UNIFORM-IN-TIME ERROR ESTIMATE OF THE RANDOM BATCH METHOD FOR THE CUCKER-SMALE MODEL

SEUNG-YEAL HA, SHI JIN, DOHEON KIM, AND DONGNAM KO

Abstract. We present a uniform-in-time (and in particle numbers as well) error estimate for the random batch method (RBM) [25] to the Cucker-Smale model. The uniform-in-time error estimates of the RBM have been obtained for various interacting particle systems, when corresponding flow generates a contraction semigroup. In this paper, we derive a uniform-in-time error estimate for RBM-approximation to the Cucker-Smale model in which the corresponding flow does not generate contractive semigroup. To derive uniform error estimate, we use asymptotic flocking estimate of the RBM-approximated Cucker-Smale model which yields the decay of relative velocities to zero, at least in the order of $\exp(-Ct^{1-\beta})$, while velocities of the original system decay exponentially. Here, $\beta \in [0, 1)$ is the decay rate of the communication weight with respect to the distance between particles in the Cucker-Smale model. We also provide several numerical simulations to confirm the analytical results.

1. Introduction

Collective behaviors of self-propelled particles (agents) are ubiquitously found in nature, for example, synchronous flashing of fireflies [5, 30] and pacemaker cells [33], swarming of fish [37], flocking of birds [12, 23, 24, 32] and herding of sheep. We refer to [1, 2, 6, 7, 17, 20, 27, 34, 35, 36, 37, 39, 40] for survey articles and related literature. Recently, the modeling of flocking dynamics has received lots of attention due to recent applications of driverless cars, drones and robots.

In 2007, Cucker and Smale introduced a Newton type second-order model for the flocking phenomena motivated by Vicsek's model [38]. To fix the idea, we begin with the Cucker-Smale (CS) model [12]. Let x_i and v_i be the position and velocity of the i-th CS particle, respectively. Then, their temporal evolution is given by the following Cauchy problem:

$$
\begin{cases}
\dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i = 1, \dots, N, \\
\dfrac{dv_i}{dt} = \dfrac{\kappa}{N-1} \displaystyle\sum_{j: j \neq i} \psi(|x_j - x_i|)(v_j - v_i), \\
(x_i(0), v_i(0)) = (x_i^{in}, v_i^{in}),
\end{cases}
\tag{1.1}
$$

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where κ is the nonnegative coupling strength and $\psi:\ \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the communication weight measuring mutual interactions, which is assumed to satisfy positivity, boundedness, Lipschitz continuity and monotonicity conditions: there exists a positive constant $\psi_M > 0$ such that

$$\begin{aligned} &0 \leq \psi(r) \leq \psi_M, \quad \forall\ r \geq 0, \quad \|\psi\|_{\mathrm{Lip}} < \infty, \\ &(\psi(r_1) - \psi(r_2))(r_1 - r_2) \leq 0, \quad r_1,\ r_2 \in \mathbb{R}_+. \end{aligned} \tag{1.2}$$

The emergent dynamics of (1.1) has been extensively studied in literature [11, 13, 19, 21, 23, 24] from various angles. Among others, we are mainly interested in the error estimate of the "Random Batch Method (RBM) [25, 26]" for the CS model (1.1).

Note that each particle interacts with all other $N-1$ particles, thus the computational cost to solve (1.1) numerically follows $\mathcal{O}(N^2)$ per time step. For large $N \gg 1$, numerical implementation of (1.1) needs a huge computational cost. This is why mean-field and continuum approximations for a large particle system (1.1) are often employed in literature for (1.1) with $N \gg 1$. However, in this work, we do not take any mean-field approximation or continuum approximation to reduce the complexity of (1.1), instead we employ the RBM for (1.1) at the particle level to reduce computational complexities, and we study its associated error estimate under suitable setting.

The RBM is a fast algorithm to approximate time-evolution of a large interacting particle system by decomposing the whole system into small, randomly decoupled subsystems for each time step with a fixed size. It can be viewed as a generalization of the Nanbu algorithm developed in [4] for the mean-field equation. We also refer to [6, 7] for the modeling and simulation of a swarming group as a composition of subsystems which characterized via a topological metric such as the nearest neighbor interactions. To reduce the complexity, instead of computing all the interactions, the RBM approximates the given system by randomly decoupled subsystems, which are batches consisting of (not more than) $P(\ll N)$ individuals. Each agent only interacts with agents in the same batch. Then, the number of interactions at each time step reduces to $\mathcal{O}(NP)$. Of course, the choice of batches is random and only used for a small duration of time, in order to average the random effect in the time-evolution. Therefore, the RBM(-approximated) system for (1.1) becomes a (randomly) switching networked system along time (see [14] for the CS flocking model).

The RBM has shown good performances in numerical simulations, starting from examples in [25]: nonlinear opinion dynamics, quantum dynamics [18] and Poisson-Boltzmann equation [31]. Moreover, the consensus-based optimization method [9, 22] and the collective behavior models [8, 29] show a quite accurate long-time behavior, where it turns out to be effective to find stable equilibria.

Next, we begin with a brief discussion on the RBM-approximation for (1.1) following the work [25]. Let τ be the time step between every two random selections of batches, and we decompose the infinite time domain $[0,\infty)$ as a union of disjoint finite-time intervals: for $\tau_m := (m-1)\tau$ where $m = 1, 2, \ldots$,

$$[0,\infty) = \bigcup_{m=1}^{\infty} [\tau_{m-1}, \tau_m).$$

At the m-th time interval $[\tau_{m-1}, \tau_m)$, we choose a partition of $\{1, 2, \ldots, N\}$ randomly, $\mathcal{B} = \{\mathcal{B}_1^m, \ldots, \mathcal{B}_n^m\}$ of $n = \lceil \frac{N}{P} \rceil$ batches with sizes at most $P > 1$, in the following way:

$$\{1, \ldots, N\} = \mathcal{B}_1^m \cup \mathcal{B}_2^m \cup \cdots \cup \mathcal{B}_n^m, \quad |\mathcal{B}_i^m| = P, \quad i = 1, \cdots, n-1, \quad |\mathcal{B}_n^m| \leq P.$$

In each sub-time interval $[\tau_{m-1}, \tau_m)$, we set $[i]_m \in \mathcal{B}$ to be the batch containing i. Finally, the RBM-(approximated) system of (1.1) becomes a switching network system:

$$
\begin{cases}
\dfrac{dx_i^R}{dt} = v_i^R, \quad t \in [\tau_{m-1}, \tau_m), \; m = 1, 2, \cdots, \\
\dfrac{dv_i^R}{dt} = \dfrac{\kappa}{P-1} \displaystyle\sum_{j \in [i]_m} \psi(|x_j^R - x_i^R|)(v_j^R - v_i^R), \\
(x_i^R(0), v_i^R(0)) = (x_i^{in}, v_i^{in}), \quad i = 1, \dots, N.
\end{cases} \tag{1.3}
$$

Note that the velocity equations experience the effect of the RBM directly, while the position equations are the same as in (1.1).

From the RBM system (1.3), the following questions naturally arise:

- (Q1): Does the RBM system (1.3) exhibit flocking dynamics?
- (Q2): If then, how well does (1.3) approximate the full system (1.1)?

In this paper, we address these two questions. First, we present an improved flocking estimate for the RBM-approximation (1.3). System (1.3) can be cast as a special case of the CS model with randomly switching topology, whose emergent dynamics has been studied in [13, 14] in continuous and discrete dynamical system settings. Thus, our presented flocking estimates can be regarded as an improved flocking estimate over [14]. For the flocking estimate, we assume that ψ is long-ranged:

$$
1/\psi(r) = \mathcal{O}(r^\beta) \quad \text{as} \quad r \to \infty \quad \text{for some } \beta \in [0, 1). \tag{1.4}
$$

For example, we can take

$$
\psi(r) = \frac{1}{(1 + r^2)^{\beta/2}}, \quad \beta \in [0, 1).
$$

This function ψ satisfies conditions (1.2) and (1.4). With a long-ranged ψ, the CS model (1.1) exhibits a global flocking independent of initial data (see Proposition 2.1). Under the same assumption on ψ, our first main result is the emergence of a global flocking (see Theorem 3.1 for detailed description): there exist positive constants x_∞^R and C such that

$$
\begin{aligned}
&\sup_{0 \le t < \infty} \mathbb{E}\Big(\frac{1}{N^2} \sum_{i,j=1}^N |x_i^R - x_j^R|^2\Big) < x_\infty^R \quad \text{and} \\
&\mathbb{E}\Big(\frac{1}{N^2} \sum_{i,j=1}^N |v_i^R - v_j^R|^2\Big) \le C \exp\Big[-\frac{C(P-1)}{(N-1)(1+\tau)} t(1+t)^{-\beta}\Big],
\end{aligned} \tag{1.5}
$$

where C depends only on ψ, β, κ and the diameter of the initial data.

From the symmetry in the CS model, in both systems (1.1) and (1.3), velocities tend to the common initial average velocity asymptotically (see Remark 2.1). Hence, we may conclude that the error also follows the same estimation (see Corollary 3.1) from (1.5):

$$
\mathbb{E}\Big(\frac{1}{N} \sum_{i=1}^N |v_i^R(t) - v_i(t)|^2\Big) \le C \exp\left(-\frac{C(P-1)}{(N-1)(1+\tau)} t(1+t)^{-\beta}\right). \tag{1.6}
$$

The estimate (1.6) only uses the convergence to the same equilibrium, which is not related to consistency between the systems (1.1) and (1.3).

Our second result suggests provide a uniform-in-time error estimate that depends on the batch size P and the size of time step τ (see Theorem 3.2 for details): when ψ has a positive lower bound ψ_0,

$$\psi(r) \geq \psi_0 \quad \text{for} \quad r \geq 0, \tag{1.7}$$

then we have

$$\mathbb{E}\Big(\frac{1}{N}\sum_{i=1}^N |v_i^R(t) - v_i(t)|^2\Big) \leq C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\tau^2 + C(1+\tau)\exp(-\kappa\psi_0 t), \tag{1.8}$$

where (x_i, v_i) and (x_i^R, v_i^R) are solutions to (1.1) and (1.3), respectively, and the dependency of the constant C is the same as in (1.5).

Note that the positive lower bound assumption (1.7) is the case of $\beta = 0$ in the long-ranged communication (1.4). However, the third time-decaying term in the right-hand side of (1.8) is independent of P and N, which is better than the estimate given by (1.6). Moreover, we expect that the arguments used to derive (1.8) could be extended to the models with non-symmetric interactions (for example, [15, 16, 32]), in which the asymptotic velocity limit can be affected by the choices of random batches.

In order to prove a uniform-in-time error estimate, one of the main assumptions of the RBM-approximation in [25] is the contraction property of the original system. However, the contraction property does not hold for the Cucker-Smale model. We overcome this difficulty by using asymptotic flocking estimates of the system and its RBM system. By splitting the interaction term into the self-contracting part and the time-decaying part, we obtain errors of the RBM-approximation that is uniform-in-time, in addition to the uniformity in N.

The rest of this paper is organized as follows. In Section 2, we review the flocking dynamics of the CS model (1.1)–(1.2) and basic lemmas of the RBM model (1.3). In Section 3, we briefly describe two main results, the flocking estimate of the RBM system (1.3) and its uniform error estimate (1.8). In Section 4, we prove the asymptotic flocking estimate for the RBM system (1.3) under the assumption of the long-ranged communication weight. In Section 5, using the flocking estimate, we derive the uniform-in-time error estimate (1.8). In Section 6, we provide several numerical simulations for the RBM system (1.3) to compare it with the analytical results. Finally, Section 7 is devoted to a brief summary of our main results and some remaining issues to be explored in a future work.

