l^1-error estimates on the immersed interface upwind scheme for linear convection equations with piecewise constant coefficients: a simple proof *

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Abstract

A linear convection equation with discontinuous coefficients arises in wave propagation through interfaces. An interface condition is needed at the interface to select a unique solution. An upwind scheme that builds this interface condition into its numerical flux is called the immersed interface upwind scheme. An l^1 error estimate of such a scheme was first established in [X. Wen and Shi Jin, *J. Comp. Math.* 26, 1-22, 2008]. In this paper, we provide a simple analysis on the l^1 error estimate. The main idea is to formulate the solution of the underline initial-value problem into the sum of solutions of two convection equations with *constant* coefficients, which can then be estimated using classical methods for the initial or boundary value problems.

1 Introduction

We are interested in the linear convection equation

$$\begin{cases} u_t + (c(x)u)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

with a piecewise constant coefficient

$$c(x) = \begin{cases} c^- > 0, & x < 0, \\ c^+ > 0, & x > 0. \end{cases} \tag{2}$$

Such equations arise in modeling wave propagation through interfaces, where jumps in $c(x)$ correspond to interfaces between different media. This is also the simplest example of hyperbolic conservations laws with discontinuous coefficients, a subject that has generated lots of mathematical interests (see for example [1, 2, 6, 21]). On the other hand, bearing the physical background in mind, an interface condition is needed at $x = 0$:

$$u(0^+, t) = \rho\, u(0^-, t), \tag{3}$$

where $\rho = 1$ corresponds to conservation of mass (u) or $\rho = c^-/c^+$ for the conservation of flux. This is the approach that will be taken in this paper, as it was in [31, 33]. Such a condition will guarantee the well-posedness of the initial value problem to (1), see [12].

When numerically solving (1)-(2), one can take the *immersed interface* approach, namely, to build the interface condition (3) into the numerical flux [31, 33] (see also [19, 17, 22] for immersed boundary or

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interface methods for general problems, and the Hamiltonian-preserving methods for Liouville equations with discontinuous potentials [11, 12]). This paper is interested in the error estimates of such a scheme.

For hyperbolic conservation laws with Lipschitz continuous coefficients, there were numerous works on convergence rate estimates for numerical methods. Half-order optimal convergence rate for monotone type or viscosity type methods were established in [3, 16, 18, 23, 25, 26, 27, 28]. For hyperbolic equations with singular coefficients, or conservation laws with discontinuous flux functions, many authors have studied the convergence of the numerical methods [7, 13, 14, 15, 20, 24, 29]. In particular, for equation (1) with a general $c(x)$ on indefinite sign changes, the convergence of a class of finite difference schemes to the duality solutions was proved in [8]. For problem (1)-(2), the duality solution is the one corresponding to the interface condition (3) with $\rho = 1$. The convergence rate of these schemes, however, are much less studied, except in a recent work where Wen and Jin established the half-order l^1 error estimate [31]. Their proof relies on the expression of the solution at later time by the initial data, following (the discrete version) of the method of characteristics subject to the interface condition (3). Some inequalities on binomial coefficients [30] were needed to complete the proof.

In this paper, we provide a simple l^1 error estimate on the immersed upwind method for (1)-(3), to be described now. Let the spatial mesh be $x_i = i\Delta x$, where $i \in \mathbb{Z}$, the set of all integers, and Δx is the mesh size. Let $t^n = n\Delta t$ be the discrete time where Δt is the time step. Let $U_i^n = U(x_i, t^n)$ be the numerical approximation of $u(x_i, t^n)$. The immersed upwind scheme proposed in [31] for the convection equation (1)-(2) with the interface condition (3) is

$$\begin{cases} U_i^{n+1} = (1-\lambda^-)U_i^n + \lambda^- U_{i-1}^n, & \text{if } i \le 0, \\ U_i^{n+1} = (1-\lambda^+)U_i^n + \lambda^+ \rho U_{i-1}^n, & \text{if } i = 1, \\ U_i^{n+1} = (1-\lambda^+)U_i^n + \lambda^+ U_{i-1}^n, & \text{if } i \ge 2, \end{cases} \tag{S1}$$

where $\lambda^\pm = c^\pm \frac{\Delta t}{\Delta x}$.

