Quantum Simulation of Partial Differential Equations via Schrödingerization

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We present a novel new way—called *Schrödingerization*—to simulate general (quantum and nonquantum) systems of linear ordinary and partial differential equations (PDEs) via quantum simulation. We introduce a new transform, referred to as *the warped phase transformation*, where any linear—including nonautonamous—system of ordinary or partial differential equation can be recast into a system of Schrödinger's equations, in real time, in a straightforward way. This approach is not only applicable to PDEs for classical problems but is also useful for quantum problems, including the preparation of quantum ground states and Gibbs thermal states, the simulation of quantum states in random media in the semiclassical limit, simulation of Schrödinger's equation in a bounded domain with artificial boundary conditions, and other non-Hermitian physics. This formulation is versatile enough to be applicable in a simple way to both digital quantum simulation as well as to analog quantum simulation, and using either qubits or continuous-variable quantum systems (qumodes).

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Introduction—Quantum simulation is one of the most natural tasks for a quantum device by preparing outputs of Schrödinger's equations directly via its own evolution [1]. However, what can we say about the suitability of quantum simulation for other more general dynamical laws, typically in the form of ordinary or partial differential equations (ODEs or PDEs) that are not Schrödinger type equations or unitary dynamics? Many of these differential equations, often due to their high dimensionality and/or the presence of multiple time and spatial scales, are prohibitively expensive for classical computers; thus it is highly desirable to develop quantum algorithms to solve them. This will also greatly broaden the application areas of quantum computing in science and engineering problems.

While the most obvious applications are for classical problems that are difficult for classical computers, simulating classical dynamics is also important for *quantum problems*. The most well-known include the preparation of quantum ground states [2,3] and Gibbs states [4], which benefit from the simulation of nonunitary dynamics and are particularly important in areas like quantum chemistry [5] and optimization.

We know quantum simulation of classical dynamics is in principle possible, if we accept reductionism and that the world is fundamentally quantum mechanical. All classical dynamical laws are, with the exception of relativity, in principle derivable from underlying Schrödinger's equations. From a computational perspective, we also know that any classical gate can be trivially embedded into a quantum gate, so we expect quantum gates to be able to simulate classical ones. However, the quantum degrees of freedom involved could typically be significantly larger than that of the classical systems one wishes to simulate. An essential question is, then, are there less resource-intensive ways to represent classical dynamics with Schrödinger's equations?

Consider the case of numerical solutions to linear PDEs, which, upon spatial and temporal discretizations, become a system of linear algebraic equations. Quantum algorithms in solving a system of linear algebraic equations [6,7] can subsequently lead to possible polynomial or superpolynomial speedups in solving PDEs [8,9]. For their nonlinear counterparts see [10–12]. However, here quantum simulation only plays the role of an algorithmic primitive: due to discretization in time, one does not prepare solution states continuously in time t by evolution of a Schrödinger equation in time t.

To obviate the discretization in time, a main difficulty is in finding a way to represent nonunitary dynamics with a unitary one, which can be evolved by quantum simulation like the Schrödinger equation. One way is via qubitization [13,14] (or block encoding), and this has very recently been applied to linear PDEs [15]. This involves unitary dilation methods (e.g., [16]) and has origins in quantum signal processing [17]. While this fairly general formalism can in principle approximate the action of any nonunitary operator, it relies heavily on building linear combinations of

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quantum states and finding suitable polynomial approximations to the nonunitary operator. This is not always simple to describe or to implement in practice with qubits and cannot take advantage of continuous-variable quantum systems. Another method, limited to the heat equation, is the imaginary time evolution approach, where the heat equation can be converted to a Schrödinger equation through a change to *imaginary* time $t \to it$ [18]. However, the state obeying Schrödinger's equation and its imaginary time counterpart do not have the same evolution. This means that, except for the steady state solution, extra resources are necessary to map between the solution in the unitarily evolving system to the original one. For instance, one requires tomographic measurements of quantum states at each small time step in the unitary evolution [19]. A conceptual alternative is required.

