# UNIFORM ERROR ESTIMATES FOR THE RANDOM BATCH METHOD TO THE FIRST-ORDER CONSENSUS MODELS WITH ANTI-SYMMETRIC INTERACTION KERNELS

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Abstract. We propose a random batch method (RBM) for a contractive interacting particle system on a network which can be formulated as a first-order consensus model with heterogeneous intrinsic forcing and convolution type consensus force. In [30], the RBM was proposed for general interacting particle systems with conservative external forces, with particle-number independent error estimate established under suitable regularity assumptions on the external force and interacting kernel. Unlike the interacting particle system in [30], our consensus model has two competing dynamics, namely "*dispersion*" (generated by heterogeneous intrinsic dynamics) and "*concentration*" (generated by consensus forcing). In a close-to-consensus regime, we present a uniform error estimate for a modified RBM in which random batch algorithm is also applied to the part of intrinsic dynamics, not only to the interaction terms. We prove that the error depends on the batch size $P$ and the time step $\tau$, uniformly in particle number and time, namely, the $L^2$-error is of $\mathcal{O}(\sqrt{\tau/P})$. Thus the computational cost per time step is $\mathcal{O}(NP)$, where $N$ is the number of particles and one typically chooses $P \ll N$, while the direct summation would cost $\mathcal{O}(N^2)$. Our analytical error estimate is further verified by numerical simulations.

## 1. Introduction

Collective behaviors of interacting particle (or multi-agent) systems appear ubiquitously in nature, e.g., aggregation of bacteria [33, 45, 46], flocking of birds [13], swarming of fish [44], synchronization of fireflies and pacemaker cells [8, 41], etc. (see [1, 2, 4, 5, 12, 16, 20, 42, 43, 47, 48] for brief surveys and related papers). Recently, due to possible engineering applications to the control of robots, driverless cars and drones, collective dynamics models have also received lots of attention from control theory communities. Such models often take the form of interacting particle systems with all-to-all interactions. Thus, the computational complexity in the numerical integration is proportional to square of the system size per time step, which is very expensive to compute in real applications. From this reason, we look for a low fidelity model with less complexity by taking only parts of interactions, not the whole ones. This is the main topic to be addressed in this paper.

*Date*: August 21, 2020.

2010 *Mathematics Subject Classification.* 37M99, 37N30, 65P99.

*Key words and phrases.* consensus, interacting particle system, random batch.

**Acknowledgment.** The work of S.-Y. Ha was supported by National Research Foundation of Korea (NRF-2020R1A2C3A01003881), and the work of S. Jin was supported by NSFC grant No. 11871297, and the work of D. Kim was supported by a KIAS Individual Grant (MG073901) at Korea Institute for Advanced Study.

**Data availability statement.** The data that supports the findings of this study are available in the supplementary material of this article.

Consider an ensemble of interacting particles with heterogeneous intrinsic dynamics and interacting force. To motivate the idea, we begin with a first-order consensus model. Let $q = q(\cdot) \in \mathbb{R}^d$ be a quantifiable observable to seek a consensus, whose meaning may depend on the context under consideration. In what follows, we assume that the temporal evolution of the state vector $Q = (q_1, \cdots, q_N) \in \mathbb{R}^{Nd}$ is governed by the Cauchy problem to the following first-order consensus model:

$$\begin{cases} \dfrac{dq_i}{dt} = \nu_i + \dfrac{\kappa}{N-1} \displaystyle\sum_{j \neq i} a_{ij} \Gamma(q_j - q_i), \quad t > 0, \\ q_i(0) = q_i^{in}, \quad i = 1, \cdots, N, \end{cases} \tag{1.1}$$

where $\kappa$ is the nonnegative coupling strength and $\nu_i$ is the intrinsic velocity of the $i$-th agent of which we may assume, without loss of generality, that the total sum is zero (see Section 3.1):

$$\sum_{i=1}^N \nu_i = 0,$$

and the adjacency matrix $(a_{ij})_{i,j=1}^N$ represents the network structure for interactions between agents satisfying symmetry and nonnegative conditions:

$$a_{ij} = a_{ji} \geq 0, \quad 1 \leq i, j \leq N. \tag{1.2}$$

Here $\Gamma$ is the interaction kernel which is assumed to be a function of relative states and satisfies the following properties: there exists $r_0 > 0$ such that

$$\Gamma \in \mathrm{Lip}(B_{r_0}(0)), \quad \Gamma(-q) = -\Gamma(q), \qquad \forall\ q \in \overline{B_{r_0}(0)}. \tag{1.3}$$

Here we denote by $B_r(x)$ the open ball centered at $x$ with radius $r$.

Note that for the interaction kernel, relative state dependence and its anti-symmetric property $(1.3)_2$ are natural for physics-based consensus models with translation invariance, e.g., particle Keller-Segel model, Kuramoto model, one-dimensional Cucker-Smale model, etc. However, anti-symmetry property in (1.3) may break down for some nonlinear consensus models with anti-symmetric social interactions [23, 32, 38] and geometric state constraints [7, 10, 14, 26, 27, 37]. We observe that the first term on the R.H.S. of (1.1) induces the *"dispersion effect"* due to the heterogeneity of $\nu_i$, and its corresponding one-body potential $V = V(Q)$ is given by a linear potential:

$$V(Q) = -\sum_{i=1}^N \nu_i q_i.$$

This potential $V$ is not strongly convex, which is one of the difficulties in the error analysis. Thus, the error estimates in [30, 31] cannot be applied for the system (1.1) directly. In contrast, the second term in the R.H.S. of (1.1) generates *"concentration effect"* and it is modeled by the convolution type consensus forces. Therefore, the overall dynamics of (1.1) is determined by the competition between dispersion and concentration. This issue will be elaborated in Section 3.1. Now, we turn to computational complexity issue of (1.1). For all-to-all interactions (for example $a_{ij} = 1$), computational complexity of the interaction terms in (1.1) is of $\mathcal{O}(N^2)$ per time step. Thus, for $N \gg 1$, it becomes unaffordable to compute the state $Q$. Thus, designing a good approximate algorithm with low computational complexity is an important and challenging problem in the era of big data. Recently, as a designing principle of such low fidelity models associated with (1.1), the random batch method (RBM) was proposed in [30] and uniform error analysis was done under suitable assumptions on external force and interaction kernel. For the mean-field limit of the Cucker-Smale model, a similar binary random algorithm was introduced in [3], following the approach of direct simulation Monte-Carlo method.

The RBM is a fast algorithm to approximate the time-evolution of a large interacting particle system. To reduce the complexity, instead of computing all the interactions, the RBM approximates the given system as decoupled subsystems at each time step, which are batches consisting of (not more than) $P$ ($P \ll N$) individuals. Hence, each agent only interacts with agents in the same batch. Then, the number of interactions we consider at each time instant reduces to the order of $\mathcal{O}(NP)$. Of course, the choice of batches is random and is only used for a small duration of time, in order to average the random effect in the time course. After one time step, one randomly rearranges the new batch combination. Therefore, the RBM–approximation becomes a (randomly) switching networked system along time (see [15] for the Cucker-Smale flocking model).