Our main idea is: 1) converting problem (1)-(3) into two initial-value problems of convection equations with *constant coefficients*, one of which uses a stretched variable for $x < 0$, and 2) formulating the convection equation with stretched variable to a boundary problem. Then we can use classical error estimate results for initial and boundary value problems of linear convections equations with *continuous* coefficients.

Let

$$\Gamma(a) = 2\sqrt{a\Delta x(1 - a\frac{\Delta t}{\Delta x})t} + \Delta x\,. \tag{4}$$

and the variation of u

$$\|u\|_{BV(A)} = \sup_{|h|\neq 0} \frac{1}{|h|} \|u(\cdot + h) - u(\cdot)\|_{L_1(A)}\,.$$

We also define the l^1-norm of a vector $\mathbf{b} = (b_j)_{j\in\mathbb{Z}} = \sum_{j\in\mathbb{Z}} |b_j|$.

Our main result is summarized in the following theorem.

Theorem 1. *Let $u_0(x)$ be a function of bounded variation. Then $\forall \rho > 0$ in the interface condition (3), the immersed interface upwind difference scheme (S1), under the CFL condition $0 < \lambda^\pm < 1$, has the following l^1-error bound:*

$$\begin{aligned} &\|U^n - u(\cdot, t_n; u_0)\|_{l^1} \\ &\le \left[\|u_0\|_{BV(\mathbb{R}^-\cup\{0\})} + \left(2\rho \|u_0\|_{BV(\mathbb{R}^-\cup\{0\})} + B\right)\left(\frac{c^+}{c^-}\right)\right]\Gamma(c^-) + \left[\rho \|u_0\|_{BV(\mathbb{R}^-\cup\{0\})} + \|u_0\|_{BV(\mathbb{R}^+)}\right]\Gamma(c^+), \end{aligned} \tag{5}$$

where

$$B = \left|\rho u_0(0^+) - u_0(0^-)\right|. \tag{6}$$

Our proof will use well-established error estimates, such as the following l^1-error estimate proved in [25], for constant $c(x)$:

Theorem 2. *The l^1-error of the upwind scheme for solving (1) with $c(x) \equiv a > 0$ is*

$$\|U^n - u(\cdot, t_n; u_0)\|_{l^1} \le \|u_0(x)\|_{BV}\,\Gamma(a)$$

Compare to the error estimates of [31], we obtain the same convergence rate but with larger constants in the bound (5). Our proof is significantly simpler than that of [31] which relies on complex combinatorial inequalities [30]. Although we only treated the case of one discontinuity, it is straightforward to extend to the case of finite number of isolated discontinuities (see Remark at the end of Section 3). It also has the potential to be applied to more complicated problems in wave propagations through interfaces. One example is the Liouville equation with discontinuous potentials [32] which has important applications in computational high frequency waves in heterogeneous media, and semiclassical modelling of quantum dynamics with potential barriers, see [11, 12, 10, 9].

2 Error bounds for boundary value problems

Before we prove theorem 1, we need the following lemmas for boundary-value problems, where the boundary conditions have discontinuities.

Lemma 3. *For the half-plane problem*

$$\begin{cases} u_t + au_x = 0, & t > 0, x \in [0,\infty), a > 0 \\ u(x,0) = 0, & x \in [0,\infty), \\ u(0,t) = \begin{cases} 0, & 0 < t < \Delta t, \\ Z, & t \geq \Delta t, \end{cases} \end{cases} \tag{B1}$$

if the upwind method is used,

$$\begin{cases} U_i^{n+1} = (1-\lambda)U_i^n + \lambda U_{i-1}^n, & i \geq 1,\ n \geq 0 \\ U_i^0 = 0, & i \geq 1, \\ U_0^n = \begin{cases} 0, & n = 0, \\ Z, & n \geq 1, \end{cases} \end{cases} \tag{BS1}$$

where $\lambda = a(\Delta t/\Delta x) \leq 1$, then the l^1-error $\|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^+)}$ is bounded by $|Z|\Gamma(a)$ for $t \geq \Delta t$, where $\Gamma(a)$ is defined in (4).