Thus we have an important question: *Is there a simpler and generic way of obtaining Schrödinger's equations naturally from any given linear dynamical system?*

We propose a new paradigm—based on a novel transformation called the *warped phase transformation*—that can map *any* linear PDE and ODE (including dissipative time-irreversible and nonautonomous ones), to Schrödinger equations in *real* time, by going to one higher dimension followed by a Fourier transform. The quantum simulation of this Schrödinger dynamics in time t makes it possible to prepare solutions of the original PDEs/ODEs at any time t, in the form of a quantum state. We call this the *Schrödingerization* approach.

This new method is not limited to the quantum simulation of any linear PDE or ODE that might arise from classical problems. It applies to many quantum problems like the preparation of quantum ground states, Gibbs thermal states, and the simulation of quantum systems in random media in the semiclassical limit and with artifical boundary conditions [20]. It is also useful for discrete dynamical systems and linear algebra [21]. Furthermore, the formulation makes it straightforward to allow both discrete and analog quantum simulation, with qubits or continuous-variable quantum systems or a hybrid or even qudits [22,23]. This versatility is a feature not currently shared with other methods.

Background—When $\boldsymbol{u}(t)$ evolves under a unitary evolution that is generated by a (Hermitian) Hamiltonian $\boldsymbol{H}(t) = \boldsymbol{H}^\dagger(t)$, it satisfies the linear differential equation

$$i\frac{d\boldsymbol{u}(t)}{dt} = \boldsymbol{H}(t)\boldsymbol{u}(t), \qquad \boldsymbol{u}(t=0) = \boldsymbol{u}(0). \tag{1}$$

The canonical description of Schrödinger's equation is a linear PDE in which $u(x,t)$ is the wave function with normalization constraint on the l_2 norm $\|\boldsymbol{u}(t)\|_2^2 = \int |u(t,x)|^2 dx = 1$, invariant in time, where $x \in \mathbb{R}^d$ is the position in d dimensions. Here $\boldsymbol{H}(t)\boldsymbol{u}(t) = \int (-\nabla_x^2 + V(t,x)) u(x,t)|x\rangle dx$, where $\boldsymbol{u}(t) = \int u(t,x)|x\rangle dx$ is the continuous-variable quantum state of d qumodes with the orthonormal basis set $\{|x\rangle\}_{x\in\mathbb{R}^d}$, ∇_x^2 is the Laplace operator with respect to x, and $V(t,x)$ is the potential function. To find numerical solutions to this problem, one can discretize in x, with uniform mesh sizes along each dimension $\Delta x = 2/M$ where M is a positive even integer. Then we use a discrete-variable representation, where $\boldsymbol{u}(t) = \sum_{i=1}^{M^d} u(t,x_i)|i\rangle$ is a M^d-dimensional vector whose entries are $u(t,x_i)$ at grid points with $i = 1,\ldots,M^d$, satisfying $\|\boldsymbol{u}(t)\|_2^2 = \sum_{i=1}^{M^d} |u(t,x_i)|^2 = 1$, where $\{|i\rangle\}$ is an orthonormal basis set. When given a state spanned by $\{|0\rangle, |1\rangle\}$ over the field $\mathbb{C}$, this is called a qubit. Thus $\boldsymbol{u}(t)\rangle$ can be described by a system of $d\log_2(M)$ qubits. In this case, $\boldsymbol{H}(t)$ is a $M^d \times M^d$ Hermitian matrix that results from a suitable discretization of the Schrödinger Hamiltonian $-\nabla_x^2 + V(t,x)$ in x. One can also use different discrete representations where $|i\rangle$ are the energy eigenstates of $\boldsymbol{H}(t)$, which are sums of products of Pauli operators, as is typical in quantum chemistry applications.