Notation: We set

$$\begin{aligned} &x_i := (x_i^1, \cdots, x_i^d), \quad v_i := (v_i^1, \cdots, v_i^d) \in \mathbb{R}^d, \quad i = 1, \ldots, N, \\ &X := (x_1, \cdots, x_N), \quad V := (v_1, \cdots, v_N) \in \mathbb{R}^{dN}, \\ &X^R := (x_1^R, \cdots, x_N^R), \quad V^R := (v_1^R, \cdots, v_N^R) \in \mathbb{R}^{dN}, \end{aligned}$$

and norms for $x \in \mathbb{R}^d$ and $X \in \mathbb{R}^{dN}$ as follows.

$$|x| := \Big(\sum_{i=1}^N |x^i|^2\Big)^{\frac{1}{2}}, \quad \mathcal{D}(X) := \max_{i,j} |x_i - x_j| \quad \text{and} \quad \mathcal{D}(V) := \max_{i,j} |v_i - v_j|. \tag{1.9}$$

We also use $X^{in} = (x_1^{in}, \dots, x_N^{in})$ and $V^{in} = (v_1^{in}, \dots, v_N^{in})$ to denote the initial data of systems (1.1) and (1.3). For the simplicity of presentation, we use the following handy notation:

$$\sum_i := \sum_{i=1}^N, \qquad \sum_{i,j} := \sum_{i=1}^N \sum_{j=1}^N, \qquad \max_i := \max_{1\le i\le N}, \qquad \max_{i,j} := \max_{1\le i,j\le N}.$$

2. Preliminaries

In this section, we briefly review exponential flocking estimates for the Cucker-Smale model and study basic properties of the RBM system (1.3).

2.1. The Cucker-Smale model. First, we recall temporal evolution of velocity moments for the Cucker-Smale model.

Lemma 2.1. [12, 24] *Let (X, V) be a solution to system (1.1)–(1.2). Then, the first velocity moment is conserved:*

$$\sum_i v_i(t) = \sum_i v_i^{in}, \quad t \ge 0. \tag{2.1}$$

Remark 2.1. *From the conservation of the first velocity moment, if velocity alignment occurs asymptotically, then particle's velocity tends to the initial average velocity:*

$$\lim_{t\to\infty} v_i(t) = v_c(0), \quad i = 1, \cdots, N,$$

where $v_c(0) = \dfrac{1}{N}\sum_j v_j^{in}$.

Next, we recall the asymptotic flocking estimate of the CS model (1.1)–(1.2).

Lemma 2.2. [3] *Let (X, V) be a solution to (1.1)–(1.2) with a zero-sum condition:*

$$\sum_i v_i(t) = 0, \quad t \ge 0.$$

Then, the functionals defined in (1.9) satisfy a system of differential inequalities:

$$\begin{cases} \left| \dfrac{d}{dt}\mathcal{D}(X) \right| \le \mathcal{D}(V), \\ \dfrac{d}{dt}\mathcal{D}(V) \le -\kappa\psi(\mathcal{D}(X))\mathcal{D}(V), \quad \text{a.e., } t > 0. \end{cases} \tag{2.2}$$

Remark 2.2. *If ψ has a positive lower bound ψ_0, i.e.,*

$$\inf_{0\le r<\infty} \psi(r) \ge \psi_0 > 0,$$

then the differential inequality $(2.2)_2$ yields the exponential decay:

$$\mathcal{D}(V) \le \mathcal{D}(V^{in})e^{-\kappa\psi_0 t}, \quad t \ge 0.$$

In the following proposition, we present a result on the flocking dynamics of system (1.1)–(1.2) which provides sufficient conditions leading to exponential flocking estimates.

Proposition 2.1. [3] *Suppose that coupling strength and initial data (X^{in}, V^{in}) satisfy*

$$\sum_i v_i^{in} = 0, \quad \mathcal{D}(V^{in}) < \frac{\kappa}{2}\int_{\mathcal{D}(X^{in})}^{\infty} \psi(s)ds,$$

and let (X,V) be a solution to (1.1)–(1.2). Then, there exists a positive constant x_∞ such that

$$\sup_{0\le t<\infty} \mathcal{D}(X) \le x_\infty \quad \textit{and} \quad \mathcal{D}(V) \le \mathcal{D}(V^{in})e^{-\kappa\psi(x_\infty)t}, \quad t \ge 0.$$

Proof. The proof follows from the Lyapunov functional approach together with (2.2). Detailed arguments can be found in [3, 23]. □

2.2. **The RBM-approximation.** In this subsection, we study basic properties of the RBM-approximate system (1.3) which is parallel to Lemma 2.1.

Lemma 2.3. *Let (X^R, V^R) be a solution to (1.2)–(1.3). Then, for any $k \in \{1, \cdots, N\}$ and $t \in [\tau_{m-1}, \tau_m]$, the total sum of velocities is conserved in each batch:*

$$\sum_{i\in[k]_m} v_i^R(t) = \sum_{i\in[k]_m} v_i^R(\tau_{m-1}).$$

Proof. We use (1.2) and take a sum $(1.3)_2$ over $i \in [k]_m$ to get

$$\begin{aligned}\frac{d}{dt}\sum_{i\in[k]_m} v_i^R &= \frac{\kappa}{P-1}\sum_{i,j\in[k]_m}\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) = -\frac{\kappa}{P-1}\sum_{i,j\in[k]_m}\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R)\\ &= 0.\end{aligned}$$

Now we integrate the above relation from τ_{m-1} to t to get the desired estimate. □

Remark 2.3. *The results of Lemma 2.3 yield the conservation of the first velocity moment as in Lemma 2.1 from the induction on m:*

$$\sum_i v_i^R(t) = \sum_i v_i^{in}, \quad t \ge 0. \tag{2.3}$$

It follows from the same arguments as in Lemma 2.2 that we can obtain estimations on the spatial and velocity diameters.

Lemma 2.4. *Let (X^R, V^R) be a solution of the RBM model (1.2)–(1.3) with initial data satisfying the following relations:*

$$\sum_i v_i^{in} = 0, \quad \mathcal{D}(X^{in}) + \mathcal{D}(V^{in}) < \infty.$$

Then, one has

$$\mathcal{D}(V^R(t)) \le \mathcal{D}(V^{in}), \quad \mathcal{D}(X^R(t)) \le \mathcal{D}(X^{in}) + \mathcal{D}(V^{in})t, \quad \forall\ t \ge 0.$$

Proof. (i) For the first assertion, we claim that the relative velocities are non-increasing in time. Let $t \in [\tau_{m-1}, \tau_m)$ be given. Then, we choose time-dependent indices k and l such that

$$|v_k^R(t) - v_l^R(t)| = \max_{i,j} |v_i^R(t) - v_j^R(t)| = \mathcal{D}(V^R(t)).$$

Then, for such k and l, we have

$$
\begin{aligned}
\frac{d}{dt}|v_k^R(t)-v_l^R(t)|^2 &= 2(v_k^R(t)-v_l^R(t))\cdot\frac{d}{dt}(v_k^R(t)-v_l^R(t)) \\
&= \frac{2\kappa}{P-1}\sum_{j\in[k]_m}\psi(x_j^R-x_k^R)(v_j^R-v_k^R)\cdot(v_k^R-v_l^R) \\
&\quad -\frac{2\kappa}{P-1}\sum_{j\in[l]_m}\psi(x_j^R-x_l^R)(v_j^R-v_l^R)\cdot(v_k^R-v_l^R).
\end{aligned}
\tag{2.4}
$$

In order to show that the right-hand side of (2.4) is not positive, we use the maximality of $|v_k^R(t)-v_l^R(t)|$ at time t. Since

$$
|v_k^R(t)-v_l^R(t)|\ge|v_j^R(t)-v_l^R(t)|,\quad j=1,\dots,N,
$$

we have

$$
\begin{aligned}
(v_j^R(t)-v_k^R(t))\cdot(v_k^R(t)-v_l^R(t)) &= -((v_k^R-v_l^R)-(v_j^R-v_l^R))\cdot(v_k^R-v_l^R) \\
&\le -|v_k^R-v_l^R|^2+|v_j^R-v_l^R||v_k^R-v_l^R|\le 0,\quad j=1,\dots,N.
\end{aligned}
$$

Similarly, we have

$$
(v_j^R(t)-v_l^R(t))\cdot(v_k^R(t)-v_l^R(t))\ge 0,\quad j=1,\dots,N.
$$

Hence, $|v_k^R(t)-v_l^R(t)|^2$ is non-increasing in time.

(ii) By definition of $\mathcal{D}(X^R(t))$, $\mathcal{D}(V^R(t))$ and the first assertion (i), one has

$$
\frac{d}{dt}\mathcal{D}(X^R(t))\le\mathcal{D}(V^R(t))\le\mathcal{D}(V^R(0)),\quad \text{a.e. } t>0.
$$

This yields the second assertion on $\mathcal{D}(X^R)$. □

3. Description of the main results

In this section, we briefly present two main results of this paper whose proofs will be presented in the next two sections.

3.1. **Flocking dynamics of the RBM-CS model.** In this subsection, we provide an asymptotic flocking dynamics of RBM-CS model with a long-ranged communications. First, we present the concepts of asymptotic flocking and stochastic flocking for the full system (1.1) and the RBM system (1.3), respectively.

Definition 3.1. *(1) Let $Z:=(X,V)$ be a solution to* (1.1)*. Then Z exhibits asymptotic flocking if the following relations hold.*

$$
\sup_{0\le t<\infty}|x_i(t)-x_j(t)|<\infty,\quad \lim_{t\to\infty}|v_i(t)-v_j(t)|=0,\quad 1\le i,j\le N.
$$

(2) Let $Z^R:=(X^R,V^R)$ be a solution to (1.3)*. Then Z^R exhibits asymptotic stochastic flocking if the following estimates hold.*

$$
\sup_{0\le t<\infty}\mathbb{E}\Big(|x_i(t)-x_j(t)|^2\Big)<\infty,\quad \lim_{t\to\infty}\mathbb{E}\Big(|v_i(t)-v_j(t)|^2\Big)=0,\quad 1\le i,j\le N.
$$

Our first result deals with the emergence of flocking dynamics to the RBM system (1.3).

Theorem 3.1. (Stochastic flocking dynamics) *Suppose the communication weight function ψ satisfies*

$$1/\psi(r) = \mathcal{O}(r^\beta) \quad as \quad r \to \infty, \tag{3.1}$$

for some $\beta \in [0,1)$, and let (X^R, V^R) be a solution to (1.2)–(1.3). *Then, there exists a positive constant $x^R_\infty = x^R_\infty(\kappa, P, N, \tau, \psi, \beta, \mathcal{D}(X^{in}), \mathcal{D}(V^{in}))$ such that*

$$(i)\ \sup_{t>0} \mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j} |x^R_i(t) - x^R_j(t)|^2\Big) < x^R_\infty,$$

$$(ii)\ \mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j} |v^R_i(t) - v^R_j(t)|^2\Big) \le C\exp\left(-\frac{C(P-1)}{(N-1)(1+\tau)}t(1+t)^{-\beta}\right),$$

for some constant C depending on ψ, κ, β, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

Proof. We leave its proof to Section 4. □

Since we also have the flocking estimates for the CS model (Proposition 2.1) and every velocities from both systems (1.1) and (1.3) converge to the same mean velocity, the uniform boundedness of errors can be derived directly as follows. This argument was previously used for the uniform-in-time mean-field limit presented in [3].