Before we discuss our main result on the uniform error estimate of RBM, we begin with a brief discussion on the original RBM–approximation for (1.1) following the work [30]. Let $\tau$ be a positive small time step, and we decompose the infinite time-horizon $[0, \infty)$ as a union of disjoint finite-time intervals: For $\tau_m := m\tau$ for $m = 0, 1, 2, \dots,$

$$[0, \infty) = \bigcup_{m=0}^{\infty} [\tau_m, \tau_{m+1}).$$

At $(m+1)$-th time interval $[\tau_m, \tau_{m+1})$, we choose a partition of $\{1, 2, \dots, N\}$ randomly, (its precise meaning will be clarified in the proof of Lemma 6.2) $\mathcal{C} = \{C_1^m, \dots, C_n^m\}$ of $n = \lceil \frac{N}{P} \rceil$ batches with sizes at most $P > 1$, in the following way:

$$\{1, \dots, N\} = C_1^m \cup C_2^m \cup \cdots \cup C_n^m, \quad |C_i^m| = P, \quad i = 1, \cdots, n-1, \quad |C_n^m| \le P.$$

In each time interval $[\tau_m, \tau_{m+1})$, we set $[i]_m \in \mathcal{C}$ to be the batch containing $i$. Then, it would be natural to consider an RBM-approximation $Q^R = (q_1^R, \cdots, q_N^R)$ given by the Cauchy problem as the following low-fidelity model:

$$\begin{cases} \dfrac{dq_i^R}{dt} = \nu_i + \dfrac{\kappa}{P-1} \displaystyle\sum_{j \in [i]_m} a_{ij} \Gamma(q_j^R - q_i^R), \quad t \in [\tau_m, \tau_{m+1}), \\ q_i^R(0) = q_i^{in}, \quad i = 1, \dots, N, \ m = 0, 1, 2, \dots. \end{cases} \tag{1.4}$$

Note that the RBM is applied only for the interacting part without changing the part of the intrinsic dynamics. Then, the particle trajectories for (1.4) can be drastically different from the original trajectory as can be seen explicitly in the Kuramoto model (see Section 2.2). Therefore, the uniform bound of $\|Q^R - Q\|$ is not true in general for the original RBM to approximate (1.1). Heuristically, this can be explained as follows. In the strong coupling regime with $\kappa \gg \max_{i,j} |\nu_i - \nu_j|$, the relative states $q_i - q_j$ for (1.1) can be uniformly bounded, i.e., concentration is dominant compared to the dispersion. However, if we take only part of interactions as in the original RBM-approximation (1.4), the effect of concentration will be mitigated so that the dispersion effect can be more dominant in the worst-case scenario. This is why the relative approximate state $q_i^R - q_j^R$ can be unbounded even if the original relative state $q_i - q_j$ is uniformly bounded (see Section 2.2 for explicit example). Thus to balance dispersion and interaction in the RBM, we also need to apply the RBM in the dispersion part as well. This is where the novelty of this work lies. Now, we need to look for an alternative RBM (approximate) system and a sufficient framework leading to the uniform boundedness of relative states. To do this, we introduce suitable decomposition of the dispersion term $\nu_i$ as a sum of $N$-dispersion terms $\bar{\nu}_{ij}$ (see Section 2.2):

$$\nu_i = \frac{\kappa}{N-1}\sum_{j=1}^{N}\bar{\nu}_{ij}, \quad \bar{\nu}_{ij} = -\bar{\nu}_{ji}, \quad i,j = 1,\dots,N. \tag{1.5}$$

Then, the original Cauchy problem (1.1) is equivalent to the following problem:

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\kappa}{N-1}\displaystyle\sum_{j\neq i}\Big(\bar{\nu}_{ij} + a_{ij}\Gamma(q_j - q_i)\Big), \quad t>0, \\ q_i(0) = q_i^{in}, \quad i = 1,\cdots,N. \end{cases} \tag{1.6}$$

Now, we apply the RBM to the above equivalent problem (1.6) to sample dispersions and interactions proportionally and obtain the modified RBM model for (1.1):

$$\begin{cases} \dfrac{dq_i^R}{dt} = \dfrac{\kappa}{P-1}\displaystyle\sum_{j\in[i]_m}\big(\bar{\nu}_{ij} + a_{ij}\Gamma(q_j^R - q_i^R)\big), \quad t\in(\tau_m,\tau_{m+1}), \\ q_i^R(0) = q_i^{in}, \quad i = 1,\dots,N, \ m = 0,1,2,\dots. \end{cases} \tag{1.7}$$

In this paper, we are interested in the following question:

> "How well the RBM system (1.7) can approximate the full system (1.1)? More precisely, is there a time-independent upper bound for the displacement $\|Q^R - Q\|$ in a suitable norm?"

For an external conservative force with a strongly convex potential, uniform error analysis between the full system and the RBM-approximated one was done in [30] using a strong convexity assumption for the external potential. In fact, the explicit dissipation structure in the coupling kernel $\Gamma$ was not utilized there. In the current work, we use this dissipation estimate resulting from the interaction kernel to suppress errors from the external force.

Below, we state our main result for the one-dimensional case. For the desired error analysis, we assume that the coupling kernel $\Gamma$ is strongly dissipative in the sense that

$$(\Gamma(\tilde{q}) - \Gamma(q))\cdot(\tilde{q} - q) \approx |\tilde{q} - q|^2, \quad \forall\ q,\tilde{q}\in[-r_0,r_0],$$

and we also assume that the full system (1.1) has an equilibrium $\Phi = (\phi_1,\cdots,\phi_N) \in (-r_0,r_0)^N$ and the initial data is sufficiently close to $\Phi$ (see Section 3 for a detailed discussion). Our main result is the following uniform error estimate, under the condition that the underlying network topology is connected strongly enough (see Theorem 5.1):

$$\sup_{0\le t<\infty}\Big[\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|q_i^R(t) - q_i(t)|^2\Big] \lesssim \Big[\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + \tau^2\Big].$$

For the multi-dimensional setting with $q\in\mathbb{R}^d$, the same error analysis can be obtained under one more extra assumption $(\mathcal{A}_0)$, which guarantees that the state $Q$ and $Q^R$ are confined in the ball $B_{r_0}(0)$ (see Remark 5.1 (4) and Section 6.3).