Quantum simulation addresses the question of how to prepare the state $\boldsymbol{u}(t)$ given access to $\boldsymbol{H}(t)$ and initial state $\boldsymbol{u}(0)$, and an estimation of the resources required. Quantum simulation falls very roughly into two categories: discretized time (digital) or continuous time (analog) [24]. In the analog case, one finds a quantum system that naturally realizes $\boldsymbol{H}(t)$ so time can run continuously without splitting the unitary evolution into many smaller pieces, for example using Trotter splitting. Here both qubit-based and continuous-variable quantum systems can be employed, but the latter has more potential for near-term implementation. However, if such an analog quantum system cannot be easily found or controlled, one can wait for farther-term technology involving digital quantum simulation methods. In this case, one counts resources as the number of queries to some given black boxes (oracles) and the number of two-qubit gates needed. This is referred to as the query and gate complexities, respectively. For simplicity here, we cite the result for time-independent $\boldsymbol{H}$, but this can be extended to time-dependent Hamiltonians (for example, [25]). Let s be the sparsity of $\boldsymbol{H}$ (maximum number of nonzero entries in each row) and $\|\boldsymbol{H}\|_{\max}$ be its max-norm (value of largest entry in absolute value). We denote the $(i,j)^{\text{th}}$ entry to $\boldsymbol{H}$ as H_{ij}. A common set of black boxes used in Hamiltonian simulation is known as the sparse access.

Definition 1—Sparse access to Hermitian matrix $\boldsymbol{H}$ refers to two unitary black boxes O_M and O_F such that $O_M|j\rangle|k\rangle|z\rangle = |j\rangle|k\rangle|z \oplus H_{jk}\rangle$ and $O_F|j\rangle|l\rangle = |j\rangle|F(j,l)\rangle$. Here the function F takes the row index j and a number $l = 1,2,\ldots,s$ and outputs the column index of the l^{th} nonzero elements in row j. There are quantum simulation protocols in terms of query complexity that scale linearly in t [13] using sparse access or linearly in t up to logarithmic factors [26].

Lemma 1—[26] Let $\tau = st\|\boldsymbol{H}\|_{\max}$. Then $\exp(-i\boldsymbol{H}t)$ acting on m_H qubits can be simulated to within error ε with query complexity $\mathcal{O}\big(\tau\log(\tau/\varepsilon)/[\log\log(\tau/\varepsilon)]\big)$ and gate complexity $\mathcal{O}\big(\tau[m_H+\log^{2.5}(\tau/\varepsilon)]\log(\tau/\varepsilon)/[\log\log(\tau/\varepsilon)]\big)$.

Throughout the Letter, we use $\tilde{\mathcal{O}}$ to denote $\mathcal{O}$ where logarithmic terms are ignored.

Schrödingerization for general ODEs or PDEs—More general linear ODEs or PDEs for $\boldsymbol{u}(t)$ that is first order in time $t \geq 0$ can be written

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{A}(t)\boldsymbol{u}, \qquad \boldsymbol{u}(t=0) = \boldsymbol{u}(0), \tag{2}$$

where $\boldsymbol{A}$ is a linear operator for ODEs and a linear differential operator for PDEs. In general the evolution operator $\mathcal{T}e^{\int_0^t \boldsymbol{A}(\tau)d\tau}$ (where $\mathcal{T}$ is the time-ordering operator) is not unitary, thus the system is not directly suitable for quantum simulation. We first decompose $\boldsymbol{A}(t)$ into an Hermitian part and an anti-Hermitian part: $\boldsymbol{A}(t) = \boldsymbol{H}_1(t) - i\boldsymbol{H}_2(t)$, where $\boldsymbol{H}_1(t) = (\boldsymbol{A}(t)+\boldsymbol{A}^\dagger(t))/2 = \boldsymbol{H}_1^\dagger(t)$ and $\boldsymbol{H}_2(t) = i(\boldsymbol{A}(t) - \boldsymbol{A}^\dagger(t))/2 = \boldsymbol{H}_2^\dagger(t)$ are both Hermitian and we assume $-\boldsymbol{H}_1$ to be positive semidefinite (so the original system is stable).

It is our aim to transform any equation of the form in Eq. (2) to that of Schrödinger's form in Eq. (1).

We begin by introducing a real one-dimensional variable $\xi > 0$ and define

$$\boldsymbol{w}(t,\xi) = e^{-\xi}\boldsymbol{u}(t). \tag{3}$$

This transformation—which we call the *warped phase transformation*—is a crucial ingredient. A similar transformation was used in [27] for a completely different purpose: to develop efficient quantum computing algorithms for uncertainty quantification problems in PDEs.