Corollary 3.1. (Boundedness of errors) *Under the same assumptions in Theorem 3.1, let (X,V) and (X^R, V^R) be the solutions to* (1.1)–(1.2) *and* (1.2)–(1.3), *respectively. Then, there exists a positive constant $\tilde{x}^R_\infty = \tilde{x}^R_\infty(\kappa, P, N, \tau, \psi, \beta, \mathcal{D}(X^{in}), \mathcal{D}(V^{in}))$ such that*

$$(i)\ \sup_{t>0} \mathbb{E}\Big(\frac{1}{N^2}\sum_{i} |x^R_i(t) - x_i(t)|^2\Big) < \tilde{x}^R_\infty,$$

$$(ii)\ \mathbb{E}\Big(\frac{1}{N}\sum_{i} |v^R_i(t) - v_i(t)|^2\Big) \le C\exp\left(-\frac{C(P-1)}{(N-1)(1+\tau)}t(1+t)^{-\beta}\right),$$

for some positive constant C depending on ψ, κ, β, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

Proof. Without loss of generality, we may assume that the initial mean velocity is zero so that the mean velocities of both (X,V) and (X^R,V^R) are zero. The proof begins with the triangle inequality:

$$\frac{1}{N}\sum_i |v^R_i - v_i|^2 \le \frac{2}{N}\sum_i (|v^R_i|^2 + |v_i|^2).$$

For v^R_i, we use the zero mean velocity to get

$$\frac{2}{N}\sum_i |v^R_i|^2 = \frac{1}{N}\sum_i |v^R_i|^2 + \frac{1}{N}\sum_j |v^R_j|^2 = \frac{1}{N^2}\sum_{i,j}(|v^R_i|^2 + |v^R_j|^2) = \frac{1}{N^2}\sum_{i,j}|v^R_i - v^R_j|^2.$$

Hence, it follows from the flocking estimate Theorem 3.1 that

$$\mathbb{E}\Big(\frac{2}{N}\sum_i |v^R_i|^2\Big) \le C\exp\left(-\frac{C(P-1)}{(N-1)(1+\tau)}t(1+t)^{-\beta}\right).$$

Similarly, since the flocking estimate Proposition 2.1 holds for v_i, we have

$$\mathbb{E}\Big(\frac{2}{N}\sum_i |v_i|^2\Big) = \mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i - v_j|^2\Big) \le C\exp\left(-\kappa\psi(x_\infty)t\right).$$

Therefore, we conclude

$$\mathbb{E}\Big(\frac{1}{N}\sum_i |v_i^R - v_i|^2\Big) \le C\exp\left(-\frac{C(P-1)}{(N-1)(1+\tau)}t(1+t)^{-\beta}\right),$$

for some constant C depending on ψ, κ, β, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

The first assertion can be derived from the following differential inequality:

$$\frac{d}{dt}\Big(\mathbb{E}\sqrt{\frac{1}{N}\sum_i |x_i^R - x_i|^2}\Big) \le \mathbb{E}\sqrt{\frac{1}{N}\sum_i |v_i^R - v_i|^2} \le \sqrt{\mathbb{E}\Big(\frac{1}{N}\sum_i |v_i^R - v_i|^2\Big)}.$$

□

Remark 3.1. *From the boundedness of velocities in Lemmas 2.2 and 2.4, we can derive that the velocity differences are bounded and the position error grows at most linearly. Corollary 3.1 says that the expectation of the velocity error indeed decays (at least) like* $\mathcal{O}(\exp(-Ct^{1-\beta}))$*, and hence, the expectation of the position error is uniformly bounded.*

3.2. **Uniform-in-time error estimate of the RBM.** In this subsection, we state our second main result on the uniform error estimate resulting from the RBM system (1.3):

$$\begin{cases} \dfrac{dx_i^R}{dt} = v_i^R, \quad t \in [\tau_{m-1}, \tau_m), \ m = 1, 2, \cdots, \\ \dfrac{dv_i^R}{dt} = \dfrac{\kappa}{P-1}\displaystyle\sum_{j\in[i]_m} \psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) \end{cases}$$

to the CS model (1.1):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i = 1, \dots, N, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N-1}\displaystyle\sum_{j:j\neq i} \psi(|x_j - x_i|)(v_j - v_i). \end{cases}$$

For the error estimate in [25], a crucial ingredient is the contraction property: for solutions $(X, V) = (x_1, \dots, x_N, v_1, \dots, v_N)$ and $(\widetilde{X}, \widetilde{V}) = (\tilde{x}_1, \dots, \tilde{x}_N, \tilde{v}_1, \dots, \tilde{v}_N)$ to the original full system (1.1) generated from different initial data (X^{in}, V^{in}) and $(\widetilde{X}^{in}, \widetilde{V}^{in})$, we want

$$\frac{d}{dt}\sum_i |v_i - \tilde{v}_i|^2 \le -C\sum_i |v_i - \tilde{v}_i|^2, \quad t \ge 0, \tag{3.2}$$

for some constant $C > 0$. From this analysis, the errors resulting from the random effect decay over time, which helps to deduce a uniform error estimate.

However, the Cucker-Smale model does not satisfy the contracting property (3.2). Since the communication weight ψ is a function on the relative position, a small difference on velocities may nullify the constant C. Hence, the approach in [25] cannot be applied to the CS model to derive a uniform error analysis as it is.

Thus, we instead use the flocking estimates of the Cucker-Smale model to derive a uniform-in-time error estimate. Under suitable conditions on the communication weight, a collective behavior emerges from Lemma 2.2 and Remark 2.2:

$$\sup_{0\le t<\infty} \mathcal{D}(X(t)) < \infty \quad \text{and} \quad \mathcal{D}(V(t)) \le \mathcal{D}(V^{in})e^{-\kappa\psi_0 t}, \quad t \ge 0.$$

Using this flocking estimate, we can decompose the dynamics of the velocity equations into two parts, a contracting part and an exponentially decaying term (see Lemma 5.1). This is the key ingredient proving the uniform-in-time properties.

Theorem 3.2. (Uniform error estimate) *Suppose that the communication weight ψ has a positive lower bound,*

$$\psi(r) \geq \psi_0 > 0, \quad \forall\, r \geq 0,$$

and let (X, V) and (X^R, V^R) be the solutions to (1.1)–(1.2) and (1.2)–(1.3), respectively. Then, we have

$$\mathbb{E}\Big(\frac{1}{N}\sum_i |v_i^R(t) - v_i(t)|^2\Big) \leq C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\tau^2 + C(1+\tau)\exp(-\kappa\psi_0 t), \tag{3.3}$$

where C is a constant depending on ψ, κ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

Proof. We leave its proof to Section 5. □

Remark 3.2. *Below, we provide several comments on the result of Theorem 3.2.*

1. *The assumption of Theorem 3.2 corresponds to the case of $\beta = 0$ in Corollary 3.1. Note that the decay rate in Corollary 3.1 depends on P and N unlike* (3.3)*.*
2. *Using the triangle inequality as in Corollary 3.1, Theorem 3.2 also deduces a flocking estimate on the RBM system* (1.3)*:*

$$\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j} |v_i^R(t) - v_j^R(t)|^2\Big) \leq C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\tau^2 + C(1+\tau)\exp(-\kappa\psi_0 t).$$

3. *For first-order interacting particle systems, uniform error estimates have been studied in* [28, 25, 26]*.*
4. *In Appendix A of* [25]*, the authors considered a Hamiltonian system and its RBM approximation:*

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad \dfrac{dv_i}{dt} = F(x_i) + \dfrac{1}{N-1}\displaystyle\sum_{j:j\neq i} \Gamma(x_i - x_j), \quad t > 0 \\ \dfrac{dx_i^R}{dt} = v_i^R, \quad \dfrac{dv_i^R}{dt} = F(x_i^R) + \dfrac{1}{P-1}\displaystyle\sum_{j\in[i]_m} \Gamma(x_i^R - x_j^R), \quad t \in (\tau_m, \tau_{m+1}), \\ (x_i(0), v_i(0)) = (x_i^R(0), v_i^R(0)) = (x_i^{in}, v_i^{in}), \end{cases}$$

where F and Γ are C^1-force fields that are bounded Lipschitz continuous:

$$\|F\|_{L^\infty} + \|\nabla F\|_{L^\infty} < \infty, \quad \|\Gamma\|_{L^\infty} + \|\Gamma\|_{Lip} < \infty,$$

and the initial data (x_i^{in}, v_i^{in}) are independently sampled from a common distribution. Then, the authors derived a finite-time error estimate: for $T \in (0, \infty)$,

$$\sup_{0\leq t\leq T}\sqrt{\mathbb{E}\Big\{\frac{1}{N}\sum_i \Big(|x_i^R(t) - x_i(t)|^2 + |v_i^R(t) - v_i(t)|^2\Big)\Big\}} \leq C(T)\sqrt{\frac{\tau}{P-1}}.$$

Note that the upper bound in the R.H.S. of (4) *depends on time T. In Theorem 3.2, asymptotic flocking estimates of the RBM-approximation results in a uniform error analysis which is independent of T.*

4. Flocking dynamics of the RBM system

In this section, we prove the flocking estimate stated in Theorem 3.1. For this, we study spatial and velocity diameters for the given configuration $\{(x_i^R, v_i^R)\}$ as in Proposition 2.1:

$$\mathcal{D}(X^R(t)) := \max_{i,j} |x_i^R(t) - x_j^R(t)| \quad \text{and} \quad \mathcal{D}(V^R(t)) := \max_{i,j} |v_i^R(t) - v_j^R(t)|.$$

We split the derivation of the stochastic flocking estimate into two steps:

- Step A (Emergence of velocity alignment): the diameters of positions and velocities are estimated in Lemma 2.4:

$$\mathcal{D}(X^R(t)) \le \mathcal{D}(X^{in}), \quad \mathcal{D}(X^R(t)) \le \mathcal{D}(X^{in}) + \mathcal{D}(V^{in})t, \quad t \ge 0$$

  Using these rough estimates, we derive

$$\frac{d}{dt}\Big(\frac{1}{P^2}\sum_{i,j\in[k]_m} |v_i^R - v_j^R|^2\Big) \le -\frac{2\kappa P\psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})t)}{P-1}\frac{1}{P^2}\sum_{i,j\in[k]_m} |v_i^R - v_j^R|^2$$

  and

$$\frac{d}{dt}\Big(\frac{1}{P^2}\sum_{\substack{i\in[k]_m, j\in[l]_m,\\ k\neq l}} |v_i^R - v_j^R|^2\Big) \le 0.$$

  We can not take expectation directly to estimate random permutation, but we may use the values at τ_{m-1} due to the following relationship,

$$\mathbb{E}\Big(\sum_{\substack{1\le i,j\le N,\\ [i]_m\neq[j]_m}} |v_i^R(\tau_{m-1}) - v_j^R(\tau_{m-1})|^2\Big) = \frac{N-P}{N-1}\mathbb{E}\Big(\sum_{i,j} |v_i^R(\tau_{m-1}) - v_j^R(\tau_{m-1})|^2\Big)$$

  and

$$\mathbb{E}\Big(\sum_{\substack{1\le i,j\le N,\\ [i]_m=[j]_m}} |v_i^R(\tau_{m-1}) - v_j^R(\tau_{m-1})|^2\Big) = \frac{P-1}{N-1}\mathbb{E}\Big(\sum_{i,j} |v_i^R(\tau_{m-1}) - v_j^R(\tau_{m-1})|^2\Big).$$

  Combining above relations and by induction on m, we get

$$\begin{aligned}\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(t) - v_j^R(t)|^2\Big) &\le \mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(0) - v_j^R(0)|^2\Big)\\ &\times \exp\left(-\frac{2\kappa P}{N-1}\frac{1}{1+\frac{2P\tau\psi_M}{P-1}}\int_0^t \psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})s)ds\right), \quad t > 0.\end{aligned}$$

  Since $1/\psi(r) = \mathcal{O}(r^\beta)$ as $r\to\infty$, for sufficiently large $t > 0$, we have

$$\int_0^t \psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})s)ds \ge Ct(1+t)^{-\beta},$$

  where C is a positive constant depending only on ψ, κ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$. Hence, we obtain velocity alignment estimate:

$$\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(t) - v_j^R(t)|^2\Big) \le Ce^{-\frac{C(P-1)}{(N-1)(1+\tau)}t(1+t)^{-\beta}}.$$

- Step B (Emergence of spatial cohesion): By using (1.3), one has

$$\frac{d}{dt}\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(t)-x_j^R(t)|^2} \le \sqrt{\frac{1}{N^2}\sum_{i,j}|v_i^R(t)-v_j^R(t)|^2}, \quad \text{a.e. } t>0.$$

This also implies

$$\begin{aligned}&\mathbb{E}\Big(\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(t)-x_j^R(t)|^2}\Big)\\ &\qquad \le \mathbb{E}\Big(\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(0)-x_j^R(0)|^2}\Big)+\int_0^t\sqrt{\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(s)-v_j^R(s)|^2\Big)}ds.\end{aligned}$$

We use velocity alignment estimate to find

$$\int_0^t\sqrt{\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(s)-v_j^R(s)|^2\Big)}ds \le C\Big(\frac{(N-1)(1+\tau)}{(P-1)}\Big)^{\frac{1}{1-\beta}}.$$

Finally, we combine the above two estimates to get desired spatial cohesion.