The rest of this paper is organized as follows. In Section 2, we briefly discuss how the nonlinear consensus model (1.1) can be derived from the well-studied three collective models, and present an example for the unbounded RBM trajectories to the Kuramoto model with $N = 4$. In Section 3, we introduce a modified RBM approximate system generated by sampling in the natural velocity part and interaction parts, and then discuss how the boundedness of RMB trajectories for the example introduced in Section 2 can be guaranteed. In Section 4, we study a sufficient framework for (1.1) leading to the unique existence of relative equilibrium. In Section 5, we present our main result on the uniform boundedness of error estimate between the full system and the RBM approximate system, and then briefly compare our result with the previous result in [30]. In Section 6, we present a proof of Theorem 5.1. In Section 7, we apply the error estimate (Theorem 5.1) to the prescribed three consensus models. In Section 8, we present several numerical simulations to check the order of error along time-evolution. Finally, Section 9 is devoted to a brief summary of our results and some remaining questions related to the RBM-approximation for future work. In Appendix A and Appendix B, we provide proofs for Lemma 6.2 and Lemma 6.4.

## 2. Preliminaries

In this section, we present how the first-order consensus model (1.1) can be derived from three prototype models for collective behaviors and then we present an example in which relative RBM trajectories to (1.4) can be unbounded via the explicit Kuramoto model. This possible unboundedness of relative states illustrates that the naive RBM-approximation (1.4) for (1.1) does not yield a uniform error estimate.

2.1. **Universality of collective behaviors.** Let $q_i = q_i(t) \in \mathcal{M}$ be a quantifiable observable of the $i$-th agent at time $t$ whose dynamics is governed by the first-order consensus model (1.1). In what follows, we consider three explicit examples in which the first-order consensus model with all-to-all couplings appears as a governing system. In each case, we can see how the generalized position $q_i$ and state manifold $\mathcal{M}$ can be interpreted.

• (Particle Keller-Segel model): Let $x_i = x_i(t) \in \mathbb{R}^d$ be the position process of the $i$-th Keller-Segel particle [33]. Then, its dynamics is governed by the system of stochastic ordinary differential equations:

$$dx_i = -\frac{\kappa}{N-1}\sum_{j\neq i}\nabla\varphi(x_j - x_i)dt + \sigma dB_i, \tag{2.1}$$

where $-\nabla_{x_i}\varphi$ and $B_i(t)$ are the self-consistent aggregation force and the standard $d$-dimensional Brownian motion, respectively. In the absence of stochastic noise, i.e., $\sigma = 0$, the system (2.1) becomes the first-order nonlinear consensus model:

$$\frac{dx_i}{dt} = -\frac{\kappa}{N-1}\sum_{j\neq i}\nabla\varphi(x_j - x_i). \tag{2.2}$$

Note that system (2.2) corresponds to the first-order consensus model (1.1) via the following obvious correspondence:

$$q_i \quad \longleftrightarrow \quad x_i, \quad \Gamma(q) \quad \longleftrightarrow \quad -\nabla\varphi(q).$$

Here $\Gamma$ is singular at the origin [33]. Hence, although the K-S model (2.2) is a consensus model, it does not satisfy our regularity assumption (1.3).

• (1D Cucker-Smale model): Let $x_i$ and $v_i$ be the position and velocity of the $i$-th Cucker-Smale particle with unit mass on the real line [21]. Then, their dynamics is governed by the system of ordinary differential equations:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N-1} \displaystyle\sum_{j \neq i} \psi(x_j - x_i)(v_j - v_i), \end{cases} \tag{2.3}$$

where $\psi$ is a nonnegative communication weight function. Now, we introduce anti-derivative of $\psi$:

$$\Psi(x) = \int_0^x \psi(y)dy, \qquad \text{i..e,} \quad \Psi'(x) = \psi(x).$$

Note that for a nonnegative communication weight $\psi \geq 0$, $\Psi$ is monotonically increasing. Then, $(2.3)_2$ can be rewritten as

$$\frac{dv_i}{dt} = \frac{d}{dt}\Big(\frac{\kappa}{N-1}\sum_{j \neq i}\int_0^{x_j - x_i} \psi(y)dy\Big) = \frac{d}{dt}\Big(\frac{\kappa}{N-1}\sum_{j \neq i}\Psi(x_j - x_i)\Big),$$

or equivalently

$$\frac{d}{dt}\Big(v_i - \frac{\kappa}{N-1}\sum_{j \neq i}\Psi(x_j - x_i)\Big) = 0. \tag{2.4}$$

By integrating the above relation (2.4), we have

$$v_i(t) = v_i^{in} - \frac{\kappa}{N-1}\sum_{j \neq i}\Psi(x_j^{in} - x_i^{in}) + \frac{\kappa}{N-1}\sum_{j \neq i}\Psi(x_j(t) - x_i(t)). \tag{2.5}$$

If we set

$$\nu_i := v_i^{in} - \frac{\kappa}{N-1}\sum_{j \neq i}\Psi(x_j^{in} - x_i^{in}),$$

then, from $(2.3)_1$ and (2.5), we get

$$\frac{dx_i}{dt} = \nu_i + \frac{\kappa}{N-1}\sum_{j \neq i}\Psi(x_j - x_i). \tag{2.6}$$

Again, system (2.6) falls down to the first-order consensus model (1.1) via the following obvious correspondence:

$$q_i \quad \longleftrightarrow \quad x_i, \quad \Gamma(q) \quad \longleftrightarrow \quad \int_0^q \psi(r)dr.$$

Note that $\Gamma$ is an increasing function. For the Cucker-Smale model and its variant, we refer to [13, 25, 28, 39].

• (The Kuramoto model): Let $\theta_i$ be the phase of the $i$-th Kuramoto oscillator. Then its dynamics is governed by the following system of first-order equations [1, 22, 35]:

$$\frac{d\theta_i}{dt} = \nu_i + \frac{\kappa}{N-1}\sum_{j \neq i}\sin(\theta_j - \theta_i), \tag{2.7}$$

where $\nu_i$ is the natural frequency of the $i$-th oscillator. Clearly, system (2.7) exactly falls down to the consensus model (1.1) with the correspondence:

$$q_i \quad \longleftrightarrow \quad \theta_i, \quad \Gamma(q) \quad \longleftrightarrow \quad \sin(q).$$

Note that $\Gamma$ is a $2\pi$-periodic function.