One can recover the solution to the original equation for $\boldsymbol{u}(t)$ using $\boldsymbol{u}(t) = \int_0^\infty \boldsymbol{w}(t,\xi)\mathrm{d}\xi$. Alternatively, one can also recover $\boldsymbol{u}$ via $\boldsymbol{u}(t) = e^{\xi}\boldsymbol{w}(t,\xi)$ for any $\xi > 0$.

Clearly, $\boldsymbol{w}$ solves the following PDE:

$$\partial_t \boldsymbol{w} = -\boldsymbol{H}_1(t)\partial_\xi \boldsymbol{w} - i\boldsymbol{H}_2(t)\boldsymbol{w}.$$

We now extend the domain of ξ to $(-\infty,\infty)$, with evenly extended initial condition $\boldsymbol{w}(0,\xi) = \exp(-|\xi|)\boldsymbol{u}_0$. Let $\tilde{\boldsymbol{w}} = \tilde{\boldsymbol{w}}(t,\eta)$ be the Fourier transform of $\boldsymbol{w}$ in ξ and $\eta \in \mathbb{R}$ be the Fourier mode. Then $\tilde{\boldsymbol{w}}$ satisfies a system of *uncoupled* Schrödinger-like equations

$$i\partial_t \tilde{\boldsymbol{w}} = (\eta\boldsymbol{H}_1(t) + \boldsymbol{H}_2(t))\tilde{\boldsymbol{w}}, \tag{4}$$

one for each η! Clearly $\eta\boldsymbol{H}_1(t) + \boldsymbol{H}_2(t)$ *is* Hermitian. We call this the *Schrödingerized* equation for $\boldsymbol{u}$. In the case without discretization of any parameter, we can consider the state $|\tilde{w}(t)\rangle \equiv (1/\|\tilde{\boldsymbol{w}}(t)\|)\int_{-\infty}^{\infty}\tilde{\boldsymbol{w}}(t,\eta)|\eta\rangle d\eta$ and then Eq. (4) becomes the Schrödinger-like equation

$$i\partial_t|\tilde{w}(t)\rangle = \boldsymbol{H}(t)|\tilde{w}(t)\rangle, \qquad \boldsymbol{H}(t) = \boldsymbol{H}^\dagger(t), \tag{5}$$

where $\boldsymbol{H}(t) = \boldsymbol{H}_1(t)\otimes\hat{\eta} + \boldsymbol{H}_2(t)\otimes\mathbf{1}$ and $\hat{\eta}|\eta\rangle = \eta|\eta\rangle$. See [22,23] for more details on this continuous-variable analog simulation approach and examples of some PDEs which are already amenable to near-term implementation without the need for digital quantum simulation. In these cases, queries and gate complexities are not relevant resources. For a d-dimensional linear PDE problem, we only require $d+1$ continuous-variable quantum modes (qumodes).

Alternatively, we can also solve these equations numerically, where digital quantum simulation is necessary. We discretize the system in x and ξ but not in t. We choose uniform mesh sizes $\Delta x = 2/M$ for the position variable in each dimension, $\Delta\xi = 2L/N$ for the ξ variable, $\Delta\eta = 2L/N$ for the η variable, where M and N are even positive integers and $L > 0$. We can define the quantum state $|\tilde{w}(t)\rangle = (1/\|\tilde{\boldsymbol{w}}(t)\|)\sum_{i=1}^{M^d}\sum_{j=-N/2}^{N/2}\tilde{w}(t,x_i,\eta_j)|i,j\rangle$. Here $\tilde{w}(t,x_i,\eta)$ with $i = 1,\ldots,M^d$ labels the x grid points and $\eta_j = j\Delta\eta$ with $j = -N/2,\ldots,N/2$. Then the discretization of the Schrödingerized equation means that Eq. (5) is now a system of M^dN ODEs with

$$\boldsymbol{H}(t) = \boldsymbol{H}_1(t)\otimes\boldsymbol{D} + \boldsymbol{H}_2(t)\otimes\mathbf{1}, \tag{6}$$

where $\boldsymbol{H}_1(t)$ and $\boldsymbol{H}_2(t)$ involve the discretization of differential operators and the matrix $\boldsymbol{D} = \mathrm{diag}(\mu_1,\ldots,\mu_N)$ is also a diagonal matrix, with entries $\mu_j = \pi(j - N/2)$.