These two steps will be elaborated in the following two subsections.

4.1. **Velocity alignment.** In next lemma, we provide an elementary estimate to be used to simplify the decay rate of the relative velocities.

Lemma 4.1. *Let $0\le a\le 1$, $b>0$ be given. Then,*

$$a+(1-a)e^{-x}\le \exp\left(-\frac{1-a}{1+b}x\right), \quad \textit{for } x\in[0,b]. \tag{4.1}$$

Proof. Note that the inequality (4.1) is equivalent to

$$\log\Big(a+(1-a)e^{-x}\Big)\le -\frac{1-a}{1+b}x. \tag{4.2}$$

Note that the left-hand side of (4.2) is convex from its nonnegative second derivative. Then, one can bound it with a linear function:

$$\begin{aligned}&\log\Big(a+(1-a)e^{-x}\Big)\\ &\le \frac{\log(a+(1-a)e^{-t})|_{t=b}-\log(a+(1-a)e^{t})|_{t=0}}{b-0}x+\log(a+(1-a)e^{t})\Big|_{t=0}\\ &=\frac{\log(a+(1-a)e^{-b})}{b}x \quad \text{for} \quad x\in[0,b].\end{aligned} \tag{4.3}$$

Next, we use elementary inequalities

$$\log x\le (x-1) \quad \text{and} \quad e^{-x}\le 1/(1+x)$$

to see

$$\frac{1}{b}\log(a+(1-a)e^{-b})\le\frac{1}{b}((a+(1-a)e^{-b})-1)=-(1-a)\frac{(1-e^{-b})}{b}\le-\frac{1-a}{1+b}. \tag{4.4}$$

Finally, we combine (4.3) and (4.4) to get the desired estimate (4.2). ☐

Now we are ready to prove the first part of Theorem 3.1 on the emergence of velocity alignment.

Proof of the first part: Let (X^R, V^R) be a solution of the RBM model (1.2)–(1.3). For a fixed $m \in \{1, 2, \cdots\}$, let $t \in (\tau_{m-1}, \tau_m)$, and suppose that ψ is long-ranged so that

$$1/\psi(r) = \mathcal{O}(r^\beta) \quad \text{as} \quad r \to \infty, \quad 0 \le \beta < 1.$$

Then, we claim:

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(t) - v_j^R(t)|^2\Big) \le 0, \\
&\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(t) - v_j^R(t)|^2\Big) \le Ce^{-\frac{C(P-1)}{(N-1)(1+\tau)}t(1+t)^{-\beta}},
\end{aligned} \tag{4.5}$$

where C is a positive constant depending on ψ, κ, β, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

• (Derivation of the first estimate in (4.5)): By Lemma 2.4, one has

$$\frac{d}{dt}\sum_{i\in[k]_m} v_i^R = 0.$$

Then, for any two batches $[k]_m$ and $[l]_m$, we have

$$\frac{d}{dt}\Big(\frac{1}{P^2}\sum_{i\in[k]_m}\sum_{j\in[l]_m}|v_i^R - v_j^R|^2\Big) = \frac{1}{P}\frac{d}{dt}\Big(\sum_{i\in[k]_m}|v_i^R|^2 + \sum_{i\in[l]_m}|v_i^R|^2\Big).$$

From the dynamics of the RBM model, one has

$$\frac{1}{P}\frac{d}{dt}\Big(\sum_{i\in[k]_m}|v_i^R|^2\Big) = -\frac{\kappa}{P(P-1)}\sum_{i,j\in[k]_m}\psi(|x_j^R - x_i^R|)|v_j^R - v_i^R|^2 \le 0. \tag{4.6}$$

Hence, we get

$$\frac{d}{dt}\Big(\frac{1}{P^2}\sum_{i\in[k]_m}\sum_{j\in[l]_m}|v_i^R - v_j^R|^2\Big) \le 0. \tag{4.7}$$

Similar to (4.6) and (4.7), for any batch $[k]_m$, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{P^2}\sum_{i,j\in[k]_m}|v_i^R - v_j^R|^2\Big) \\
&= \frac{2}{P}\frac{d}{dt}\sum_{i\in[k]_m}|v_i^R|^2 = -\frac{2\kappa}{P(P-1)}\sum_{i,j\in[k]_m}\psi(|x_j^R - x_i^R|)|v_j^R - v_i^R|^2 \\
&\le -\frac{2\kappa\psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})t)}{P(P-1)}\sum_{i,j\in[k]_m}|v_i^R - v_j^R|^2 \le 0,
\end{aligned} \tag{4.8}$$

where we used Lemma 2.4 in the last inequality. Therefore, we combine (4.7) and (4.8) to conclude

$$\frac{d}{dt}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(t) - v_j^R(t)|^2\Big) \le 0.$$

• (Derivation of the second estimate in (4.5)): Next, we consider the decay rate in expectation. Note that we can estimate the decay in (4.8), when two particles are in the same batch. For any $t \in [\tau_{m-1}, \tau_m]$, we have

$$
\begin{aligned}
&\sum_{i,j} |v_i^R(t) - v_j^R(t)|^2 \\
&\le \sum_{\substack{1\le i,j\le N,\\ [i]_m \neq [j]_m}} |v_i^R(t) - v_j^R(t)|^2 + \sum_{\substack{1\le i,j\le N,\\ [i]_m = [j]_m}} |v_i^R(t) - v_j^R(t)|^2 \\
&\le \sum_{\substack{1\le i,j\le N,\\ [i]_m \neq [j]_m}} |v_i^R(\tau_{m-1}) - v_j^R(\tau_{m-1})|^2 \\
&\quad + \sum_{\substack{1\le i,j\le N,\\ [i]_m = [j]_m}} |v_i^R(\tau_{m-1}) - v_j^R(\tau_{m-1})|^2 \exp\left(-\frac{2\kappa P}{P-1} \int_{\tau_{m-1}}^t \psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})s) ds \right),
\end{aligned}
$$

where we used Grönwall's lemma only for the second term containing ($[i]_m = [j]_m$).

From this rough bound estimate, we take expectation on both sides. Note that the event whether i and j are in the same batch ($[i]_m = [j]_m$) or not ($[i]_m \neq [j]_m$) is independent of the random variables $v_i^R(\tau_{m-1})$, $i = 1, \ldots, N$.

Since the choice of batches are random with the uniform distribution, the probability of the event ($[i]_m = [j]_m$) is $(P-1)/(N-1)$. This shows that, for any $t \in [\tau_{m-1}, \tau_m]$, we have

$$
\begin{aligned}
\mathbb{E}\Big(\frac{1}{N^2} \sum_{i,j} |v_i^R(t) - v_j^R(t)|^2 \Big) &\le \mathbb{E}\Big(\frac{1}{N^2} \sum_{1\le i,j\le N} |v_i^R(\tau_{m-1}) - v_j^R(\tau_{m-1})|^2 \Big) \\
&\quad \times \left(\frac{N-P}{N-1} + \frac{P-1}{N-1} \exp\left(-\frac{2\kappa P}{P-1} \int_{\tau_{m-1}}^t \psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})s) ds \right) \right).
\end{aligned}
$$

Next, we use Lemma 4.1 to simplify the decay rate. We set

$$
x = \frac{2\kappa P}{P-1} \int_{\tau_{m-1}}^t \psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})s) ds, \quad a = \frac{N-P}{N-1} \quad \text{and} \quad b = \frac{2P\tau\psi_M}{P-1}.
$$

Then, one has

$$
\begin{aligned}
\mathbb{E}\Big(\frac{1}{N^2} \sum_{i,j} |v_i^R(t) - v_j^R(t)|^2 \Big) &\le \mathbb{E}\Big(\frac{1}{N^2} \sum_{1\le i,j\le N} |v_i^R(\tau_{m-1}) - v_j^R(\tau_{m-1})|^2 \Big) \\
&\quad \times \exp\left(-\frac{2\kappa P}{N-1} \frac{1}{1 + \frac{2P\tau\psi_M}{P-1}} \int_{\tau_{m-1}}^t \psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})s) ds \right).
\end{aligned}
$$

By induction on m, we get

$$
\begin{aligned}
\mathbb{E}\Big(\frac{1}{N^2} \sum_{i,j} |v_i^R(t) - v_j^R(t)|^2 \Big) &\le \mathbb{E}\Big(\frac{1}{N^2} \sum_{i,j} |v_i^R(0) - v_j^R(0)|^2 \Big) \\
&\quad \times \exp\left(-\frac{2\kappa P}{N-1} \frac{1}{1 + \frac{2P\tau\psi_M}{P-1}} \int_0^t \psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})s) ds \right), \quad t > 0.
\end{aligned}
$$

Here, the coefficient in the exponential function can be estimated as

$$-\frac{2\kappa P}{N-1}\frac{1}{1+\frac{2P\tau\psi_M}{P-1}} \le -\frac{C(P-1)}{(N-1)(1+\tau)}$$

for some constant C which depends on ψ and κ, but independent of τ, P and N.

Since ψ is bounded, non-increasing and $1/\psi(r) = \mathcal{O}(r^\beta)$ as $r \to \infty$, we have

$$\int_0^t \psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})s)ds \ge t\psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})t) \ge Ct(1+t)^{-\beta},$$

where C is a positive constant which only depends on ψ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$. Hence, we get

$$\mathbb{E}\left(\frac{1}{N^2}\sum_{i,j}|v_i^R(t) - v_j^R(t)|^2\right) \le Ce^{-\frac{C(P-1)}{(N-1)(1+\tau)}t(1+t)^{-\beta}}, \tag{4.9}$$

for C depending only on ψ, κ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

Remark 4.1. *The technical difficulty of estimating the decay of the relative velocities comes from the dependency between the batch $[i]_m$ and the approximated state $v_i^R(t)$ for $t > \tau_{m-1}$. If they were independent, from* (4.8)*, we can easily get*

$$\frac{d}{dt}\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R - v_j^R|^2\Big) \le -\frac{2\kappa P\psi(\mathcal{D}(X^{in}) + \mathcal{D}(V^{in})t)}{P-1}\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R - v_j^R|^2\Big),$$

which will deduce the conclusion of Theorem 3.1 without dependency on P, N or τ.

4.2. **Uniform spatial cohesion.** In this subsection, we study the uniform boundedness of the relative positions which is the second part of Theorem 3.1.