2.2. **Example for unboundedness of RBM trajectories.** In the explicit examples of the consensus models in the previous subsection, one can observe that the interaction function $\Gamma$ depends on the relative states, $(q_j - q_i)$. In particular, the interaction of the Kuramoto model (2.7) changes dramatically if $q_j - q_i$ is bigger than $\pi$, since $\sin(q_j - q_i)$ changes its sign. This can be a critical problem when we consider the stability of the RBM-approximated system (1.4). More precisely, we briefly explain one key difficulty for the error analysis via the Kuramoto model with $N = 4$. For definiteness, we set

$$\nu_1 = \nu_2 = -\nu_3 = -\nu_4 = 1 \quad \text{and} \quad q_1^{in} = q_2^{in} = q_3^{in} = q_4^{in} = 0.$$

In this case, the Cauchy problem to the Kuramoto model reads as follows.

$$\begin{cases} \dfrac{dq_1}{dt} = 1 + \dfrac{\kappa}{3}\Big( \sin(q_2 - q_1) + \sin(q_3 - q_1) + \sin(q_4 - q_1) \Big), \\ \dfrac{dq_2}{dt} = 1 + \dfrac{\kappa}{3}\Big( \sin(q_1 - q_2) + \sin(q_3 - q_2) + \sin(q_4 - q_2) \Big), \\ \dfrac{dq_3}{dt} = -1 + \dfrac{\kappa}{3}\Big( \sin(q_1 - q_3) + \sin(q_2 - q_3) + \sin(q_4 - q_3) \Big), \\ \dfrac{dq_4}{dt} = -1 + \dfrac{\kappa}{3}\Big( \sin(q_1 - q_4) + \sin(q_2 - q_4) + \sin(q_3 - q_4) \Big), \\ (q_1, q_2, q_3, q_4)(0) = (0, 0, 0, 0). \end{cases} \tag{2.8}$$

Then, it is easy to see that

$$q_1(t) = q_2(t), \quad q_3(t) = q_4(t), \quad t \geq 0,$$

and $q_1$ and $q_3$ satisfy the Kuramoto model for two oscillators:

$$\begin{cases} \dfrac{dq_1}{dt} = 1 + \dfrac{2\kappa}{3} \sin(q_3 - q_1), \\ \dfrac{dq_3}{dt} = -1 + \dfrac{2\kappa}{3} \sin(q_1 - q_3), \\ (q_1, q_3)(0) = (0, 0). \end{cases}$$

If follows from [11] that if $\kappa > 3$, then the system converges to the equilibrium $Q^e = (q_1^e, q_2^e, q_3^e, q_4^e)$ exponentially fast:

$$\lim_{t \to \infty} Q(t) = Q^e. \tag{2.9}$$

Now we return to the RBM-approximated system (1.4) of (2.8) with $P = 2$. In this case, one of partitions for the whole ensemble could be

$$\{1, 2, 3, 4\} = \{1, 2\} \cup \{3, 4\}.$$

As the worst case scenario, we choose this partition in all time intervals $[\tau_m, \tau_{m+1})$. Then, the RBM-approximated system (1.4) for (2.8) becomes

$$\begin{cases} \dfrac{dq_1^R}{dt} = 1 + \kappa \sin(q_2^R - q_1^R), \quad \dfrac{dq_2^R}{dt} = 1 + \kappa \sin(q_1^R - q_2^R), \quad t > 0, \\ \dfrac{dq_3^R}{dt} = -1 + \kappa \sin(q_4^R - q_3^R), \quad \dfrac{dq_4^R}{dt} = -1 + \kappa \sin(q_3^R - q_4^R), \\ (q_1, q_2, q_3, q_4)(0) = (0, 0, 0, 0). \end{cases}$$

Direct calculation yields

$$q_1^R(t) = q_2^R(t) = t \quad \text{and} \quad q_3^R(t) = q_4^R(t) = -t, \quad t \geq 0, \tag{2.10}$$

which is clearly unbounded. This RBM solution (2.10) is quite different from the full trajectory (2.9) even for $\kappa > 3$. Moreover, $Q^R$ is independent of the parameters $\kappa$. This kind of unboundedness illustrates the breakdown of the naive RBM-approximation to (2.8), although numerically the probability of this is extremely low. To guarantee the uniform boundedness of the RBM trajectories (which will be presented in Lemma 6.1), in next section we propose an alternative version of the RBM-approximation to (1.1) in a large coupling regime $\kappa \gg 1$ in which a unique existence of relative equilibrium is guaranteed in a region near $q = 0$.

## 3. The modified RBM approximate system

In this section, we propose an alternative modified RBM-approximation for (1.1) which was briefly outlined in Introduction.

### 3.1. **Decomposition of natural velocities.** Note that since $\Gamma$ is anti-symmetric and locally bounded near zero, one has

$$\Gamma(0) = 0.$$

and system (1.1) can be rewritten as

$$\begin{cases} \dfrac{dq_i}{dt} = \nu_i + \dfrac{\kappa}{N-1}\displaystyle\sum_{j=1}^{N} a_{ij}\Gamma(q_j - q_i), \quad t > 0, \\ q_i(0) = q_i^{in}, \quad i = 1, \cdots, N. \end{cases} \tag{3.1}$$

**Lemma 3.1.** *Let $\{q_i\}$ be a solution to (3.1). Then, the following assertions hold.*

*(1) (Translation-invariance): system (3.1) is invariant under the translation $\tilde{q}_i := q_i + \alpha, \quad \alpha \in \mathbb{R}$:*

$$\frac{d\tilde{q}_i}{dt} = \nu_i + \frac{\kappa}{N-1}\sum_{j=1}^{N} a_{ij}\Gamma(\tilde{q}_j - \tilde{q}_i).$$

*(2) (Conservation law): the quantity $S(t) := \sum_{i=1}^N q_i(t) - t\sum_{i=1}^N \nu_i$ is conserved along the dynamics (3.1):*

$$S(t) = S(0), \quad t > 0.$$

*Proof.* (i) The first assertion follows from the fact that the interaction $\Gamma$ is convolution type, i.e., it is expressed in terms of relative positions $q_j - q_i$.

(ii) We use the symmetry property of $a_{ij}$ and $\Gamma(-q) = -\Gamma(q)$ to see

$$\frac{\kappa}{N-1}\sum_{i,j} a_{ij}\Gamma(q_j - q_i) = -\frac{\kappa}{N-1}\sum_{i,j} a_{ji}\Gamma(q_j - q_i) = -\frac{\kappa}{N-1}\sum_{i,j} a_{ij}\Gamma(q_j - q_i).$$

Thus, one has

$$\frac{\kappa}{N-1}\sum_{i,j} a_{ij}\Gamma(q_j - q_i) = 0.$$

Now we use the above relation and (3.1) to obtain

$$\frac{d}{dt}\sum_{i=1}^N q_i = \sum_{i=1}^N \nu_i + \frac{\kappa}{N-1}\sum_{i,j} a_{ij}\Gamma(q_j - q_i) = \sum_{i=1}^N \nu_i.$$

This yields the desired estimate. $\square$

**Remark 3.1.** *Below, we comment on ramifications of Lemma 3.1.*

*1. Consider a frame moving with the average velocity* $\frac{1}{N}\sum_j \nu_j$. *We set*

$$\tilde{q}_i := q_i - \frac{t}{N}\sum_j \nu_j, \quad \tilde{\nu}_j := \nu_j - \frac{1}{N}\sum_j \nu_j. \tag{3.2}$$

