} < 1, \tag{4.23}$$

where δ is defined in Theorem 4.1 and C_π is a constant independent of K and ϵ, then the gPC error has following estimate

$$\|g^e\|^2_{H^s_x L^2_z} \le C(T)\frac{e^{-\delta t}}{K^{2r-1}}, \tag{4.24}$$

where C (linearly depending on T) and δ are constants independent of K and ε.

Proof. By Theorem 3.2 and standard estimate for truncation error of orthogonal polynomial approximations

$$\|\mathcal{R}^K\|^2_{H^s_x L^2_z} \le \|\mathcal{R}^K\|^2_{H^s_x H^r_z} \le C_\pi \frac{\|g\|^2_{H^s_x H^r_z}}{K^{2r}} \le C_\pi C_0 \frac{e^{-\delta t}}{K^{2r}}, \tag{4.25}$$

where C_π is a constant independent on K. Let the projection error be

$$\mathcal{E}^K(t,x,z) = \sum_{k=1}^{K} (\tilde{g}^k - g^k)\phi_k = \sum_{k=1}^{K} e^k(t,x)\phi_k(z),$$

where $\tilde{g}^k = \int_{I_z} g\phi_k d\mu$ and we denote $\mathbf{e} = [e^1, \dots, e^K]$.

Let

$$\mathcal{T}(f) = \partial_t f + \frac{1}{\varepsilon} V \partial_x f + \frac{1}{\varepsilon^2}\mathcal{L}(f) - \mathcal{B}(f,f)\,. \tag{4.26}$$

Since g^K is the gPC solution, then for all $k = 1, \dots, K$,

$$\langle \mathcal{T}(g^K),\ \phi_k\rangle_\mu = 0. \tag{4.27}$$

Due to $\langle \mathcal{T}(g),\ \phi_k\rangle_\mu = 0$, one can have

$$\langle \mathcal{T}(g) - \mathcal{T}(g^K),\ \phi_k\rangle_\mu = 0. \tag{4.28}$$

For the first term inside of $\mathcal{T}$, it follows

$$\langle \partial_t g - \partial_t g^K,\ \phi_k\rangle_\mu = \langle \partial_t \mathcal{R}^K,\ \phi_k\rangle_\mu + \langle \partial_t \mathcal{E}^K,\ \phi_k\rangle_\mu = \langle \partial_t \mathcal{E}^K,\ \phi_k\rangle_\mu,$$

since

$$\langle \partial_t \mathcal{R}^K,\ \phi_k\rangle_\mu = \langle \partial_t g,\ \phi_k\rangle_\mu - \left\langle \partial_t \sum_{i=1}^{\infty} \langle g,\ \phi_i\rangle_\mu \phi_i,\ \phi_k \right\rangle_\mu = \partial_t\langle g,\ \phi_k\rangle_\mu - \partial_t\langle g,\ \phi_k\rangle_\mu = 0.$$

Similarly, one can show $\left\langle \partial_x \mathcal{R}^K,\ \phi_k\right\rangle_\mu = 0$. Then (4.28) becomes

$$\begin{aligned}\langle \partial_t \mathcal{E}^K + \frac{1}{\varepsilon} V\partial_x \mathcal{E}^K + \frac{1}{\varepsilon^2}\mathcal{L}(\mathcal{E}^K),\ \phi_k\rangle_\mu &= \langle \left(\mathcal{B}(g,g) - \mathcal{B}(g^K,g^K)\right),\ \phi_k\rangle_\mu - \frac{1}{\varepsilon^2}\langle \mathcal{L}(\mathcal{R}^K),\ \phi_k\rangle_\mu \\ &= \langle \left(\mathcal{B}(g-g^K,g) + \mathcal{B}(g^K,g-g^K)\right),\ \phi_k\rangle_\mu - \frac{1}{\varepsilon^2}\langle \mathcal{L}(\mathcal{R}^K),\ \phi_k\rangle_\mu\end{aligned} \tag{4.29}$$