Proof of the second part: We claim that there exists a positive constant x_∞^R depending on ψ, κ, τ, β, P, N, $\mathcal{D}(X^{in})$ and $\mathcal{D}(X^{in})$ such that

$$\sup_{0\le t<\infty} \mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|x_i^R - x_j^R|^2\Big) < x_\infty^R.$$

From the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{d}{dt}\Big(\frac{1}{N^2}\sum_{i,j}|x_i^R(t) - x_j^R(t)|^2\Big) &= \frac{2}{N^2}\sum_{i,j}\langle x_i^R(t) - x_j^R(t), v_i^R(t) - v_j^R(t)\rangle \\ &\le 2\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(t) - x_j^R(t)|^2}\sqrt{\frac{1}{N^2}\sum_{i,j}|v_i^R(t) - v_j^R(t)|^2},\end{aligned}$$

so that

$$\frac{d}{dt}\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(t) - x_j^R(t)|^2} \le \sqrt{\frac{1}{N^2}\sum_{i,j}|v_i^R(t) - v_j^R(t)|^2}, \quad \text{a.e. } t > 0.$$

Hence, the derivative of the position norm is bounded by the velocity norm. Then, if we integrate both sides with respect to time and take an expectation, we can get

$$\begin{aligned}
&\mathbb{E}\Big(\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(t)-x_j^R(t)|^2}\Big)-\mathbb{E}\Big(\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(0)-x_j^R(0)|^2}\Big)\\
&\quad\le\int_0^t\mathbb{E}\Big(\sqrt{\frac{1}{N^2}\sum_{i,j}|v_i^R(s)-v_j^R(s)|^2}\Big)ds\le\int_0^t\sqrt{\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(s)-v_j^R(s)|^2\Big)}ds.
\end{aligned}$$

Now we use the decay of relative velocities (4.9) to get

$$\begin{aligned}
&\int_0^t\sqrt{\mathbb{E}\Big(\frac{1}{N^2}\sum_{i,j}|v_i^R(s)-v_j^R(s)|^2\Big)}ds\\
&\le C\int_0^\infty e^{-\frac{C(P-1)}{(N-1)(1+\tau)}t(1+t)^{-\beta}}dt\le C\big((P-1)^{-1}(N-1)(1+\tau)\big)^{\frac{1}{1-\beta}}\int_0^\infty e^{-t^{1-\beta}}dt\\
&\le C\big((P-1)^{-1}(N-1)(1+\tau)\big)^{\frac{1}{1-\beta}},
\end{aligned}$$

for a constant $C>0$ depending on ψ, κ, β, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$. This implies the boundedness of the position norm,

$$\begin{aligned}
&\sup_{t\ge0}\mathbb{E}\Big(\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(t)-x_j^R(t)|^2}\Big)\\
&\quad\le\sqrt{\frac{1}{N^2}\sum_{i,j}|x_i^R(0)-x_j^R(0)|^2}+C\big((P-1)^{-1}(N-1)(1+\tau)\big)^{\frac{1}{1-\beta}}\\
&\quad\le\mathcal{D}(X^{in})+C\big((P-1)^{-1}(N-1)(1+\tau)\big)^{\frac{1}{1-\beta}}.
\end{aligned}$$

This completes the second part of Theorem 3.1.

5. Uniform-in-time error estimate

In this section, we derive a uniform error estimate (Theorem 3.2) for the RBM model (1.3). The proof is based on the idea in [25], which considers stochastic first-order models. In the case of the second-order model without noise, we need to use the flocking estimate of the RBM solution described in Theorem 3.1.

Consider the CS model:

$$\begin{cases}\dfrac{dx_i}{dt}=v_i,\quad t>0,\quad i=1,\cdots,N,\\[2mm] \dfrac{dv_i}{dt}=\dfrac{\kappa}{N-1}\displaystyle\sum_{j=1}^N\psi(|x_j-x_i|)(v_j-v_i)\end{cases}\tag{5.1}$$

and its corresponding RBM model:

$$\begin{cases}\dfrac{dx_i^R}{dt}=v_i^R,\quad t\in[\tau_{m-1},\tau_m),\quad m=1,2,\cdots,\quad i=1,\cdots,N,\\[2mm] \dfrac{dv_i^R}{dt}=\dfrac{\kappa}{P-1}\displaystyle\sum_{j\in[i]_m}\psi(|x_j^R-x_i^R|)(v_j^R-v_i^R).\end{cases}\tag{5.2}$$

For the proof of Theorem 3.2, we need to measure the distance between the solution $Z = (X, V)$ and its RBM-approximation $Z^R = (X^R, V^R)$. For this, we set

$$w_i := v_i^R - v_i, \quad i = 1, \cdots, N.$$

Our goal is to construct a differential inequality for $\Big(\frac{1}{N}\sum_i |w_i|^2\Big)$. From the dynamics (5.1) and (5.2), for $t \in [\tau_{m-1}, \tau_m)$ with $m \geq 1$, w_i satisfies

$$\frac{dw_i}{dt} = \frac{\kappa}{P-1}\sum_{j\in[i]_m}\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \frac{\kappa}{N-1}\sum_{j:j\neq i}\psi(|x_j - x_i|)(v_j - v_i). \tag{5.3}$$

Note that it has two different summations on j, one is in $[i]_m$ and the other is for all $j \neq i$. Following [25], we may decompose these summations into two parts. One is for the full summation and the other is on the RBM-approximated trajectories:

$$\frac{dw_i}{dt} = \frac{\kappa}{N-1}\sum_{j:j\neq i}\big[\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \psi(|x_j - x_i|)(v_j - v_i)\big] + \kappa\chi_{m,i}(Z^R), \tag{5.4}$$

where $\chi_{m,i}(Z^R)$ is the remaining term:

$$\chi_{m,i}(Z^R) := \frac{1}{P-1}\sum_{j\in[i]_m}\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \frac{1}{N-1}\sum_{j:j\neq i}\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R). \tag{5.5}$$

By multiplying $2w_i$ to both sides and averaging over i, we have

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{N}\sum_i |w_i|^2\right] \\ &= \frac{2\kappa}{N(N-1)}\sum_i\sum_{j:j\neq i}\Big[\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \psi(|x_j - x_i|)(v_j - v_i)\Big]\cdot(v_i^R - v_i) \\ &\quad + \frac{2\kappa}{N}\sum_i w_i \cdot \chi_{m,i}(Z^R). \end{aligned} \tag{5.6}$$

Then, we take an expectation on (5.6) to get

$$\frac{d}{dt}\mathbb{E}\Big(\frac{1}{N}\sum_i |w_i|^2\Big) = \frac{2\kappa}{N}\mathcal{S} + \frac{2\kappa}{N}\sum_i \mathcal{R}_i, \tag{5.7}$$

where the functionals $\mathcal{S}(t)$ and $\mathcal{R}_i(t)$ are defined as follows:

$$\begin{aligned} &\mathcal{S}(t) := \frac{1}{N-1}\mathbb{E}\left(\sum_i\sum_{j:j\neq i}\Big[\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \psi(|x_j - x_i|)(v_j - v_i)\Big]\cdot(v_i^R - v_i)\right), \\ &\mathcal{R}_i(t) := \mathbb{E}\left[w_i \cdot \chi_{m,i}(Z^R)\right]. \end{aligned} \tag{5.8}$$

Note that, if ψ is a constant function $\psi(\cdot) \equiv \psi_0 > 0$, then we have

$$
\begin{aligned}
\mathcal{S}(t) &= \frac{1}{N-1}\mathbb{E}\left(\sum_{i,j}\left[\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \psi(|x_j - x_i|)(v_j - v_i)\right]\cdot w_i\right) \\
&= \frac{\psi_0}{N-1}\mathbb{E}\left(\sum_{i,j}[w_j - w_i]\cdot w_i\right) = -\frac{N\psi_0}{N-1}\mathbb{E}\left(\sum_i |w_i|^2\right)
\end{aligned} \tag{5.9}
$$

from the conservation of the first velocity moment in Lemma 2.1 and Lemma 2.3. This represents the contracting property on w_i, which induces

$$
\frac{d}{dt}\mathbb{E}\left(\frac{1}{N}\sum_i |w_i|^2\right) = -\frac{2N\kappa\psi_0}{N-1}\mathbb{E}\left(\frac{1}{N}\sum_i |w_i|^2\right) + \frac{2\kappa}{N}\sum_i \mathcal{R}_i.
$$

Therefore, if we can bound $|\mathcal{R}_i(t)|$, we may get the same error estimate as in [25] for velocities.

In the following subsection, we present an estimate for $\mathcal{S}$ with nonconstant ψ.

5.1. **Estimates on the functional $\mathcal{S}$.** In this subsection, we study the estimate for the functional $\mathcal{S}$. The analysis on the RBM in [25] suggests that, in order to prove the uniform error estimates, the contraction property (as in (5.9)) is required. We do not have this property when ψ depends on the relative positions. To overcome this difficulty, we need the flocking estimate, Lemma 2.2 and Remark 2.2.

Lemma 5.1. *Suppose that the communication weight function ψ has a positive lower bound,*

$$
\psi(r) \geq \psi_0, \quad \forall\ r \geq 0,
$$

and let (X, V) and (X^R, V^R) be solutions to (5.1) and (5.2), respectively. Then, the functional $\mathcal{S}(t)$ from (5.8) satisfies

$$
\mathcal{S}(t) \leq -\psi_0\mathbb{E}\Big(\sum_i |w_i|^2\Big) + C\sqrt{N}e^{-\kappa\psi_0 t}\mathbb{E}\sqrt{\Big(\sum_i |w_i|^2\Big)},
$$

for some constant C depending on ψ_M and $\mathcal{D}(V^{in})$.

Proof. We split the ψ term into two pieces to get

$$
\begin{aligned}
\mathcal{S}(t) &= \frac{1}{N-1}\mathbb{E}\Big(\sum_{i,j}\left[\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \psi(|x_j - x_i|)(v_j - v_i)\right]\cdot(v_i^R - v_i)\Big) \\
&= \frac{1}{N-1}\mathbb{E}\Big(\sum_{i,j}\psi(|x_j^R - x_i^R|)((v_j^R - v_i^R) - (v_j - v_i))\cdot(v_i^R - v_i)\Big) \\
&+ \frac{1}{N-1}\mathbb{E}\Big(\sum_{i,j}\left[(\psi(|x_j^R - x_i^R|) - \psi(|x_j - x_i|))(v_j - v_i)\right]\cdot w_i\Big) \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \tag{5.10}
$$

• (Estimate of $\mathcal{I}_1$): We interchange i and j and use the symmetry property of ψ to get

$$
\begin{aligned}
\mathcal{I}_1 &= \frac{1}{N-1}\mathbb{E}\Big(\sum_{i,j}\psi(|x_j^R - x_i^R|)((v_j^R - v_i^R) - (v_j - v_i))(v_i^R - v_i)\Big) \\
&= \frac{1}{N-1}\mathbb{E}\Big(\sum_{i,j}\psi(|x_j^R - x_i^R|)((v_j^R - v_j) - (v_i^R - v_i))(v_i^R - v_i)\Big) \\
&= -\frac{1}{2(N-1)}\mathbb{E}\Big(\sum_{i,j}\psi(|x_j^R - x_i^R|)|(v_j^R - v_j) - (v_i^R - v_i)|^2\Big) \\
&\le -\frac{\psi_0}{2(N-1)}\mathbb{E}\Big(\sum_{i,j}|(v_j^R - v_j) - (v_i^R - v_i)|^2\Big) \\
&= -\frac{\psi_0}{N-1}\mathbb{E}\Big(\sum_{i,j}|v_i^R - v_i|^2\Big) \\
&= -\frac{N\psi_0}{N-1}\mathbb{E}\Big(\sum_i |w_i|^2\Big) \le -\psi_0\mathbb{E}\Big(\sum_i |w_i|^2\Big).
\end{aligned} \tag{5.11}
$$