Proof. We define an initial problem

$$\begin{cases} v_t + av_x = 0, & t > \Delta t, x \in (-\infty,\infty), \\ v(x,\Delta t) = \begin{cases} Z, & x \leq 0, \\ 0, & x > 0. \end{cases} \end{cases} \tag{B2}$$

Clearly, $u(x,t) = v(x,t)$, for $t \geq \Delta t$, $x \in [0,\infty)$.

If one uses the upwind method

$$\begin{cases} V_i^{n+1} = (1-\lambda)V_i^n + \lambda V_{i-1}^n, & n \geq 2, \\ V_i^1 = \begin{cases} Z, & i \leq 0, \\ 0, & i > 0, \end{cases} \end{cases} \tag{BS2}$$

Then clearly $U_i^n = V_i^n$, for $n \geq 1$, $i \geq 1$. By Theorem 2, when $t \geq \Delta t$,

$$\|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^+)} = \|V^n - v(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^+)} = \|V^n - v(\cdot, t_n; u_0)\|_{l^1} \leq |Z|\Gamma(a),$$

where $\mathbb{Z}^+ = \{n | n \in \mathbb{Z}, n > 0\}$. □

For more general piecewise constant boundary conditions, we just need to break them into simple ones like those in Lemma 3. So we have the following lemma.

Lemma 4. *For the initial-boundary value problem*

$$\begin{cases} u_t + a u_x = 0, & t > 0, x > 0, a > 0 \\ u(x,0) = b_{-1}, & x > 0, \\ u(0,t) = b_n, & n\Delta t \le t < (n+1)\,\Delta t, \end{cases} \tag{B3}$$

where b_k, $k = -1, 0, \cdots, n \cdots$, are constant, if the upwind method is used,

$$\begin{cases} U_i^{n+1} = (1-\lambda) U_i^n + \lambda U_{i-1}^n, & i \ge 1,\ n \ge 0 \\ U_i^0 = b_{-1}, & i > 0, \\ U_0^n = b_n, & n \ge 0, \end{cases} \tag{BS3}$$

then under the CFL condition $\lambda \le 1$, the l^1-error $\|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^+)}$ is bounded by $\left(\sum |b_n - b_{n-1}|\right)\Gamma(a)$.

Proof. We define a series of problems:

$$\begin{cases} (v_0)_t + a\,(v_0)_x = 0, & t > 0, x \in (-\infty, \infty), \\ v_0(x,0) = \begin{cases} b_0, & x \le 0, \\ b_{-1}, & x > 0. \end{cases} \end{cases} \tag{I1}$$

$$\begin{cases} (v_k)_t + a\,(v_k)_x = 0, & t > k\Delta t, x \in (-\infty, \infty), \\ v_k(x, k\Delta t) = \begin{cases} b_k - b_{k-1}, & x \le 0, \\ 0, & x > 0. \end{cases} \end{cases} \tag{B4}$$

Using the upwind method

$$\begin{cases} (V_0)_i^{n+1} = (1-\lambda)\,(V_0)_i^n + \lambda\,(V_0)_{i-1}^n\,, & i \ge 1, n \ge 0, \\ (V_0)_i^0 = \begin{cases} b_0, & i \le 0, \\ b_{-1}, & i > 0, \end{cases} \end{cases} \tag{IS1}$$

$$\begin{cases} (V_k)_i^{n+1} = (1-\lambda)\,(V_k)_i^n + \lambda\,(V_k)_{i-1}^n\,, & i \ge 1, n \ge k, \\ (V_k)_i^k = \begin{cases} b_k - b_{k-1}, & i \le 0, \\ 0, & i > 0, \end{cases} \end{cases} \tag{BS4}$$

one can see that scheme (IS1) corresponds to problem (I1) and for each k and $t \ge k\Delta t$, scheme (BS4) corresponds to problem (B4). $v_0(x,t)$, $v_k(x,t)$,$(V_0)_i^n$ and $(V_k)_i^n$ are defined so that $u(x,t) = \sum_{l=0}^k v_l(x,t)$, for $x \ge 0$ and $k\Delta t \le t < (k+1)\,\Delta t$, and $U_i^k = \sum_{l=0}^k (V_l)_i^k$, for $i \ge 1$. Thus, by Lemma 3, for $\forall k \ge 0$, $k\Delta t \le t < (k+1)\,\Delta t$,