Then (4.29) becomes an equation for $e^k(t,x)$,

$$\begin{aligned}\partial_t e^k + \frac{1}{\varepsilon} V e^k_x + \frac{1}{\varepsilon^2}\mathcal{L}_k(e^k) &= \left(\mathcal{B}_k(g-g^K,g) + \mathcal{B}_k(g^K,g-g^K)\right) - \frac{1}{\varepsilon^2}\mathcal{L}_k(\mathcal{R}^K) \\ &\triangleq \mathcal{B}^g_k(t,x) + \frac{1}{\varepsilon^2}\mathcal{L}_k(\mathcal{R}^K).\end{aligned} \tag{4.30}$$

Since

$$\mathcal{L}_k(\mathcal{R}^K) = \int_{I_z} \left(\sigma_0 + \varepsilon^2\sigma_1(z)\right) L\left(\sum_{i=k+1}^{\infty} \langle g,\ \phi_i\rangle_\mu\right)\phi_i(z)d\mu = \varepsilon^2\int_{I_z} \sigma_1 z L\left(\sum_{i=k+1}^{\infty} \langle g,\ \phi_i\rangle_\mu\right)\phi_i(z)d\mu,$$

(4.30) is indeed

$$e_t^k + \frac{1}{\varepsilon} V e_x^k + \frac{1}{\varepsilon^2}\mathcal{L}_k(e^k) = \mathcal{B}_k^g(t,x) + \int_{I_z} \sigma_1 z \mathcal{L}(\mathcal{R}^K)\phi_k(z) d\mu \tag{4.31}$$

Let

$$\mathcal{G}_k = -\frac{1}{\varepsilon} V\partial_x - \frac{1}{\varepsilon^2}\mathcal{L}_k.$$

Then by similar analysis in Theorem 4.1, one has

$$e^k(t,x) = e^{\mathcal{G}_k t} e^k(0,x) + \int_0^t e^{\mathcal{G}_k(t-\tau)} \mathcal{B}_k^g(\tau,x) d\tau + \int_0^t e^{\mathcal{G}_k(t-\tau)} \int_{I_z} \sigma_1 z \mathcal{L}(\mathcal{R}^K)\phi_k(z) d\mu d\tau. \tag{4.32}$$

Taking L_x^2 norm and summing them from $k=1$ to $k=K$, it follows

$$\sum_{k=1}^K \|e^k\|_{L_x}^2 \le 3 e^{-\delta t}\|e^k(0)\|_{L_x}^2 + 3\int_0^t e^{-\delta(t-\tau)} \sum_{k=1}^K \|\mathcal{B}_k^g(\tau)\|_{L_x}^2 d\tau + 3\int_0^t e^{-\delta(t-\tau)} \sum_{k=1}^K \left\| \int_{I_z} \sigma_1 z \mathcal{L}(\mathcal{R}^K)\phi_k(z) d\mu \right\|_{L_x}^2 d\tau. \tag{4.33}$$