*Then, it is easy to see that* $(\tilde{q}_j, \tilde{\nu}_j)$ *satisfies the same system:*

$$\frac{d\tilde{q}_i}{dt} = \tilde{\nu}_i + \frac{\kappa}{N-1}\sum_{j=1}^N a_{ij}\Gamma(\tilde{q}_j - \tilde{q}_i),$$

*with the constraint:*

$$\sum_j \tilde{\nu}_i = 0.$$

*Thus, imposing the zero sum condition* $\sum_j \nu_j$ *can be made without loss of generality. In what follows, the solution* $Q = (q_1, \cdots, q_N)$ *to*

$$\nu_i + \frac{\kappa}{N-1}\sum_{j=1}^N a_{ij}\Gamma(q_j - q_i), \quad i = 1, \cdots, N, \qquad \sum_i \nu_i = 0 \tag{3.3}$$

*will be called a relative equilibrium.*

*2. For the case in which* $\Gamma$ *is bounded on the state space, say*

$$|\Gamma(q)| \le \Gamma^\infty,$$

*the interaction part in the R.H.S. of* (3.1) *is uniformly bounded:*

$$\Big|\frac{\kappa}{N-1}\sum_{j\neq i} a_{ij}\Gamma(q_j - q_i)\Big| \le \kappa\Gamma^\infty \max_{i,j}|a_{ij}|.$$

*Hence, if the coupling strength* $\kappa$ *is sufficiently small such that*

$$|\nu_i| > \kappa\Gamma^\infty \max_{i,j}|a_{ij}|,$$

*then the R.H.S. of* (3.1) *cannot be zero. Thus, system* (3.1) *cannot have a relative equilibrium (or an equilibrium in moving frame* (3.2)*). Therefore, relative equilibriums can exist only in a large coupling strength regime.*

Now, we are ready to derive the precise formulae for $\bar{\nu}_{ij}$ in (1.5) based on the relative equilibrium (3.3). Let $\Phi = (\phi_1, \cdots \phi_N)$ be the unique relative equilibria for (1.1) in the ball $(B_{r_0}(0))$ appearing in (1.3). Then, from (3.3), one can see that natural velocity $\nu_i$ can be expressed as a sum of some quantities depending on the relative equilibrium $\Phi$:

$$\nu_i = -\frac{\kappa}{N-1}\sum_{j=1}^N a_{ij}\Gamma(\phi_j - \phi_i) = \frac{\kappa}{N-1}\sum_{j=1}^N\Big[-a_{ij}\Gamma(\phi_j - \phi_i)\Big].$$

Hence, if we set

$$\bar{\nu}_{ij} := \bar{\nu}_{ij}(\phi_1, \cdots, \phi_N) := -a_{ij}\Gamma(\phi_j - \phi_i), \quad i, j = 1, \cdots, N, \tag{3.4}$$

then it is easy to check that $\bar{\nu}_{ij}$ satisfies relation (1.5). Finally, system (1.1) and its effective RBM approximate system can be rewritten as

$$
\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\kappa}{N-1} \displaystyle\sum_{j=1}^{N} \Big( \bar{\nu}_{ij} + a_{ij}\Gamma(q_j - q_i) \Big), \quad t > 0, \\ \dfrac{dq_i^R}{dt} = \dfrac{\kappa}{P-1} \displaystyle\sum_{j \in [i]_m} \Big( \bar{\nu}_{ij} + a_{ij}\Gamma(q_j^R - q_i^R) \Big), \quad t \in [\tau_m, \tau_{m+1}), \quad m = 0, 1, 2, \cdots . \end{cases} \tag{3.5}
$$

3.2. **Effectiveness of the modified RBM approximate system.** In this subsection, we explain how the modified RBM approximate system can recover the uniform boundedness of relative state, unlike the naive RBM approximate system (1.4), via the same example introduced in Section 2.2.

Note that the RBM approximate model $(3.5)_2$ reads as

$$
\begin{cases} \dfrac{dq_i^R}{dt} = \kappa \displaystyle\sum_{j \in [i]_m} \Big( \sin(q_j^R - q_i^R) - \sin(\phi_j - \phi_i) \Big), \quad t \in [\tau_m, \tau_{m+1}), \quad m = 0, 1, 2, \cdots , \\ q_i(0) = 0, \quad i = 1, 2, 3, 4, \end{cases}
$$

where $(\phi_1, \phi_2, \phi_3, \phi_4)$ satisfies

$$
\begin{cases} 0 = 1 + \dfrac{\kappa}{3} \Big( \sin(\phi_2 - \phi_1) + \sin(\phi_3 - \phi_1) + \sin(\phi_4 - \phi_1) \Big), \\ 0 = 1 + \dfrac{\kappa}{3} \Big( \sin(\phi_1 - \phi_2) + \sin(\phi_3 - \phi_2) + \sin(\phi_4 - \phi_2) \Big), \\ 0 = -1 + \dfrac{\kappa}{3} \Big( \sin(\phi_1 - \phi_3) + \sin(\phi_2 - \phi_3) + \sin(\phi_4 - \phi_3) \Big), \\ 0 = -1 + \dfrac{\kappa}{3} \Big( \sin(\phi_1 - \phi_4) + \sin(\phi_2 - \phi_4) + \sin(\phi_3 - \phi_4) \Big). \end{cases} \tag{3.6}
$$

If $\kappa$ is large enough, one solution is

$$
(\phi_1, \phi_2, \phi_3, \phi_4) = \left( \frac{1}{2} \arcsin \frac{3}{2\kappa}, \frac{1}{2} \arcsin \frac{3}{2\kappa}, -\frac{1}{2} \arcsin \frac{3}{2\kappa}, -\frac{1}{2} \arcsin \frac{3}{2\kappa} \right),
$$

which clearly satisfies (3.6). In this case, for any fixed interval $[\tau_m, \tau_{m+1})$, the modified RBM dynamics for $Q^R$ is given as follows: if $[1]_m = [2]_m \neq [3]_m = [4]_m$, then

$$
\begin{cases} \dfrac{dq_1^R}{dt} = \kappa \sin(q_2^R - q_1^R), \quad \dfrac{dq_2^R}{dt} = \kappa \sin(q_1^R - q_2^R), \\ \dfrac{dq_3^R}{dt} = \kappa \sin(q_4^R - q_3^R), \quad \dfrac{dq_4^R}{dt} = \kappa \sin(q_3^R - q_4^R), \quad t \in [\tau_m, \tau_{m+1}). \end{cases}
$$

if $[1]_m = [3]_m \neq [2]_m = [4]_m$, then

$$
\begin{cases} \dfrac{dq_1^R}{dt} = \kappa \Big( \sin(q_3^R - q_1^R) + \dfrac{3}{2\kappa} \Big), \quad \dfrac{dq_2^R}{dt} = \kappa \Big( \sin(q_4^R - q_2^R) + \dfrac{3}{2\kappa} \Big), \\ \dfrac{dq_3^R}{dt} = \kappa \Big( \sin(q_1^R - q_3^R) - \dfrac{3}{2\kappa} \Big), \quad \dfrac{dq_4^R}{dt} = \kappa \Big( \sin(q_2^R - q_4^R) - \dfrac{3}{2\kappa} \Big), \quad t \in [\tau_m, \tau_{m+1}), \end{cases}
$$