For example, for the d-dimensional heat equation with unit diffusion, $\boldsymbol{H}_2 = 0$ and $\boldsymbol{H}_1 = -\boldsymbol{P}_1^2\cdots-\boldsymbol{P}_d^2$ where $\boldsymbol{P}_l = \mathbf{1}^{\otimes l-1}\otimes P_l\otimes\mathbf{1}^{\otimes d-l}$, $l = 1,\ldots,d$ and P_l is the discretization of the momentum operator $-i\partial_x$ with respect to the l^{th} spatial variable. We define $|\tilde{w}(t)\rangle = (\mathbf{1}^{\otimes M^d}\otimes\mathcal{F}_\xi)|w(t)\rangle$, where $\mathcal{F}_\xi$ is the discrete Fourier transform with respect to variable ξ, so $|\tilde{w}(0)\rangle = (\mathbf{1}^{\otimes M^d}\otimes\mathcal{F}_\xi)|u_0\rangle\sum_{j=-N/2}^{N/2}\exp(-|\xi_j|)|j\rangle$. After evolving this initial state with respect to unitary generated by the Hamiltonian $\boldsymbol{H}$ and applying an inverse quantum Fourier transform $\mathcal{F}_\xi^{-1}$ onto the second register, we obtain $|w(t)\rangle = (\mathbf{1}^{\otimes M^d}\otimes\mathcal{F}_\xi^{-1})|\tilde{w}(t)\rangle$. Due to this unitary evolution, $\|\boldsymbol{w}(t)\| = \|\tilde{\boldsymbol{w}}(t)\|$. From $|w(t)\rangle$ one can recover the quantum state of u whose entries are proportional to the solutions of the original equation $|u(t)\rangle = (1/\|\boldsymbol{u}(t)\|)\sum_{i=1}^{M^d}u(t,x_i)|i\rangle$. We can do this by either projecting $|w(t)\rangle$ onto $\mathbf{1}\otimes\sum_{k=N/2}^{N}|k\rangle\langle k|$ (projecting only onto $\xi > 0$), or using amplitude amplification to boost the chance of retrieving $|u(t)\rangle$ to probability $\sim\|\boldsymbol{u}(0)\|/\|\boldsymbol{u}(t)\|$.

We call this the *Schrödingerization* approach to preparing $|u(t)\rangle = \boldsymbol{u}(t)/\|\boldsymbol{u}(t)\|_2$, where we only need quantum simulation applied onto $|\tilde{w}(t)\rangle$ using the Hamiltonian $\boldsymbol{H}(t)$.

For simplicity of the quantum simulation part, we can look at the case of time-independent $\boldsymbol{A}$. Let $s \sim \max(s(\boldsymbol{H}_1), s(\boldsymbol{H}_2))$ be the maximum of the sparsity of $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. The max-norm $\|\boldsymbol{H}\|_{\max} \sim \max(\|\boldsymbol{H}_1\|_{\max}/\epsilon, \|\boldsymbol{H}_2\|_{\max})$, where ϵ is the computational precision, which appears here due to the extra ∂_ξ operator. In the transport equation case, the two terms in the latter expression are of the same order (see Supplemental Material Sec. 5[28]). The cost in digital quantum simulation for Schrödingerization is then the following.

Theorem 1—Given sparse access to the $M^d \times M^d$ matrix $\boldsymbol{H}$ and the unitary U_{initial} that prepares the initial quantum state $|\boldsymbol{u}(0)\rangle$ to precision ϵ. With the Schrödingerization approach, the state $|\boldsymbol{u}(t)\rangle$ can be prepared with query complexity $\tilde{\mathcal{O}}((\|\boldsymbol{u}(0)\|/\|\boldsymbol{u}(t)\|)st\|\boldsymbol{H}\|_{\max})$ and $\tilde{\mathcal{O}}(\|\boldsymbol{u}(0)\|/\|\boldsymbol{u}(t)\|dst\|\boldsymbol{H}\|_{\max})$ additional two-qubit gates.