$$\begin{aligned} \left\|U^k - u(x,t)\right\|_{l^1(\mathbb{Z}^+)} &\le \left\| \sum_{l=0}^k (V_l)^k - \sum_{l=0}^k v_l(x,t) \right\|_{l^1(\mathbb{Z}^+)} \le \sum_{l=0}^k \left\|(V_l)^k - v_l(x,t)\right\|_{l^1(\mathbb{Z}^+)} \\ &\le \left(\sum_{l=0}^k |b_l - b_{l-1}| \right) \Gamma(a) \le \left(\sum |b_n - b_{n-1}| \right) \Gamma(a). \end{aligned}$$

□

3 Proof of the main theorem

Now we prove theorem 1 with a monotone $u_0(x)$. The main idea is to decompose u as the sum of the solutions, each of which solves a linear convection equation with a constant coefficient.

Proof. [**Proof of theorem 1 with a monotone** $u_0(x)$**].** We can assume $u_0(x)$ is monotonically decreasing. If it is monotonically increasing, the proof is similar. For reader's convenience, we write (1)-(3) here again:

$$\begin{cases} u_t + c(x)u_x = 0, & t \geq 0, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \\ c(x) = \begin{cases} c^-, \\ c^+, \end{cases} & \begin{matrix} x < 0, \\ x > 0, \end{matrix} \\ u(0^+,t) = \rho u(0^-,t), & t > 0. \end{cases} \tag{P1}$$

There are two steps to prove the theorem.

Step 1: write $u(x,t)$ as the sum of $v(x,t)$ and $w(x,t)$, where

$$\begin{cases} v_t + c^+ v_x = 0, & t > 0, x \in \mathbb{R}, \\ v(x,0) = v_0(x), & x \in \mathbb{R}, \\ v_0(x) = \begin{cases} \rho u_0\left(\frac{c^-}{c^+}x\right), \\ A, \end{cases} & \begin{matrix} x \leq 0, \\ x > 0, \end{matrix} \end{cases} \tag{P2}$$

and

$$\begin{cases} w_t + c^+ w_x = 0, & t > 0, x \in \mathbb{R}, \\ w(x,0) = w_0(x), & x \in \mathbb{R}, \\ w_0(x) = \begin{cases} 0, \\ u_0(x) - A, \end{cases} & \begin{matrix} x \leq 0, \\ x > 0, \end{matrix} \end{cases} \tag{P3}$$

where A can be any constant and later we will set it to be $u_0(0^+)$ to avoid extra errors introduced by splitting $u_0(x)$. Then the relation between $u(x,t)$, $v(x,t)$ and $w(x,t)$ is

$$u(x,t) = \begin{cases} \frac{1}{\rho} v\left(\frac{c^+}{c^-}x, t\right), & x \leq 0, \\ v(x,t) + w(x,t), & x > 0. \end{cases}$$

Next, we consider the corresponding numerical schemes for v and w.

For problem (P1), the scheme is (S1),

$$\begin{cases} U_i^{n+1} = \left(1 - c^-\left(\frac{\Delta t}{\Delta x}\right)\right) U_i^n + c^-\left(\frac{\Delta t}{\Delta x}\right) U_{i-1}^n, & \text{if } i \leq 0, \\ U_i^{n+1} = \left(1 - c^+\left(\frac{\Delta t}{\Delta x}\right)\right) U_i^n + c^+\left(\frac{\Delta t}{\Delta x}\right) \rho U_{i-1}^n, & \text{if } i = 1, \\ U_i^{n+1} = \left(1 - c^+\left(\frac{\Delta t}{\Delta x}\right)\right) U_i^n + c^+\left(\frac{\Delta t}{\Delta x}\right) U_{i-1}^n, & \text{if } i \geq 2, \end{cases}$$

with initial condition

$$U_i^0 = u_0(x_i). \tag{7}$$

The CFL condition $0 < \lambda^\pm = c^\pm\left(\frac{\Delta t}{\Delta x}\right) < 1$ should be satisfied. To match (S1), we will use the following schemes for problem (P2) and (P3) respectively. For problem (P2), the scheme is