• (Estimate of $\mathcal{I}_2$): We use the flocking estimate in Remark 2.2:

$$|v_j(t) - v_i(t)| \le \mathcal{D}(V^{in})\exp(-\kappa\psi_0 t), \quad i,j = 1,2,\dots,N,$$

to get

$$
\begin{aligned}
\mathcal{I}_2 &= \frac{1}{N-1}\mathbb{E}\Big(\sum_i\sum_{j:j\neq i}\big[(\psi(|x_j^R - x_i^R|) - \psi(|x_j - x_i|))(v_j - v_i)\big]\cdot w_i\Big) \\
&\le \frac{\psi_M}{N-1}\mathbb{E}\Big(\sum_i\sum_{j:j\neq i}|v_j - v_i||w_i|\Big) \\
&\le Ce^{-\kappa\psi_0 t}\mathbb{E}\Big(\sum_i |w_i|\Big) \le C\sqrt{N}e^{-\kappa\psi_0 t}\sqrt{\mathbb{E}\Big(\sum_i |w_i|^2\Big)}.
\end{aligned} \tag{5.12}
$$

In (5.10), we combine estimates (5.11) and (5.12) to find

$$\mathcal{S}(t) \le -\psi_0\mathbb{E}\Big(\sum_i |w_i|^2\Big) + C\sqrt{N}e^{-\kappa\psi_0 t}\sqrt{\mathbb{E}\Big(\sum_i |w_i|^2\Big)},$$

for some positive constant C depending on $\mathcal{D}(V^{in})$. □

5.2. **Estimates for $\mathcal{R}_i$.** In this subsection, we derive estimates for $\mathcal{R}_i$:

$$\mathcal{R}_i(t) = \mathbb{E}\left[w_i\cdot\chi_{m,i}(Z^R)\right].$$

First, we recall the stochastic process $\chi_{m,i}(Z^R)$ defined in (5.5):

$$\chi_{m,i}(Z^R) = \frac{1}{P-1}\sum_{j\in[i]_m}\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \frac{1}{N-1}\sum_{j:j\neq i}\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R).$$

Lemma 5.2. *$\chi_{m,i}(Z^R(\tau_{m-1}))$ satisfies the following stochastic estimates: there exists a positive constant C depending only on ψ_M and $\mathcal{D}(V^{in})$ such that*

$$\mathbb{E}[\chi_{m,i}(Z^R(\tau_{m-1}))] = 0, \quad \mathrm{Var}[\chi_{m,i}(Z^R(\tau_{m-1}))] \le C\left(\frac{1}{P-1} - \frac{1}{N-1}\right).$$

Proof. The proof follows from the same arguments in (Lemma 3.1 [25]), which says that the expectation is zero due to the independence between the randomness of the batches $[i]_m$ and the variable $Z^R(\tau_{m-1}) = (X^R(\tau_{m-1}), V^R(\tau_{m-1}))$. Moreover, one has

$$\mathrm{Var}[\chi_{m,i}(Z^R(\tau_{m-1}))] = \left(\frac{1}{P-1} - \frac{1}{N-1}\right)\Lambda_i. \tag{5.13}$$

Here, Λ_i is given by

$$\Lambda_i[Z^R] := \frac{1}{N-2}\sum_{j:j\neq i}\mathbb{E}\left|\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \frac{1}{N-1}\sum_{k:k\neq i}\psi(x_k^R - x_i^R)(v_k^R - v_i^R)\right|^2.$$

Now, it only remains to bound $\|\Lambda_i\|_\infty$. Following Lemma 2.4, we get

$$|\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R)| \le \psi_M \mathcal{D}(V^{in}).$$

Therefore, we have

$$\|\Lambda_i\|_\infty \le \frac{N-1}{N-2}(2\psi_M\mathcal{D}(V^{in}))^2 \le 8\psi_M^2\mathcal{D}(V^{in})^2. \tag{5.14}$$

Finally, we combine (5.13) and (5.14) to get the desired estimate. □

Lemma 5.3. *Let $Z = (X,V)$ and $Z^R = (X^R, V^R)$ be solutions to* (5.1) *and* (5.2)*, respectively. Then, for any i, $m \ge 1$ and $t \in [\tau_{m-1}, \tau_m)$, we have*

$$|\chi_{m,i}(Z^R(t)) - \chi_{m,i}(Z^R(\tau_{m-1}))| \le C\tau,$$

where $\tau = \tau_m - \tau_{m-1}$ and C is a positive constant depending on ψ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

Proof. Note that the difference of $\chi_{m,i}(\cdot)$ can be rearranged as follows:

$$\begin{aligned}
&\chi_{m,i}(Z^R(t)) - \chi_{m,i}(Z^R(\tau_{m-1}))\\
&\quad = \Big[\frac{1}{P-1}\sum_{j\in[i]_m}\psi(x_j^R(t) - x_i^R(t))(v_j^R(t) - v_i^R(t))\\
&\quad - \frac{1}{P-1}\sum_{j\in[i]_m}\psi(x_j^R(\tau_{m-1}) - x_i^R(\tau_{m-1}))(v_j^R(\tau_{m-1}) - v_i^R(\tau_{m-1}))\Big]\\
&\quad - \Big[\frac{1}{N-1}\sum_{j:j\neq i}\psi(x_j^R(t) - x_i^R(t))(v_j^R(t) - v_i^R(t))\\
&\quad - \frac{1}{N-1}\sum_{j:j\neq i}\psi(x_j^R(\tau_{m-1}) - x_i^R(\tau_{m-1}))(v_j^R(\tau_{m-1}) - v_i^R(\tau_{m-1}))\Big].
\end{aligned}$$

Note that each term represents a difference of some quantity between t and τ_{m-1}. By the Lipschitz continuity of ψ and Lemma 2.4, this is bounded by $C\tau$. Here, the constant C depends on the interaction kernel ψ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$. □

Lemma 5.4. *Let $Z = (X,V)$ and $Z^R = (X^R, V^R)$ be two solutions to system* (5.1) *and* (5.2)*, respectively. Then, for $t \in [\tau_{m-1}, \tau_m)$, $W = V^R - V$ satisfies the following assertions:*

(1) For any $m > 0$ and i, there exists a positive constant C depending on ψ, κ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$ such that

$$|w_i(t) - w_i(\tau_{m-1})| \leq C\tau. \tag{5.15}$$

(2) For any $m > 0$ and i, we have

$$\begin{aligned} &\mathbb{E}[(w_i(t) - w_i(\tau_{m-1})) \cdot \chi_{m,i}(Z^R(\tau_{m-1}))] \\ &\leq \frac{C\tau}{P-1} \sum_{j \in [i]_m} (|w_i| + |w_j|) + C\tau^2 + C\tau \exp(-\kappa\psi(x_\infty)t) + C\tau \left(\frac{1}{P-1} - \frac{1}{N-1} \right), \end{aligned} \tag{5.16}$$

where x_∞ is the constant defined in Proposition 2.1 and C is a constant depending on ψ, κ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

Proof. (i) The first estimate (5.15) follows from Lemma 2.2 and Lemma 2.4 that

$$\left| \frac{dw_i}{dt} \right| \leq C$$

for some constant C depending only on ψ, κ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

(ii) For the second one (5.16), we start from the relation

$$\mathbb{E}[(w_i(t) - w_i(\tau_{m-1})) \cdot \chi_{m,i}(Z^R(\tau_{m-1}))] = \mathbb{E}\left[\int_{\tau_{m-1}}^t \frac{d}{dt} w_i(s) ds \cdot \chi_{m,i}(Z^R(\tau_{m-1})) \right]. \tag{5.17}$$

Now we estimate the derivative of w_i more precisely. First, from the first assertion, we have

$$|w_i(t)| - |w_i(\tau_{m-1})| \leq C\tau. \tag{5.18}$$

On the other hand, as in (5.4), we can rearrange (5.3) with the summation on $[i]_m$:

$$\begin{aligned} \frac{dw_i}{dt} &= \frac{\kappa}{P-1} \sum_{j \in [i]_m} \left[\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \psi(|x_j - x_i|)(v_j - v_i) \right] + \kappa\chi_{m,i}(Z) \\ &=: \mathcal{I}_3 + \kappa\chi_{m,i}(Z). \end{aligned} \tag{5.19}$$

• (Estimate of $\mathcal{I}_3$): We use the conservation of the first moment (Lemma 2.3) to find

$$
\begin{aligned}
|\mathcal{I}_3| &= \frac{\kappa}{P-1}\left|\sum_{j\in[i]_m}\left[\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \psi(|x_j - x_i|)(v_j - v_i)\right]\right| \\
&\le \frac{\kappa}{P-1}\left|\sum_{j\in[i]_m}\psi(|x_j^R - x_i^R|)\left[(v_j^R - v_i^R) - (v_j - v_i)\right]\right| \\
&\quad + \frac{\kappa}{P-1}\left|\sum_{j\in[i]_m}\left[\psi(|x_j^R - x_i^R|) - \psi(|x_j - x_i|)\right](v_j - v_i)\right| \\
&\le \frac{\kappa\psi_M}{P-1}\sum_{j\in[i]_m}\left|(v_j^R - v_i^R) - (v_j - v_i)\right| + \frac{\kappa\psi_M}{P-1}\sum_{j\in[i]_m}|v_j - v_i| \\
&\le \frac{C}{P-1}\sum_{j\in[i]_m}(|w_j| + |w_i|) + C\exp(-\kappa\psi(x_\infty)t).
\end{aligned}
\tag{5.20}
$$

Then, we use (5.18) and (5.20) to obtain

$$
\begin{aligned}
\mathbb{E}(|\mathcal{I}_3|) &= \mathbb{E}\Big(\Big|\frac{\kappa}{P-1}\sum_{j\in[i]_m}\left[\psi(|x_j^R - x_i^R|)(v_j^R - v_i^R) - \psi(|x_j - x_i|)(v_j - v_i)\right]\Big|\Big) \\
&\le \mathbb{E}\Big(\frac{C}{P-1}\sum_{j\in[i]_m}(|w_j(\tau_{m-1})| + |w_i(\tau_{m-1})|)\Big) + C\exp(-\kappa\psi(x_\infty)t) + C\tau \\
&\le \frac{C}{N-1}\sum_{j:j\neq i}\Big(\mathbb{E}|w_j(\tau_{m-1})| + \mathbb{E}|w_i(\tau_{m-1})|\Big) + C\exp(-\kappa\psi(x_\infty)t) + C\tau,
\end{aligned}
$$

where we used the independence between $|w_j(\tau_{m-1})|$ and $[i]_m$. By using the first assertion again, we get

$$
\mathbb{E}(|\mathcal{I}_3|) \le \frac{C}{N-1}\sum_{j:j\neq i}\Big(\mathbb{E}|w_j(t)| + \mathbb{E}|w_i(t)|\Big) + C\exp(-\kappa\psi(x_\infty)t) + C\tau, \tag{5.21}
$$

for a constant C depending on ψ, κ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

Now it remains to apply (5.19) and (5.21) to (5.17). First, from (5.19), we have

$$
\begin{aligned}
&\mathbb{E}[(w_i(t) - w_i(\tau_{m-1}))\cdot\chi_{m,i}(Z^R(\tau_{m-1}))] \\
&\quad = \mathbb{E}\left[\int_{\tau_{m-1}}^t \frac{d}{dt}w_i(s)ds\cdot\chi_{m,i}(Z^R(\tau_{m-1}))\right] \\
&\quad = \mathbb{E}\left[\int_{\tau_{m-1}}^t (\mathcal{I}_3(s) + \kappa\chi_{m,i}(Z^R(s)))ds\cdot\chi_{m,i}(Z^R(\tau_{m-1}))\right] \\
&\quad = \mathbb{E}\Big(\int_{\tau_{m-1}}^t \mathcal{I}_3(s)ds\chi_{m,i}(Z^R(\tau_{m-1}))\Big) + \kappa\mathbb{E}\Big(\int_{\tau_{m-1}}^t \chi_{m,i}(Z^R(s))\cdot\chi_{m,i}(Z^R(\tau_{m-1}))ds\Big).
\end{aligned}
$$

Next, we use (5.21) and the boundedness of $|\chi_{m,i}(Z^R(\tau_{m-1}))|$ to bound the first term, and we can apply Lemma 5.2 to the second term as follows:

$$\begin{aligned}
&\mathbb{E}[(w_i(t)-w_i(\tau_{m-1}))\cdot\chi_{m,i}(Z^R(\tau_{m-1}))]\\
&\quad\le \frac{C\tau}{N-1}\sum_{j:j\neq i}(\mathbb{E}|w_i(t)|+\mathbb{E}|w_j(t)|)+C\tau^2+C\tau\exp(-\kappa\psi(x_\infty)t)\\
&\qquad+\int_{\tau_{m-1}}^t \kappa\sqrt{\mathbb{E}|\chi_{m,i}(Z(s))|^2}\sqrt{\mathbb{E}|\chi_{m,i}(Z^R(\tau_{m-1}))|^2}ds\\
&\quad\le \frac{C\tau}{N-1}\sum_{j:j\neq i}(\mathbb{E}|w_i|+\mathbb{E}|w_j|)+C\tau^2+C\tau\exp(-\kappa\psi(x_\infty)t)+C\tau\left(\frac{1}{P-1}-\frac{1}{N-1}\right).
\end{aligned}$$

In the last inequality, we used the boundedness of $\mathrm{Var}[\chi_{m,i}(Z(s))]$ from the corresponding arguments of Lemma 5.2.