Proof—See Supplemental Material Sec. 2 [28] and Ref. [29] for the use of amplitude amplification to reduce the above scaling from $\|\boldsymbol{u}(0)\|^2/\|\boldsymbol{u}(t)\|^2$ to $\|\boldsymbol{u}(0)\|/\|\boldsymbol{u}(t)\|$. ∎

Below we consider some applications of this approach, both for quantum and classical problems.

Ground state preparation: Suppose one wants to prepare a D-dimensional ground state $|E_0\rangle$ from a given state $|u(0)\rangle = \sum_{j=0}^{D-1} \alpha_j |E_j\rangle$, where $\alpha_j \in \mathbb{C}$, $\|\boldsymbol{u}(0)\| = 1$, and $\{|E_j\rangle\}$ are the nondegenerate orthonormal eigenstates of a $D \times D$ (positive definite Hermitian) Hamiltonian $\boldsymbol{h}$. $\boldsymbol{u}(t)$ is the vector whose entries are the amplitudes of the unnormalized $|u(t)\rangle$. If one evolves $\boldsymbol{u}(t) = \exp(-\boldsymbol{h}t)\boldsymbol{u}(0)$ according to $\partial_t \boldsymbol{u} = -\boldsymbol{h}\boldsymbol{u}$ with initial condition $\boldsymbol{u}(0)$, then one can write $|u(t)\rangle \propto \exp(-\boldsymbol{h}t)|u(0)\rangle$ where $\boldsymbol{h}$ is a $D \times D$ Hermitian matrix with sparsity s and max-norm $\|\boldsymbol{h}\|_{\max}$. Then $\boldsymbol{H}_2 = -\boldsymbol{h}$, $\boldsymbol{H}_2 = \boldsymbol{0}$. Assuming a nonzero spectral gap $\Delta = E_1 - E_0 > 0$, then the convergence to the ground state $|u(t)\rangle \to |E_0\rangle$ is exponentially fast. Then it can be shown that the total query and gate complexity costs in preparing the ground state to quantum fidelity $1 - \epsilon$ for $\epsilon \ll 1$ is $\tilde{\mathcal{O}}(s\|\boldsymbol{h}\|_{\max}/(|\alpha_0|\Delta\epsilon))$. Here the scaling in $|\alpha_0|$ and Δ is comparable to nonheuristic schemes like quantum phase estimation [2] and near-optimal lower bounds for ground state preparation [3], up to logarithmic factors. The drawback is that this scheme has an extra factor $1/\epsilon$ compared to the near-optimal schemes. This originates from $\|\boldsymbol{H}_2 \otimes \boldsymbol{D}\|_{\max} \sim \|\boldsymbol{h}\|_{\max}/\epsilon$. This can be improved to almost $\log(1/\epsilon)$ by using smoother initial data for $\boldsymbol{w}$. See [30]. See Supplemental Material Sec. 3[28] for more details, which also contains a short description of the difference to the imaginary time evolution method [19].