$$\begin{cases} V_i^{n+1} = \left[1 - c^+\left(\frac{\Delta t}{\frac{c^+}{c^-}\Delta x}\right)\right] V_i^n + c^+\left(\frac{\Delta t}{\frac{c^+}{c^-}\Delta x}\right) V_{i-1}^n, & \text{if } i \leq 0, \\ V_i^{n+1} = \left(1 - c^+\left(\frac{\Delta t}{\Delta x}\right)\right) V_i^n + c^+\left(\frac{\Delta t}{\Delta x}\right) V_{i-1}^n, & \text{if } i \geq 1, \end{cases} \tag{S2}$$

with the initial condition

$$V_i^0 = v_0(x_i'),$$

where we use a variable mesh size $\Delta x' = \frac{c^+}{c^-}\Delta x$ for $i \leq 0$ and $\Delta x' = \Delta x$ for $i > 1$, i.e.,

$$x_i' = \begin{cases} i\left(\frac{c^+}{c^-}\right)\Delta x = \left(\frac{c^+}{c^-}\right)x_i, & i \leq 0, \\ i\Delta x = x_i, & i > 0. \end{cases}$$

For problem (P3), the scheme is

$$W_i^{n+1} = \left(1 - c^+\left(\frac{\Delta t}{\Delta x}\right)\right) W_i^n + c^+\left(\frac{\Delta t}{\Delta x}\right) W_{i-1}^n, \quad n \geq 1, \ i \in \mathbb{Z}, \tag{S3}$$

with the initial condition

$$W_i^0 = w_0(x_i).$$

One can check $U_i^n = \frac{1}{\rho} V_i^n$, for $i \le 0$. Notice that for $i > 0$, the scheme for $V_i^n + W_i^n$ has the same inital condition and mesh sizes as those for U_i^n. Their boundary conditions at x_0 are also the same, i.e., $U_0^n = \frac{1}{\rho} V_0^n$. So $U_i^n = V_i^n + W_i^n$, for $i > 0$. Thus the relation between U_i^n, V_i^n and W_i^n is

$$U_i^n = \begin{cases} \frac{1}{\rho} V_i^n, & i \le 0 \\ V_i^n + W_i^n, & i > 0. \end{cases}$$

The l^1-error between U^n and u is

$$\begin{aligned} &\|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z})} \qquad (8) \\ &= \|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^- \cup \{0\})} + \|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^+)} \\ &\le \|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^- \cup \{0\})} + \|V^n + W^n - v(\cdot, t_n; v_0) - w(\cdot, t_n; w_0)\|_{l^1(\mathbb{Z}^+)} \\ &\le \|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^- \cup \{0\})} + \|V^n - v(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)} + \|W^n - w(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)}. \end{aligned}$$

From Theorem 2,

$$\|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^- \cup \{0\})} \le \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} \Gamma(c^-). \qquad (9)$$

By setting $A = u_0(x_1)$ in (P2) and (P3), one has

$$\|W^n - w(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)} \le \|u_0\|_{BV(\mathbb{R}^+)} \Gamma(c^+). \qquad (10)$$

Step 2: we formulate the upwind scheme (S2) into a scheme with uniform mesh for a boundary value problem to estimate $\|V_i^n - v(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)}$. Introduce another scheme

$$\Phi_i^{n+1} = \left[1 - c^+ \left(\frac{\Delta t}{\frac{c^+}{c^-}\Delta x}\right)\right] \Phi_i^n + c^+ \left(\frac{\Delta t}{\frac{c^+}{c^-}\Delta x}\right) \Phi_{i-1}^n, \quad n \ge 0, \ i \in \mathbb{Z}, \qquad \text{(S4)}$$

with initial condition

$$\Phi_i^0 = v_0(x_i''),$$

where $x_i'' = i\left(\frac{c^+}{c^-}\Delta x\right)$.