if $[1]_m = [4]_m \neq [2]_m = [3]_m$, then

$$
\begin{cases} \dfrac{dq_1^R}{dt} = \kappa \Big( \sin(q_4^R - q_1^R) + \dfrac{3}{2\kappa} \Big), \quad \dfrac{dq_2^R}{dt} = \kappa \Big( \sin(q_3^R - q_2^R) + \dfrac{3}{2\kappa} \Big), \\ \dfrac{dq_3^R}{dt} = \kappa \Big( \sin(q_2^R - q_3^R) - \dfrac{3}{2\kappa} \Big), \quad \dfrac{dq_4^R}{dt} = \kappa \Big( \sin(q_1^R - q_4^R) - \dfrac{3}{2\kappa} \Big), \quad t \in [\tau_m, \tau_{m+1}). \end{cases}
$$

From the uniqueness of the solution, we have

$$(q_1^R, q_2^R, q_3^R, q_4^R) = (x^R, x^R, -x^R, -x^R),$$

where $x^R$ is the solution to the following Cauchy problem:

$$\begin{cases} \dfrac{dx^R}{dt} = \begin{cases} 0 & \text{if} \quad [1]_m = [2]_m \neq [3]_m = [4]_m, \\ -\kappa \sin(2x^R) + \frac{3}{2}, & \text{otherwise}, \end{cases} & t \in [\tau_m, \tau_{m+1}), \\ x^R(0) = 0. \end{cases}$$

Since

$$0 \le x^R(t) < \frac{1}{2} \arcsin \frac{3}{2\kappa} \le \frac{\pi}{4} \quad \text{for} \quad 0 \le t < \infty,$$

the trajectory of $Q^R$ is bounded. We will prove boundedness of $Q^R$ for general $\Gamma$ in Lemma 6.1.

As can be seen in (3.4) and (3.5), the formulation of the modified RBM approximate system is crucially dependent on the existence of a unique relative equilibrium. Thus, in the following section, we study a sufficient framework which guarantees the unique existence of relative equilibrium for (1.1) in the region $B_{r_0}(0)$.

## 4. Existence of a unique relative equilibrium

In this section, we provide an existence of an equilibrium for (1.1) using the contraction mapping principle.

Recall that the relative equibrium $\Phi = (\phi_1, \cdots, \phi_N) \in \mathbb{R}^N$ satisfies

$$\nu_i + \frac{\kappa}{N-1} \sum_{k=1}^{N} a_{ik} \Gamma(\phi_k - \phi_i) = 0, \quad i = 1, \cdots, N, \quad \sum_{j=1}^{N} \nu_j = 0. \tag{4.1}$$

In next proposition, we present sufficient conditions for the unique solvability of the equation $F = 0$.

**Proposition 4.1.** *Let $r > 0$ and $M$ be a linear subspace of $\mathbb{R}^N$. Suppose that $F : M \cap \overline{B_r(0)} \to M$ is a Lipschitz continuous function satisfying the dissipativie conditions:*

$$\langle \mathbf{x}, F(\mathbf{x}) \rangle \le a\|\mathbf{x}\| - b\|\mathbf{x}\|^2, \quad \langle \mathbf{x}-\mathbf{y}, F(\mathbf{x}) - F(\mathbf{y}) \rangle \le -c\|\mathbf{x}-\mathbf{y}\|^2, \quad \forall\, \mathbf{x}, \mathbf{y} \in M \cap \overline{B_r(0)} \tag{4.2}$$

*for some $a, b, c > 0$ with $a < br$. Then, equation $F(\mathbf{x}) = 0$ has a unique root in $M \cap \overline{B_r(0)}$.*

*Proof.* Suppose that $F$ satisfies the dissipative conditions (4.2). We set

$$F_M := \sup_{\mathbf{x} \in M \cap \overline{B(r)}} \|F(\mathbf{x})\|, \quad L := \sup_{\substack{\mathbf{x}, \mathbf{y} \in M \cap \overline{B_r(0)} \\ \mathbf{x} \neq \mathbf{y}}} \frac{\|F(\mathbf{x}) - F(\mathbf{y})\|}{\|\mathbf{x} - \mathbf{y}\|}.$$

Define a parametrized family of functions $f^\varepsilon : M \cap \overline{B_r(0)} \to M$ ($\varepsilon > 0$) by

$$f^\varepsilon(\mathbf{x}) := \mathbf{x} + \varepsilon F(\mathbf{x}).$$

Then, it is easy to see that the fixed point $\mathbf{x}$ for $f^\varepsilon$ is exactly the root of $F$:

$$f^\varepsilon(\mathbf{x}) = \mathbf{x} \quad \Longleftrightarrow \quad F(\mathbf{x}) = 0.$$

For this, we claim: for a sufficiently small $\varepsilon > 0$,

$$f^\varepsilon \text{ is a contraction mapping on } M \cap \overline{B_r(0)}. \tag{4.3}$$

This can be done in two steps below.

• Step A: We check

$$f^\varepsilon(M \cap \overline{B_r(0)}) \subseteq M \cap \overline{B_r(0)} \quad \text{for } 0 < \varepsilon \ll 1.$$

For any $\varepsilon > 0$, we use (4.2) to get

$$\begin{aligned} \|f^\varepsilon(\mathbf{x})\|^2 &= \|\mathbf{x}\|^2 + 2\varepsilon\langle \mathbf{x}, F(\mathbf{x})\rangle + \varepsilon^2 \|F(\mathbf{x})\|^2 \\ &\le (1-2b\varepsilon)\|\mathbf{x}\|^2 + 2a\varepsilon\|\mathbf{x}\| + \varepsilon^2 F_M^2 =: g(\|\mathbf{x}\|), \quad \forall\, \mathbf{x} \in M \cap \overline{B_r(0)}. \end{aligned} \tag{4.4}$$

If $0 < \varepsilon < \frac{1}{2b}$, then $g$ is convex. Thus, the convexity of $g$ and (4.4) yield

$$\|f^\varepsilon(\mathbf{x})\|^2 \le \max_{0 \le s \le r} g(s) = \max\{g(0), g(r)\} = \max\left\{\varepsilon^2 F_M^2, r^2 - 2\varepsilon r(br - a) + \varepsilon^2 F_M^2\right\}.$$

For a sufficiently small $\varepsilon > 0$, the right-hand side is smaller than $r^2$.