One can follow the proof in [11] to treat the non-linear term in (4.33) as

$$\begin{aligned}
&\|\mathcal{B}_k(g-g^K, g) + \mathcal{B}_k(g^K, g-g^K)\|_{L_x}^2 \\
=& \int_{\mathbb{T}} \left(\int_{I_z} \mathcal{B}_k(g-g^K, g) + \mathcal{B}_k(g^K, g-g^K)\phi_k(z) d\mu \right)^2 dx \\
\le& \int_{\mathbb{T}} \left(\int_{I_z} [\mathcal{B}_k(g-g^K, g) + \mathcal{B}_k(g^K, g-g^K)]^2 d\mu \right) \left(\int_{I_z} \phi_k^2 d\mu \right) dx \\
\le& 2\int_{I_z} \left(\|\mathcal{B}(g-g^K, g)\|_{L_x}^2 + \|\mathcal{B}(g^K, g-g^K)\|_{L_x}^2 \right) d\mu \\
\le& 2\hat{C} \int_{I_z} \left(\|g\|_{L_x}^2 \|(g-g^K)\|_{L_x}^2 + \|g^K\|_{L_x}^2 \|g-g^K)\|_{L_x}^2 \right) d\mu \\
\le& 2\hat{C} \left(\int_{I_z} \left(\|g\|_{L_x}^2 + \|g^K\|_{L_x}^2 \right) d\mu \right) \left(\int_{I_z} \|(g-g^K)\|_{L_x}^2 d\mu \right) \\
\le& C_0 e^{-\delta t} \int_{I_z} \|(g-g^K)\|_{L_x}^2 d\mu \\
\le& C_0 e^{-\delta t} \int_{I_z} \left(\|\mathcal{R}^K\|_{L_x}^2 + \|\mathcal{E}^K\|_{L_x}^2 \right) d\mu \\
\le& C_0 e^{-\delta t} \left(\|\mathcal{R}^K\|^2 + \|\mathcal{E}^K\|^2 \right) \\
\le& C_0 e^{-\delta t} \left(\frac{C_\pi C_0 e^{-\delta t}}{K^{2r}} + \|\mathcal{E}^K\|^2 \right) \\
\le& \tilde{C}_0 e^{-\delta t} \left(\frac{e^{-\delta t}}{K^{2r}} + \|\mathcal{E}^K\|^2 \right),
\end{aligned} \tag{4.34}$$

where $\tilde{C}_0$ is a constant from initial data ($\|g^K(0)\|$ and $\|g(0)\|$) and independent of K and ε. In the above estimate, we use the Cauchy-Schwartz inequality in the first and fourth inequalities, and the fifth one is due to (3.51) and (4.20). In the last inequality, we use (4.25). Similarly,

$$\left\| \int_{I_z} \sigma_1 z L(\mathcal{R}^K)\phi_k(z) d\mu \right\|_{L_x}^2 \le \tilde{C}_0{}' \sigma_{\max}^2 \frac{e^{-\delta t}}{K^{2r}}. \tag{4.35}$$

Then (4.33) becomes

$$\begin{aligned}\sum_{k=1}^K \|e^k\|_{L_x}^2 \le\ & 3e^{-\delta t}\sum_{k=1}^K \|e^k(0)\|_{L_x}^2 + 3\int_0^t e^{-\delta(t-\tau)}\tilde{C}_0 e^{-\delta\tau} K\left(\frac{e^{-\delta\tau}}{K^{2r}} + \|\mathcal{E}^K\|^2\right)d\tau \\ & + 3\int_0^t e^{-\delta(t-\tau)} K\tilde{C}_0{}' \sigma_{\max}^2 \frac{e^{-\delta t}}{K^{2r}} d\tau \\ =\ & 3e^{-\delta t}\sum_{k=1}^K \|e^k(0)\|_{L_x}^2 + 3\int_0^t e^{-\delta(t-\tau)}\tilde{C}_0 e^{-\delta\tau} K\left(\frac{e^{-\delta\tau}}{K^{2r}} + \sum_{k=1}^K \|e^k\|_{L_x}^2\right)d\tau \\ & + 3\int_0^t e^{-\delta(t-\tau)} K\tilde{C}_0{}' \sigma_{\max}^2 \frac{e^{-\delta\tau}}{K^{2r}} d\tau.\end{aligned} \tag{4.36}$$

Set $S(t) = \sup_{0\le\tau\le t} K^{2r-1}e^{\delta\tau}\sum_{k=1}^K \|e^k\|_{L_x}^2$. Multiplying $K^{2r-1}e^{\delta t}$ to both sides of (4.36), one has

$$S(t) \le 3e^{-\delta t}K^{2r-1}S(0) + \frac{3\tilde{C}_0}{\delta} + \frac{3\tilde{C}_0}{\delta}S(t) + 3\tilde{C}_0{}'\sigma_{\max}^2 T. \tag{4.37}$$