Scheme (S4) is very much like (S3), except we use spatial step length $\Delta x'' = \frac{c^+}{c^-}\Delta x$ for the whole line. By Theorem 2,

$$\begin{aligned} &\|\Phi^n - v(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)} \le \|\Phi^n - v(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z})} \qquad (11) \\ &\le \left(\|v_0\|_{BV(\mathbb{R}^- \cup \{0\})} + \|v_0\|_{BV(\mathbb{R}^+)} + B\right) \left[2\sqrt{c^+ \left(\frac{c^+}{c^-}\right) \Delta x \left(1 - c^+ \frac{\Delta t}{\left(\frac{c^+}{c^-}\right)\Delta x}\right) t} + \left(\frac{c^+}{c^-}\right) \Delta x\right] \\ &\le \left(\rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + B\right) \left(\frac{c^+}{c^-}\right) \left[2\sqrt{c^- \Delta x \left(1 - c^- \frac{\Delta t}{\Delta x}\right) t} + \Delta x\right] \\ &= \left(\rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + B\right) \left(\frac{c^+}{c^-}\right) \Gamma(c^-), \end{aligned}$$

where B is defined in (6). Clearly for $i \le 0$, $V_i^n = \Phi_i^n$. In particular, for $i = 0$, $V_0^n = \Phi_0^n$. Since the upwind method is monotonicity-preserving, V_i^n should be monotonically decreasing in i for $\forall n \ge 0$. That means, $V_{-1}^n > V_0^n$, for $\forall n \ge 0$. By the scheme, $V_0^{n+1} = (1-\lambda)V_0^n + \lambda V_{-1}^n$, one can see, $V_0^{n+1} \ge V_0^n$ for $\forall n \ge 0$. Suppose $V_0^n = \Phi_0^n = b_n$. Then $\{b_n\}$ is increasing. In addition, $|b_n| \le \rho \sup_{x \in \mathbb{R}} u_0(x)$, for $\forall n \ge 0$. Thus, $\sum |b_n - b_{n-1}| \le \rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})}$.

If one only focuses on the right half domain, where $i \geq 0$, one can use Lemma 4, since scheme (S2) and scheme (S4) have the same boundary condition at $i = 0$, and the same constant initial data since v_0 is a constant for $x > 0$.

$$\begin{aligned}
&\|\Phi^n - V^n\|_{l^1(\mathbb{Z}^+)} \leq \|\Phi^n - \psi\|_{l^1(\mathbb{Z}^+)} + \|\psi - V^n\|_{l^1(\mathbb{Z}^+)} \qquad (12)\\
&\leq \rho \sum |b_n - b_{n-1}| \Gamma(c^+) + \rho \sum |b_n - b_{n-1}| \left[2\sqrt{c^+ \left(\frac{c^+}{c^-}\right) \Delta x \left(1 - c^+ \frac{\Delta t}{\frac{c^+}{c^-}\Delta x}\right) t} + \left(\frac{c^+}{c^-}\right) \Delta x \right]\\
&= \rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} \Gamma(c^+) + \rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} \left(\frac{c^+}{c^-}\right) \Gamma(c^-),
\end{aligned}$$

where ψ is the solution to the initial-boundary value problem defined in (B3) in Lemma 4, with $b_n = V_0^n = \Phi_0^n$. Combining all the results above, (8), (9), (10), (11) and (12), we have

$$\begin{aligned}
&\|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z})}\\
&\leq \|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^- \cup \{0\})} + \|V^n - v(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)} + \|W^n - w(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)}\\
&\leq \|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z}^- \cup \{0\})} + \|V^n - \Phi^n(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)} + \|\Phi^n - v(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)}\\
&\quad + \|W^n - w(\cdot, t_n; v_0)\|_{l^1(\mathbb{Z}^+)}\\
&\leq \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} \Gamma(c^-) + \rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} \Gamma(c^+) + \rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} \left(\frac{c^+}{c^-}\right) \Gamma(c^-)\\
&\quad + \left(\rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + B\right) \left(\frac{c^+}{c^-}\right) \Gamma(c^-) + \|u_0\|_{BV(\mathbb{R}^+)} \Gamma(c^+)\\
&\leq \left[\|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + \left(2\rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + B\right) \left(\frac{c^+}{c^-}\right)\right] \Gamma(c^-) + \left(\rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + \|u_0\|_{BV(\mathbb{R}^+)}\right) \Gamma(c^+)
\end{aligned}$$

□

Now we are ready to prove Theorem 1.