□

5.3. **Proof of Theorem 3.2.** Now, we are ready to provide the proof of Theorem 3.2. It follows from (5.7) that

$$\frac{d}{dt}\mathbb{E}\Big(\frac{1}{N}\sum_i|w_i|^2\Big)=\frac{2\kappa}{N}\mathcal{S}+\frac{2\kappa}{N}\sum_i\mathcal{R}_i. \tag{5.22}$$

By Lemma 5.1, one has

$$\mathcal{S}(t)\le-\psi_0\mathbb{E}\Big(\sum_i|w_i|^2\Big)+C\sqrt{N}e^{-\kappa\psi_0 t}\mathbb{E}\sqrt{\Big(\sum_i|w_i|^2\Big)}, \tag{5.23}$$

where C depends on $\mathcal{D}(V^{in})$.

Next, we estimate $\mathcal{R}_i(t)$. In order to use Lemma 5.2, we split the terms into the differences as follows:

$$\begin{aligned}
\mathcal{R}_i(t)&=\mathbb{E}[w_i(t)\cdot\chi_{m,i}(Z^R(t))]\\
&=\mathbb{E}[(w_i(t)-w_i(\tau_{m-1}))\cdot\chi_{m,i}(Z^R(\tau_{m-1})))]+\mathbb{E}[w_i(\tau_{m-1})\cdot\chi_{m,i}(Z^R(\tau_{m-1}))]\\
&\quad+\mathbb{E}[w_i(t)\cdot(\chi_{m,i}(Z^R(t))-\chi_{m,i}(Z^R(\tau_{m-1}))]\\
&=:\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6.
\end{aligned} \tag{5.24}$$

- (Estimate of $\mathcal{I}_4$): By Lemma 5.4, one has

$$\mathcal{I}_4\le\frac{C\tau}{N-1}\sum_{j:j\neq i}(\mathbb{E}|w_i|+\mathbb{E}|w_j|)+C\tau^2+C\tau\exp(-\kappa\psi(x_\infty)t)+C\tau\left(\frac{1}{P-1}-\frac{1}{N-1}\right).$$

- (Estimate of $\mathcal{I}_5$): Since the time is fixed at τ_{m-1}, we use Lemma 5.2 to get

$$\begin{aligned}
\mathcal{I}_5&=\mathbb{E}[w_i(\tau_{m-1})\cdot\chi_{m,i}(Z^R(\tau_{m-1}))]\\
&=\mathbb{E}\left[w_i(\tau_{m-1})\cdot\mathbb{E}[\chi_{m,i}(Z^R(\tau_{m-1}))\mid(w_i(\tau_{m-1})\text{ is given})]\right]=0.
\end{aligned}$$

- (Estimate of $\mathcal{I}_6$): The term is bounded by the growth condition, Lemma 5.3:

$$\mathcal{I}_6=\mathbb{E}[|w_i(t)||\chi_{m,i}(Z^R(t))-\chi_{m,i}(Z^R(\tau_{m-1}))|]\le C\mathbb{E}|w_i(t)|\tau.$$

In (5.24), we combine all the estimates for $\mathcal{I}_4$, $\mathcal{I}_5$ and $\mathcal{I}_6$ to get

$$\begin{aligned}\mathcal{R}_i(t) &\le C\tau\left(\mathbb{E}|w_i| + \frac{1}{N-1}\sum_{j:j\neq i}\mathbb{E}|w_j| + \left(\frac{1}{P-1} - \frac{1}{N-1}\right)\right)\\ &\quad + C\tau^2 + C\tau\exp(-\kappa\psi(x_\infty)t).\end{aligned} \tag{5.25}$$

In (5.22), we combine (5.23) and (5.25) and the Cauchy-Schwarz inequality to get

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\left(\frac{1}{N}\sum_i |w_i|^2\right) &= \frac{2\kappa}{N}\mathcal{S}(t) + \frac{2\kappa}{N}\sum_i \mathcal{R}_i(t)\\ &\le -C\mathbb{E}\left(\frac{1}{N}\sum_i |w_i|^2\right) + C\sqrt{\mathbb{E}\left(\frac{1}{N}\sum_i |w_i|^2\right)}\left(\tau + e^{-\kappa\psi_0 t}\right)\\ &\quad + C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\kappa\tau^2 + C\tau e^{-\kappa\psi_0 t}.\end{aligned}$$

From Young's inequality, we may split the second term into the contracting term and the remaining terms,

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\left(\frac{1}{N}\sum_i |w_i|^2\right) &\le -C\mathbb{E}\left(\frac{1}{N}\sum_i |w_i|^2\right) + C\left(\tau + e^{-\kappa\psi_0 t}\right)^2\\ &\quad + C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\tau^2 + C\tau e^{-\kappa\psi_0 t}\\ &\le -C\mathbb{E}\left(\frac{1}{N}\sum_i |w_i|^2\right) + C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\tau^2 + C(1+\tau)e^{-\kappa\psi_0 t}.\end{aligned}$$

Finally, we use Grönwall's Lemma (Lemma A.1 in [10]) to conclude Theorem 3.2:

$$\frac{1}{N}\mathbb{E}\sum_i |w_i|^2 \le C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\tau^2 + C(1+\tau)e^{-\kappa\psi_0 t}$$

for some constant C depending on ψ, κ, $\mathcal{D}(X^{in})$ and $\mathcal{D}(V^{in})$.

6. Numerical simulations

In this section, we present numerical simulations on the RBM-approximated trajectories to compare with the original CS trajectories. The ℓ^2-errors are our main interest from the error estimate in Theorem 3.2, where the expectation of the ℓ^2-error is proved to be bounded uniformly in time:

$$\mathbb{E}\Big(\frac{1}{N}\sum_i |v_i^R(t) - v_i(t)|^2\Big) \le C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\tau^2 + C\exp\left(-\kappa\psi_0 t\right). \tag{6.1}$$

In order to integrate the RBM system, we used the forward Euler method for a fast computation. Numerical simulations in this section are computed with the following parameters unless stated otherwise:

$$d = 2, \quad \tau = 0.1, \quad \kappa = 1, \quad P = 2, \quad N = 64 \quad \text{and} \quad \beta = 1/2.$$

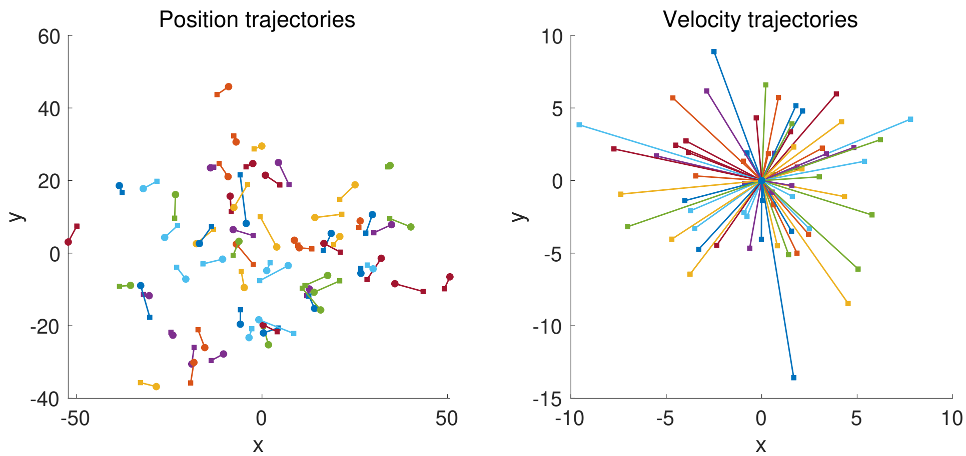

FIGURE 1. The trajectories for $t \in [0, 10]$ in the Cucker-Smale model from random initial data. The squared marks represent the initial data and the big dots denote the final data at $t = 10$.

The initial datum is drawn randomly from a Gaussian distribution and used for the whole simulations. For the communication weight, we use

$$\psi(X) := \frac{1}{(1 + |X|^2)^{1/4}}.$$

Although a positive lower bound assumption on ψ was assumed in Theorem 3.2, numerical simulations in this section show similar results to (6.1) since the relative positions in RBM-approximated trajectories are not growing fast.

6.1. **Dependence of error on the batch size P.** We first deal with numerical simulations for the dependence on P. For simplicity, we set the time step of the Euler method to be the same as the size of time step for random choice of batches, $\tau = 0.1$. Here we use various batch sizes, $P = 2, 4, 8, 16, 32$.

Figures 1 and 2 show the trajectories of the CS model (1.1) and its RBM-approximation (1.3) for $t \in [0, 10]$ from one realization of a series of random batches along time. Note that, in Figure 1, the velocities tend to the origin with nearly linear trajectories in the original CS model, where the origin is the mean velocity at time $t = 0$. As we can observe in Figure 2, the RBM-approximated velocities also converge to zero, but with zig-zag routes. This phenomenon is similar to the iterating process following the stochastic gradient descent algorithm.

In Figure 3, the ℓ^2-errors are drawn from 1000 random simulations for each P. The ℓ^2-error is calculated by

$$\ell^2\text{-error} := \sqrt{\frac{1}{N} \sum_i |v_i^R(t) - v_i(t)|^2}, \quad t \geq 0$$

for each simulated solution $V^R = (v_1^R, \ldots, v_N^R)$. The colored area corresponding to each P shows the evolution of the ℓ^2-errors over time, where 950 simulations are plotted excluding 50 bad simulations, i.e., 95% confidence interval at each time.