One can usually choose $\mathcal{E}^K(0) = 0$. Hence, one may obtain from (4.23) that

$$S(t) \le C(T), \tag{4.38}$$

that is

$$\|\mathcal{E}^K\| \le \frac{C(T)e^{-\frac{\delta}{2}t}}{K^{r-\frac{1}{2}}}, \tag{4.39}$$

where $C(T)$ (linearly depending on T) and δ are constants independent of K and ε. For higher derivatives in x, one can take H_x^s norm on equation (4.32) and sum those K equations. Then by similar analysis, one will have

$$\|\mathcal{E}^K\|_{H_x^s L_z^2}^2 \le \frac{C(T)e^{-\delta t}}{K^{2r-1}}. \tag{4.40}$$

Then combining with (4.25), we finish the proof.

□

Appendix

A Proof of Theorem 4.1

Proof. Let's consider the linearized equation.

$$\partial_t(k^q g^k) + \frac{1}{\varepsilon} V \partial_x (k^q g^k) + \frac{1}{\varepsilon^2}\mathcal{L}_k(k^q g^K) = 0. \tag{A.1}$$

Similar to the proof of the Lemma 3.5, one will arrive at

$$\begin{aligned}&\frac{1}{2}\partial_t \left\{ |j^q \hat{g}_k^j|^2 - \frac{\varepsilon \alpha^K k}{1+k^2}(i\mathcal{K} j^q \hat{g}_k^j, k^q \overline{\hat{g}^j}_k) \right\} + \frac{\alpha^K \lambda_2}{2}\frac{k^2}{1+k^2}|j^q \hat{g}_k^j|^2 \\ &+ \frac{1}{1+k^2}\left((1+k^2)\frac{\lambda_0 \sigma_{\min}}{\varepsilon^2} - \lambda_1 \sigma_{\max}\alpha^K k^2 + \frac{1}{\varepsilon^2}\frac{\lambda_1 \sigma_{\max}^2 C_M \alpha^K}{\lambda_2}\right)|P^\perp j^q \hat{g}_k^j|^2 \\ &\le \frac{\lambda_1 \sigma_{\max}^2 C_M \alpha^K}{\lambda_2}\sum_{i=1, i\neq j}^K |P^\perp j^q \hat{g}_k^i|^2 \chi_{ij} + \frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2 C_M}{\lambda_2 \alpha^K}\sum_{i=1,i\neq j}^K |P^\perp j^q \hat{g}_k^i|^2 \chi_{ij}.\end{aligned} \tag{A.2}$$

Here we need

$$\alpha^K \le \frac{\lambda_0 \sigma_{\min}}{2\sigma_{\max}^2 C_M \lambda_1 \varepsilon^2} \quad \text{and} \quad \alpha^K \le \frac{\lambda_0 \sigma_{\min}\lambda_2}{\sigma_{\max}^2 \lambda_1},$$

i.e.

$$\alpha^K \le \frac{\lambda_0 \sigma_{\min}}{2\sigma_{\max}^2 \lambda_1}\min\left\{\frac{1}{\varepsilon^2 C_M}, \lambda_2\right\}$$