• Step B: We check

$$f^\varepsilon \text{ is a contraction for } 0 < \varepsilon \ll 1.$$

For $\mathbf{x}, \mathbf{y} \in M \cap \overline{B_r(0)}$, we can get

$$\begin{aligned} \|f^\varepsilon(\mathbf{x}) - f^\varepsilon(\mathbf{y})\|^2 &= \|\mathbf{x} - \mathbf{y}\|^2 + 2\varepsilon\langle \mathbf{x} - \mathbf{y}, F(\mathbf{x}) - F(\mathbf{y})\rangle + \varepsilon^2\|F(\mathbf{x}) - F(\mathbf{y})\|^2 \\ &\le (1 - \varepsilon(2c - \varepsilon L^2))\|\mathbf{x} - \mathbf{y}\|^2. \end{aligned}$$

For a sufficiently small $\varepsilon > 0$, $(1 - \varepsilon(2c - \varepsilon L^2))$ is less than 1. Therefore, the claim (4.3) is verified. The lemma follows by the contraction mapping theorem since $f^\varepsilon$ has a fixed point in $M \cap \overline{B_r(0)}$. □

Next, we return to (4.1) and discuss a sufficient framework $(\mathcal{A})$ for the unique solvability of (4.1) and uniform error analysis.

- $(\mathcal{A}_1)$ (Bi-Lipschitz property near the origin): There exists positive constants $r_0$, $L_1$ and $L_2$ such that

  $$L_1|\tilde{q} - q|^2 \le (\Gamma(\tilde{q}) - \Gamma(q)) \cdot (\tilde{q} - q) \le L_2|\tilde{q} - q|^2, \quad \forall\, q, \tilde{q} \in \overline{B_{r_0}(0)}.$$

- $(\mathcal{A}_2)$ (Algebraic connectivity): Let $L = (\ell_{ij})$ be the *Laplacian matrix* defined by

  $$\ell_{ij} = \begin{cases} \sum_{k \ne i} a_{ik} & \text{if} \quad i = j, \\ -a_{ij} & \text{if} \quad i \ne j. \end{cases}$$

  Denote by $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ the eigenvalues of $L$. The connectivity of $G = (a_{ij})$ is equivalent to the condition $\lambda_2 > 0$. We call $\lambda_2$ the *algebraic connectivity* of the graph $G$ with weights $a_{ij}$. We assume $\lambda_2$ is strictly positive:

  $$\lambda_2 = \min_{\mathbf{x} \ne 0,\ \mathbf{x}\cdot\mathbf{1} = 0} \frac{\mathbf{x}^\top L \mathbf{x}}{\|\mathbf{x}\|^2} = N \min_{\exists\ i,j:\ x_i \ne x_j} \frac{\sum_{i,j} a_{ij}(x_i - x_j)^2}{\sum_{i,j}(x_i - x_j)^2} > 0. \tag{4.5}$$

  For more details on the Laplacian matrices, see [17]. In addition, we assume that $0 \le a_{ij} \le C'$ for some constant $C' > 0$.

- $(\mathcal{A}_3)$ (Existence of unique relative equilibrium): there exists a state $\phi = (\phi_1, \cdots, \phi_N) \in \mathbb{R}^N$ such that

$$\nu_i + \frac{\kappa}{N-1}\sum_{j=1}^N a_{ij}\Gamma(\phi_j - \phi_i) = 0, \quad i = 1, \cdots, N, \quad \sum_j \nu_j = 0, \quad \max_{i,j}|\phi_i - \phi_j| < r_0.$$

As a direct application of Proposition 4.1, one has the unique existence of relative equilibrium for (1.1) on $\overline{B_{r_0}(0)}$.

**Corollary 4.1.** *Let the system parameters $a, b$ and $M$ be given by*

$$a = \|\nu\|, \quad b = \frac{\kappa L_1 \lambda_2}{N-1} \quad \textit{and} \quad M := (\mathrm{span}\{\mathbf{1}\})^\perp \subset \mathbb{R}^N.$$

*Suppose conditions $(\mathcal{A}_1)$–$(\mathcal{A}_2)$ hold, and*

$$\sum_{j=1}^N \nu_j = 0, \qquad \frac{a}{b} = \frac{(N-1)\|\nu\|}{\kappa L_1 \lambda_2} < \frac{r_0}{\sqrt{2}}.$$

*Then, relative equilibrium equations* (4.1) *have a unique solution $\phi = (\phi_1, \ldots, \phi_N) \in \mathbb{R}^N$ satisfying condition $(\mathcal{A}_3)$.*

*Proof.* We set

$$\mathbf{x} = (x_1, \cdots, x_N) \quad \text{and} \quad \mathbf{y} = (y_1, \cdots, y_N).$$

Let $r$ be any positive real number satisfying $\frac{a}{b} < r < \frac{r_0}{\sqrt{2}}$, and define $F = (F_1, \cdots, F_N) : M \cap \overline{B_r(0)} \to M$ as

$$F_i(\mathbf{x}) := \nu_i + \frac{\kappa}{N-1}\sum_{j=1}^N a_{ij}\Gamma(x_j - x_i), \quad i = 1, \cdots, N. \tag{4.6}$$

In the sequel, we check the conditions in (4.2) one by one, adopting the approach using the algebraic connectivity in [40].

⋄ (The first condition): We use (4.5) and (4.6) to get

$$\begin{aligned}
\langle \mathbf{x}, F(\mathbf{x})\rangle &= \sum_{i=1}^N \nu_i x_i + \frac{\kappa}{N-1}\sum_{i,j=1}^N a_{ij} x_i \Gamma(x_j - x_i) \\
&= \sum_{i=1}^N \nu_i x_i - \frac{\kappa}{2(N-1)}\sum_{i,j=1}^N a_{ij}(x_j - x_i)\Gamma(x_j - x_i) \\
&\le \sum_{i=1}^N \nu_i x_i - \frac{\kappa L_1}{2(N-1)}\sum_{i,j=1}^N a_{ij}(x_j - x_i)^2 \\
&= \nu^\top \mathbf{x} - \frac{\kappa L_1}{N-1}\mathbf{x}^\top L \mathbf{x} \\
&\le \|\nu\|\|\mathbf{x}\| - \frac{\kappa L_1 \lambda_2}{N-1}\|\mathbf{x}\|^2.
\end{aligned} \tag{4.7}$$

$\diamond$ (The second condition): It follows from (4.6) that

$$F_i(\mathbf{x}) - F_i(\mathbf{y}) = \frac{\kappa}{N-1}\sum_{j=1}^N a_{ij}\Big(\Gamma(x_j - x_i) - \Gamma(y_j - y_i)\Big).$$