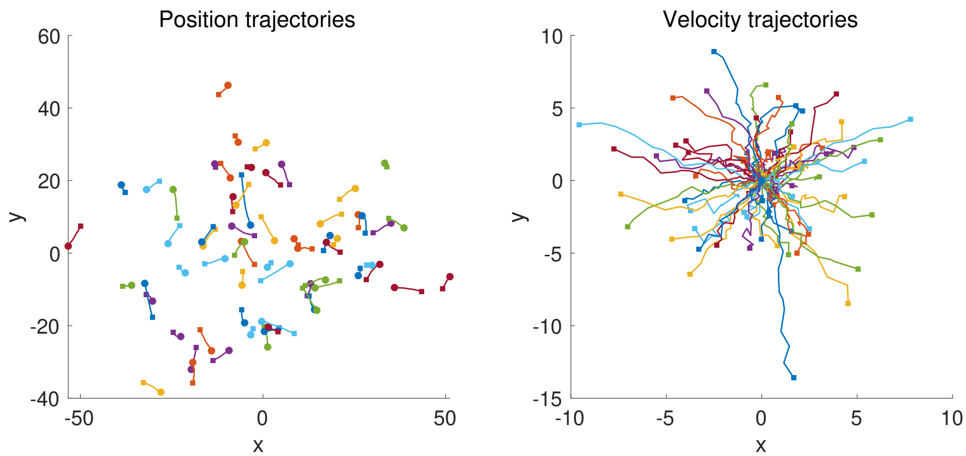

FIGURE 2. The trajectories for $t \in [0, 10]$ in the RBM system with $P = 2$ from the same initial data as in Figure 1. The squared marks represent the initial data and the big dots denote the final data at $t = 10$.

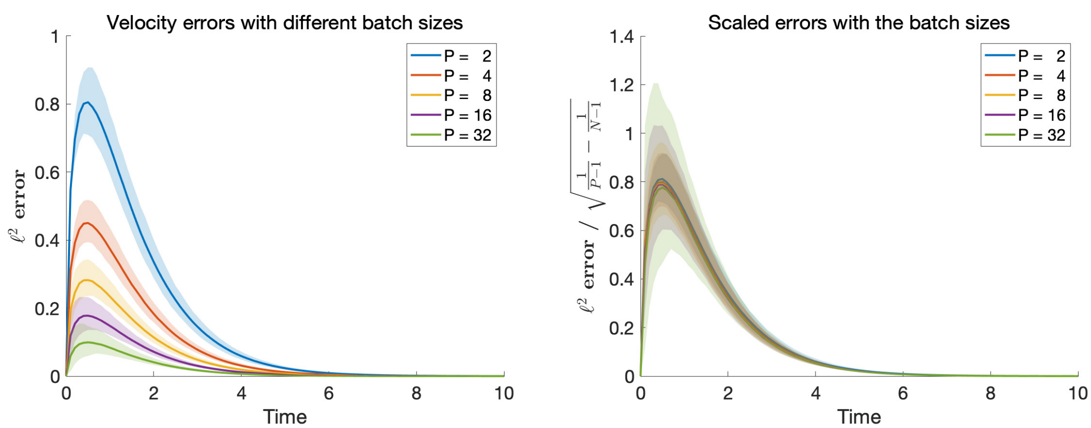

FIGURE 3. Left: The ℓ^2-errors of velocities from 1000 simulations, computed with different P. For each P, a colored region is drawn with 95% of the total simulations excluding the 5% bad ones. The thick lines in the middle represent the median errors. Right: Scaled error by the term $\sqrt{\frac{1}{P-1} - \frac{1}{N-1}}$. The scaled errors from different P show similar values along time.

The right plot of Figure 3 shows the scaled error proportional to $\sqrt{\frac{1}{P-1} - \frac{1}{N-1}}$, which is the order of error proved in Theorem 3.2. Note that all the scaled errors from different P have similar median along time. From this, one can expect that the error estimate (6.1) gives the right order with respect to P.

Figure 4 shows the same plot of Figure 3 with the ℓ^2-error of the positions instead of the velocities. We need to mention that, as we discussed in Corollary 3.1, the errors on velocities decay to zero. Since this convergence is fast enough, the errors on the positions

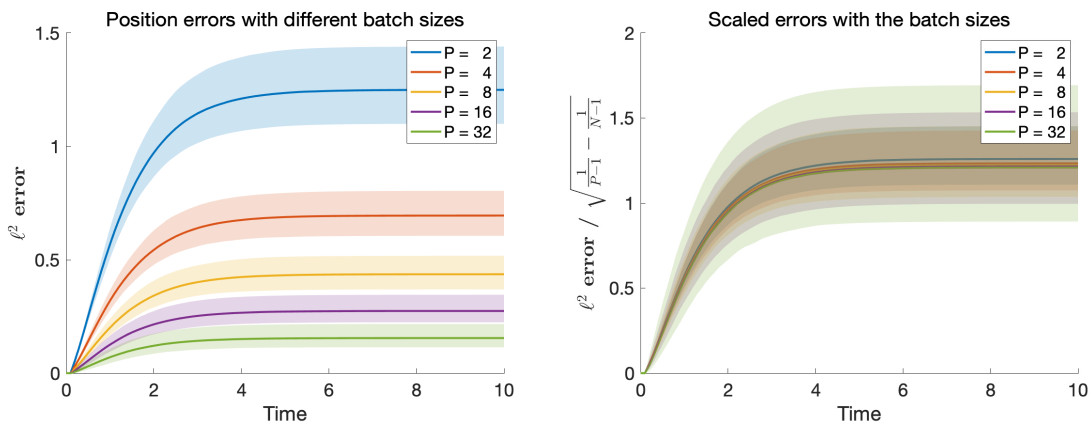

FIGURE 4. Left: The ℓ^2-errors of positions from 1000 simulations, computed with different P. For each P, a colored region is drawn with 95% of the total simulations excluding the 5% bad ones. The thick lines in the middle represent the median errors. Right: Scaled error by the term $\sqrt{\frac{1}{P-1} - \frac{1}{N-1}}$. The scaled errors from different P show similar values along time.

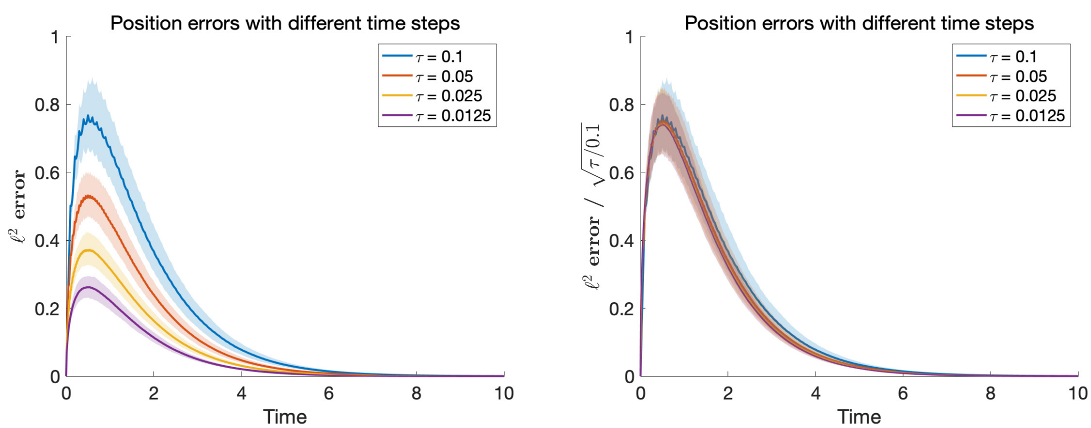

FIGURE 5. Left: The ℓ^2-errors of velocities from 1000 simulations, computed with different τ. For each τ, a colored region is drawn with 95% of the total simulations excluding the 5% bad ones. The thick lines in the middle represent the median errors. Right: Scaled error by the term $\sqrt{\tau/0.1}$. The scaled errors from different P show similar values along time.

are uniformly bounded as we can see in Figure 4. We can also observe that the same scaling property holds with the ratio $\sqrt{\frac{1}{P-1} - \frac{1}{N-1}}$ in Figure 4.

6.2. **Dependence of error on the size of time step τ.** Next, we fix $P = 2$ but instead test various τ. In order to compare different τ under the same circumstance, we set the time step of the Euler method to be $\Delta t = 0.0125$, and test $\tau = 0.1$, 0.05, 0.025 and 0.0125. The other parameters and the drawing of graphs are the same as in Section 6.1.

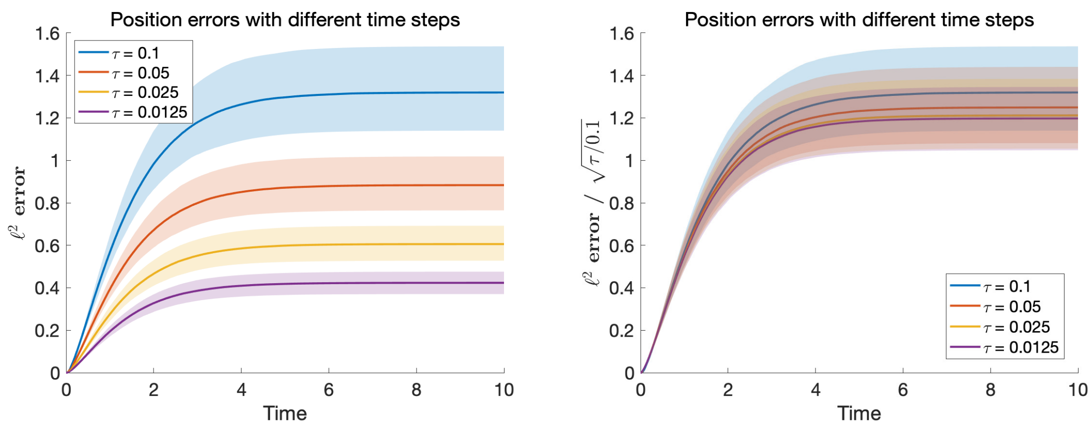

FIGURE 6. Left: The ℓ^2-errors of positions from 1000 simulations, computed with different τ. For each τ, a colored region is drawn with 95% of the total simulations excluding the 5% bad ones. The thick lines in the middle represent the median errors. Right: Scaled error by the term $\sqrt{\tau/0.1}$. The scaled errors from different P show similar values along time.

Figures 5 and 6 are results corresponding to Figures 3 and 4. We can also observe that the decreasing ratio of the error on τ is approximately to the order of $\sqrt{\tau}$ as we expected from the error estimate (6.1).

One notable point is that the velocity error shows a saw-like pattern in Figure 5, which is due to the difference between the short duration time τ and the time-discretization Δt. At each time interval $[\tau_{m-1}, \tau_m)$, the error from the random batches accumulates linearly. From the new choice of the batches at τ_m, the error gets stabilized and the approximated trajectories experience stiffer changes of acceleration. If τ and Δt are the same, we cannot recognize it (as in Figure 3) since it happens at every time step of the Euler method.

7. CONCLUSION

In this paper, we have analyzed the RBM-approximation for the Cucker-Smale model, which is one of prototype models for deterministic second-order collective behavior systems. To establish a time independent error estimate, one needs the contraction property of the original system [25]. However, this contraction property does not hold for the Cucker-Smale model. We overcome this difficulty by using asymptotic flocking estimates of the system and its RBM system. By splitting the interaction term into the self-contracting part and the time-decaying part, we obtain errors from the RBM-approximation which is uniform in time, in addition to its independence on N, the number of particles.

In the convergence analysis of the RBM system, our arguments rely on the understanding of the deterministic flocking models with switching network. From this observation, we expect that a similar uniform error estimate can be derived for general second-order models (for example, N-body systems) if we could analyze the long-time behavior of the interacting particle systems.

In the current work, we considered the Cucker-Smale model on the complete and undirected network. However, one can also consider the Cucker-Smale model on a general directed network, which makes coupling non-symmetric so that there is no known conserved quantity. This will be left as a future work.

The stochastic Cucker-Smale model, where each individual is influenced from external noise such as Brownian motion, is also an interesting system for the error analysis of the RBM. As simulated in [9], optimization problems using interacting particle systems utilize stochastic effect to improve efficiency of algorithms. Since we used the flocking estimate (Theorem 3.1) for the error estimate (Theorem 3.2), the expectation of the positions and velocities need to be analyzed for the stochastic RBM systems. We leave this case in a future work as well.

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