Proof. [Proof of Theorem 1]. Since $u_0(x)$ is a function of bounded variation, it can be split into two functions of bounded variation, $u_0(x) = a_0(x) + b_0(x)$, such that $a_0(x)$ is increasing, $b_0(x)$ is decreasing and $\|a_0\|_{BV(\mathbb{R})} + \|b_0\|_{BV(\mathbb{R})} = \|u_0\|_{BV(\mathbb{R})}$. Suppose $a(x,t)$ and $b(x,t)$ evolve according to (1)-(3) with initial values $a_0(x)$ and $b_0(x)$, and we use the same scheme for $a(x,t)$ and $b(x,t)$ as we did for $u(x,t)$.

Then by theorem 3,

$$\begin{aligned}
&\|U^n - u(\cdot, t_n; u_0)\|_{l^1(\mathbb{Z})}\\
&= \|(A^n + B^n) - (a(\cdot, t_n; a_0) + b(\cdot, t_n; b_0))\|_{l^1(\mathbb{Z})}\\
&\leq \|A^n - a(\cdot, t_n; a_0)\|_{l^1(\mathbb{Z})} + \|B^n - b(\cdot, t_n; b_0)\|_{l^1(\mathbb{Z})}\\
&\leq \left[\|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + \left(2\rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + B\right) \left(\frac{c^+}{c^-}\right)\right] \Gamma(c^-) + \left(\rho \|u_0\|_{BV(\mathbb{R}^- \cup \{0\})} + \|u_0\|_{BV(\mathbb{R}^+)}\right) \Gamma(c^+).
\end{aligned}$$

□

Noticing if the discretization of $b(t)$ is $b_n = b(n\Delta t)$, then $\|b_n - b(t)\|_{l^1(\mathbb{Z}^+ \cup \{0\})} \leq \|b\|_{BV(\mathbb{R}^+ \cup \{0\})} \Delta t$, one can also get the following corollary from the proofs of Lemma 4 and Theorem 3.

Corollary 5. *For the initial-boundary value problem*

$$\begin{cases}
u_t + a u_x = 0, & t \geq 0, x \geq 0,\\
u(x,0) = f(x), & x \geq 0,\\
u(0,t) = b(t), & t \geq 0,\\
b(0) = f(0), &
\end{cases}$$

if the upwind method is used,

$$\begin{cases} U_i^{n+1} = (1-\lambda)U_i^n + \lambda U_{i-1}^n, & i \geq 1,\ n \geq 0, \\ U_i^0 = f(x_i), & i \geq 1, \\ U_0^n = b_n, & n \geq 0, \end{cases}$$

then under CFL condition $\lambda \leq 1$, one has

$$\|U^n - u(\cdot, t_n)\|_{l^1(\mathbb{Z}^+)} \leq \left(\|b\|_{BV(\mathbb{R}_t^+ \cup \{0\})} + \|f\|_{BV(\mathbb{R}_x^+ \cup \{0\})}\right)\Gamma(a) + \|b\|_{BV(\mathbb{R}_t^+ \cup \{0\})}\,\Delta t.$$

Remark: If there are more than one (say $N \geq 2$) isolated discontiunities in the coefficient $c(x)$, the above analysis can be extended in a straightforward way. The key step in the proof of Theorem 1 is to split $u(x,t)$ as the sum of $v(x,t)$ and $w(x,t)$. If there is another discontinuity, one can further split $v(x,t)$ and $w(x,t)$ in the same way and still carry out the rest of the analysis. The final error will depend on a factor of 2^N for N discontinuities. Of course, this constant is not sharp due to the technicality used in this paper.

4 Conclusion

In this paper, we obtained an l^1 error bound for the immersed upwind scheme for the convection equation with a discontinuous coefficient. Our error bound has the same sharp convergence rate as that in [31] with a larger coefficient, but the method of proof is much simpler. The main idea is to formulate the problem into linear superposition of convection equations with constant coefficients, and then use classical estimates for initial and boundary value problems with constant coefficient. The simplicity of the method offers the possibility of extending it for more complicated problems of wave propagation in heterogeneous media, in particular, to Liouville equation with discontinuous potentials which arises in the semiclassical modelling of quantum dynamics with potential barriers and geometric optics through interfaces [11, 12, 10].

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