to get

$$\begin{aligned}&\frac{1}{2}\partial_t \left\{ |j^q \hat{g}_k^j|^2 - \frac{\varepsilon \alpha^K k}{1+k^2}(i\mathcal{K} j^q \hat{g}_k^j, k^q \overline{\hat{g}^j}_k) \right\} + \frac{\alpha^K \lambda_2}{2}\frac{k^2}{1+k^2}|j^q \hat{g}_k^j|^2 + \frac{\lambda_0 \sigma_{\min}}{2\varepsilon^2}|P^\perp j^q \hat{g}_k^j|^2 \\ &\le \frac{\lambda_1 \sigma_{\max}^2 C_M \alpha^K}{\lambda_2}\sum_{i=1,i\neq j}^K |P^\perp j^q \hat{g}_k^i|^2 \chi_{ij} + \frac{1}{\varepsilon^2}\frac{2\sigma_{\max}^2 C_M}{\lambda_2 \alpha^K}\sum_{i=1,i\neq j}^K |P^\perp j^q \hat{g}_k^i|^2 \chi_{ij},\end{aligned} \tag{A.3}$$

where χ_{ij} is the indicator function of the set of indexes (i,j) such that $\tilde{S}_{ij} \neq 0$, namely,

$$\chi_{ij} = \begin{cases} 0, & \tilde{S}_{ij} = 0, \\ 1, & \tilde{S}_{ij} \neq 0. \end{cases} \tag{A.4}$$

Since σ is linear in z and ϕ_k is $(k-1)^{th}$ degree polynomials, $\tilde{S}_{ij}$ is 0 when $(i-1)+1 < (j-1)$. Thus there are only three choices for i:

$$i = j+1, \ j, \ \text{or} \ j-1,$$

and equivalently $j = i-1, \ i, \ \text{or} \ i+1$, which implies

$$\frac{j}{2} \le \frac{i+1}{2} \le i. \tag{A.5}$$

In this case, we assume $\sigma(z) = \sigma_0 + \varepsilon^{3/2}\sigma_1 z$ as stated before, one will arrive at

$$\begin{aligned}&\frac{1}{2}\partial_t \left\{ |j^q \hat{g}_k^j|^2 - \frac{\varepsilon \alpha^K k}{1+k^2}(i\mathcal{K} j^q \hat{g}_k^j, k^q \overline{\hat{g}^j}_k) \right\} + \frac{\alpha^K \lambda_2}{4}\frac{k^2}{1+k^2}|j^q \hat{g}_k^j|^2 + \frac{\lambda_0 \sigma_{\min}}{2\varepsilon^2}|P^\perp j^q \hat{g}_k^j|^2 \\ \le& \frac{\lambda_1 \sigma_{\max}^2 C_M \alpha^K 2^q}{\lambda_2}\left(|P^\perp (j-1)^q \hat{g}_k^{j-1}|^2 + |P^\perp (j+1)^q \hat{g}_k^{j+1}|^2\right) \\ +& \frac{1}{\varepsilon^2}\frac{\sigma_{\max}^2 C_M 2^{q+1}}{\lambda_2 \alpha^K}\left(|P^\perp (j-1)^q \hat{g}_k^{j-1}|^2 + |P^\perp (j+1)^q \hat{g}_k^{j+1}|^2\right) \\ =& \kappa \left(|P^\perp (j-1)^q \hat{g}_k^{j-1}|^2 + |P^\perp (j+1)^q \hat{g}_k^{j+1}|^2\right),\end{aligned} \tag{A.6}$$

where

$$\kappa = \frac{\lambda_1 \sigma_{\max}^2 C_M \alpha^K 2^q}{\lambda_2} + \frac{1}{\varepsilon^2} \frac{\sigma_{\max}^2 C_M 2^{q+1}}{\lambda_2 \alpha^K}.$$

If one gathers (A.6) from $j = 1$ to $j = K$, it follows

$$\begin{aligned}
&\sum_{j=1}^{K} \left\{ \frac{1}{2} \partial_t \{ |j^q \hat{g}_k^j|^2 - \frac{\varepsilon \alpha^K k}{1+k^2} (i\mathcal{K} j^q \hat{g}_k^j, k^q \overline{\hat{g}^j}_k) \} + \frac{\alpha^K \lambda_2}{4} \frac{k^2}{1+k^2} |j^q \hat{g}_k^j|^2 \right\} + \sum_{j=1}^{K} \frac{\lambda_0 \sigma_{\min}}{2\varepsilon^2} |P^\perp j^q \hat{g}_k^j|^2 \\
\leq &\kappa \left(\sum_{j=1}^{K-1} |P^\perp j^q \hat{g}_k^j|^2 + \sum_{j=2}^{K} |P^\perp j^q \hat{g}_k^j|^2 \right) \qquad \text{(A.7)} \\
\leq &2\kappa \left(\sum_{j=1}^{K} |P^\perp j^q \hat{g}_k^j|^2 \right).
\end{aligned}$$