Gibbs thermal state preparation: To create the Gibbs thermal state at temperature T corresponding to the same $D \times D$ Hamiltonian $\boldsymbol{h}$, we can similarly use the Schrödingerization approach to prepare the normalized pure state $|\Psi(\beta)\rangle = \sum_k \sqrt{\exp(-\beta E_k)/Z}|E_k E_k\rangle$, where $Z = \mathrm{Tr}(\exp(-\beta\boldsymbol{h}))$ is the partition function for $\boldsymbol{h}$ with $\beta = 1/(k_B T)$. In this case we make use of Schrödingerization with $\boldsymbol{H}_2 = -\boldsymbol{h} \otimes \boldsymbol{1}$. Then by tracing out one register we obtain the Gibbs state $\rho_{Gibbs}(\beta) = \sum_{j=0}^{D-1} \exp(-\beta E_j)|E_j\rangle\langle E_j| = \mathrm{Tr}_1(|\Psi(\beta)\rangle\langle\Psi(\beta)|)$. This quantum simulation method differs from previous methods [4] that make use of quantum phase estimation to prepare $|\Psi(\beta)\rangle$. Following Theorem 3, it can be shown that we can prepare $\rho_{Gibbs}(\beta)$ to precision ϵ with query and gate complexity $\tilde{\mathcal{O}}(s\|\boldsymbol{h}\|_{\max}\beta\sqrt{D/Z}/\epsilon)$. Heuristic methods aside, this coincides with the best-known scaling, to our knowledge, with respect to D and Z [4,14]. The ϵ scaling can be similarly improved to almost $\log(1/\epsilon)$ by using smoother initial data for $\boldsymbol{w}$; see [30]. See Supplemental Material Sec. 4[28] for more details.

Quantum PDEs: In practice, the simulation of quantum dynamics is done in a bounded domain. Here one employs artificial boundary conditions that may absorb outgoing wave packets and keep the size of the computational domain to a minimum. These methods include complex absorbing potential, which involves including an imaginary component to the potential. It also includes perfectly matched layers and a Dirichlet-to-Neumann map. All these can be tackled with Schrödingerization [20]. In fact, our methods can be applied more broadly to more general non-Hermitian physics, which will be an area of further exploration [31]. Another example is the application of Schrödingerization to solving linear transport problems, which can be used to simulate quantum states in random media in the semiclassical limit [32,33]. See Supplemental Material Sec. 5 [28] for more details, and see [34] and [35] therein for why the transport problem can be treated similarly to the ground state preparation problem, with observables extracted using, for instance, [36].

Classical ODEs and PDEs: There are many other applications where Schrödingerization has been applied—general PDEs including Liouville, Fokker Planck, Vlaslov-Fokker-Planck, and Black-Scholes' equations [37], Maxwell's equations [39], nonlinear Hamilton-Jacobi PDEs, nonlinear scalar hyperbolic PDEs and nonlinear ODEs [22,38], and PDEs with physical boundary conditions and interface conditions [40]. Schrödingerization is also suitable for nonautonomous PDEs [25]. It can be shown that Schrödingerization also provides a new stable numerical method for ill-posed PDEs [30]. Our method can also be extended to discrete dynamical systems [21].

Linear algebra problems: The cost to solve a quantum linear system of equations using Schrödingerization, unlike using methods based on solving a system of linear algebraic equations by matrix inversion [6], *has no direct dependence on the condition number of the matrices* [21]. This makes these class of methods distinct from previous quantum linear systems algorithms.

Summary—We have introduced a conceptually new method, called Schrödingerization, which makes it possible—in a simple yet general way—to simulate solutions of *any*

linear ODE or PDE using quantum simulation. This method can be framed in the traditional language of (continuous-variable) dynamical equations. The connection between classical dynamics and its corresponding Schrödinger's equations can be seen easily at the level of the dynamical equations, without approximations and with the addition of only a single extra dimension. For deeper connections between this method and the mathematics of dilation, see [41]. It is also suitable for both the continuous-variable or analogue and digital frameworks alike [22].

Note added—After completion of this article, the authors were informed by the authors of [42] that they were preparing a paper which presented a complementary perspective on a similar algorithm. Their approach, as shown in Theorem 1 in [42], directly represents the solution—in integral form of the Fourier variable—of our Schrödingerized PDEs, then expresses it as a linear combination of Hamiltonian simulation problems. This would be more appealing to the linear combination of unitaries community. We directly take the Fourier transform and write it into a PDE form, the Schrödingerized equation for $\tilde{\boldsymbol{w}}$. Ours is more attractive to the PDE community and more convenient to the use of the well-established computational PDE techniques to develop quantum algorithms. Another important feature of Schrödingerization is that, since it is a continuous variable formulation, it is naturally suited for analog quantum computing, which could be realized in the nearer term compared with digital quantum computation [22,23].

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