In this way, if there exists α^K such that

$$2\kappa = \frac{\lambda_1 \sigma_{\max}^2 C_M \alpha^K 2^{q+1}}{\lambda_2} + \frac{1}{\varepsilon^2} \frac{\sigma_{\max}^2 C_M 2^{q+2}}{\lambda_2 \alpha^K} \leq \frac{\lambda_0 \sigma_{\min}}{4\varepsilon^2}, \qquad \text{(A.8)}$$

then the RHS can be bounded by the term $\frac{\lambda_0 \sigma_{\min}}{4\varepsilon^2} \sum_{j=1}^{K} |P^\perp j^q \hat{g}_k^j|^2$. Rewrite (A.8), it follows

$$\lambda_1 \sigma_{\max}^2 C_M 2^{q+1} (\alpha^K)^2 - \frac{\lambda_2 \lambda_0 \sigma_{\min}}{4\varepsilon^2} \alpha_K + \frac{\sigma_{\max}^2 C_M 2^{q+2}}{\varepsilon^2} \leq 0. \qquad \text{(A.9)}$$

Since $\varepsilon \leq \sqrt{\frac{\lambda_2^2 \lambda_0^2 \sigma_{\min}^2}{\sigma_{\max}^4 C_M^2 \lambda_1 2^{2q+9}}}$, that is $\left(\frac{\lambda_2 \lambda_0 \sigma_{\min}}{4\varepsilon}\right)^2 - 4\lambda_1 \sigma_{\max}^2 2^{q+1} \frac{\sigma_{\max}^2 2^{q+2}}{\varepsilon^2} > 0$, there is indeed an α_K satisfying (A.8). This leads to the exponential decay of the solution of the linearized equation (A.1). Let

$$\mathcal{G}_k = -\frac{1}{\varepsilon} V \partial_x - \frac{1}{\varepsilon^2} \mathcal{L}_k k^q.$$

Then with Lemma 4.1, one has

$$k^q g^k(t,x) = e^{\mathcal{G}_k t} k^q g^k(0,x) + \int e^{\mathcal{G}_k (t-\tau)} k^q \mathcal{B}_k(g^K, g^K) d\tau. \qquad \text{(A.10)}$$

Taking H_x^s norm and summing them from $k = 1$ to $k = K$, it follows

$$\begin{aligned}
\sum_{k=1}^{K} \|k^q g^k(t)\|_{H_x^s}^2 &\leq \sum_{k=1}^{K} \|e^{\mathcal{G}_k t} k^q g^k(0)\|_{H_x^s}^2 + \int \sum_{k=1}^{K} \|e^{\mathcal{G}_k (t-\tau)} k^q \mathcal{B}_k(g^K, g^K)\|_{H_x^s}^2 d\tau \\
&\leq e^{-\delta t} \sum_{k=1}^{K} \|k^q g^k(0)\|_{H_x^s}^2 + \int e^{-\delta(t-\tau)} \left(\sum_{k=1}^{K} \|k^q g^k(t)\|_{H_x^s}^2 \right)^2 d\tau.
\end{aligned} \qquad \text{(A.11)}$$

Then with similar argument in Theorem 3.2, one obtains uniform exponential decay for $E^K(t) \triangleq \sum_{k=1}^{K} \|k^q g^k\|_{H_x^s}^2$. □

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