Then, we have

$$\begin{aligned}
\langle F(\mathbf{x}) - F(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle &= \sum_{i=1}^N (F_i(\mathbf{x}) - F_i(\mathbf{y}))(x_i - y_i)\\
&= \frac{\kappa}{N-1}\sum_{i,j=1}^N a_{ij}\Big(\Gamma(x_j - x_i) - \Gamma(y_j - y_i)\Big)(x_i - y_i)\\
&= -\frac{\kappa}{N-1}\sum_{i,j=1}^N a_{ij}\Big(\Gamma(x_j - x_i) - \Gamma(y_j - y_i)\Big)(x_j - y_j) \quad \text{by } i \leftrightarrow j\\
&= \frac{\kappa}{2(N-1)}\sum_{i,j=1}^N a_{ij}\Big(\Gamma(x_j - x_i) - \Gamma(y_j - y_i)\Big)\Big((x_i - y_i) - (x_j - y_j)\Big)\\
&= -\frac{\kappa}{2(N-1)}\sum_{i,j=1}^N a_{ij}\Big(\Gamma(x_j - x_i) - \Gamma(y_j - y_i)\Big)\Big((x_j - x_i) - (y_j - y_i)\Big)\\
&\le -\frac{\kappa L_1}{2(N-1)}\sum_{i,j=1}^N a_{ij}\Big|(x_j - x_i) - (y_j - y_i)\Big|^2.
\end{aligned}$$

From (4.5) in the assumption $(\mathcal{A}_2)$, we have

$$-\sum_{i,j=1}^N a_{ij}\Big|(x_j - x_i) - (y_j - y_i)\Big|^2 \le -\frac{\lambda_2}{N}\sum_{i,j=1}^N \Big|(x_j - y_j) - (x_j - y_i)\Big|^2.$$

Hence, it yields

$$\begin{aligned}
\langle F(\mathbf{x}) - F(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle &\le -\frac{\kappa\lambda_2 L_1}{2N(N-1)}\sum_{i,j=1}^N \Big|(x_j - y_j) - (x_j - y_i)\Big|^2\\
&\le -\frac{\kappa\lambda_2 L_1}{(N-1)}\sum_{i=1}^N |x_i - y_i|^2 = -\frac{\kappa\lambda_2 L_1}{(N-1)}\|\mathbf{x}-\mathbf{y}\|^2.
\end{aligned} \tag{4.8}$$

If we set

$$a = \|\nu\|, \quad b = \frac{\kappa L_1 \lambda_2}{N-1} \quad \text{and} \quad c = \frac{\kappa\lambda_2 L_1}{(N-1)},$$

then it follows from (4.7) and (4.8) that conditions (4.2) holds. Thus, by Proposition 4.1, system (4.1) has a unique solution $\phi = (\phi_1, \ldots, \phi_N) \in \mathbb{R}^N$ in $M \cap \overline{B_r(0)}$, which is a subset of $M \cap B_{r_0/\sqrt{2}}(0)$. Finally, we have

$$\max_{i,j} |\phi_i - \phi_j| \le \max_{i\neq j}(|\phi_i| + |\phi_j|) \le \max_{i\neq j}\sqrt{2(|\phi_i|^2 + |\phi_j|^2)} \le \sqrt{2}\|\phi\| < r_0$$

so that the equilibrium satisfies condition $(\mathcal{A}_3)$. $\square$

**Remark 4.1.** *(1) From the second assertion in Corollary 4.1, the coefficients depend on $N$. However, this is not surprising according to (4.5): if $a_{ij} = 1$ for all $i$ and $j$ (the all-to-all network), then we get $\lambda_2 = N$ and the $N$-dependency is natural.*

*(2) Proposition 4.1 can be generalized to the multi-dimensional case, $\phi = (\phi_1, \dots, \phi_N) \in (\mathbb{R}^d)^N$, with the same argument. Instead of choosing $M = (span\{\mathbf{1}\})^\perp$, we set*

$$M = \left\{ \mathbf{x} = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N \;\middle|\; \sum_{i=1}^N x_i = 0 \right\}^\perp \quad \textit{with} \quad \|\mathbf{x}\| := \sqrt{\sum_{i=1}^N |x_i|^2}.$$

*Hence, the corresponding result of Corollary 4.1 also holds for the multi-dimensional case.*

Proposition 4.1 can be applied to a wide range of collective behavior models with the contracting property, including several examples presented in Subsection 2.1.

## 5. Description of the main result

In this section, we briefly discuss our main result on the uniform error analysis for the RBM-approximation and a brief comparison with earlier work [30]. For simplicity, we assume that the dimension $d$ is 1. As noted in [30], key ingredients for the error analysis to the RBM-approximation require some regularity assumptions on $\Gamma$ and the external potential.

### 5.1. Main results.

Next we state our main result on the uniform error analysis.

**Theorem 5.1.** (Uniform error estimate) *For $d = 1$, suppose system (1.1) satisfies assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, and initial data is a small perturbation of the given relative equilibrium $\Phi = (\phi_1, \cdots, \phi_N)$ in $(\mathcal{A}_3)$:*

$$\max_{1\le i,j\le N} |(q_i^{in} - \phi_i) - (q_j^{in} - \phi_j)| + \max_{1\le i,j\le N} |\phi_i - \phi_j| < r_0,$$

*where $r_0$ is a positive constant appearing in $(\mathcal{A}_1)$. Let $Q$ and $Q^R$ be solutions of (1.1) and (1.7) with the same initial data $Q^{in}$, respectively. Then, the discrepancy $Q^R - Q$ satisfies a uniform error estimate:*

$$\sup_{0\le t<\infty} \left[ \frac{1}{N} \sum_{i=1}^N \mathbb{E}|q_i^R(t) - q_i(t)|^2 \right] \le C \left[ \frac{N}{\lambda_2}\tau \left( \frac{1}{P-1} - \frac{1}{N-1} \right) + \left( \frac{N}{\lambda_2}\tau \right)^2 \right], \tag{5.1}$$

*for some constant $C$ independent of $\tau$, $P$, $N$, $t$ and $\lambda_2$.*

*Proof.* Section 6 is fully devoted to the proof. □

**Remark 5.1.** *In the sequel, we give several comments on the result of Theorem 5.1*

*(1) Consider a sequence of graphs $\{G^N\}_{N=1,2,\dots}$, with each term associated to the $N \times N$ adjacency matrix $(a_{ij}^N)_{1\le i,j\le N}$, and denote $\lambda_2^N$ by its algebraic connectivity. If $a_{ij}^N = 1$, $\forall\ i, j$, $\forall\ N$, then $\lambda_2^N \equiv N$, so the upper bound in (5.1) is independent of $N$, although it looks dependent on $N$ at first glance. More generally, if the graphs $G^N$ are connected strongly enough so that $\liminf_{N\to\infty} \frac{\lambda_2^N}{N} > 0$, then the upper bound in the R.H.S. of (5.1) is independent of $N$.*

(2) *In Section 7, we will see that the three concrete examples of first-order consensus models, namely the linear consensus model, the Kuramoto model and the 1D Cucker-Smale model, satisfy the uniform RBM error estimates by verifying the framework* $(\mathcal{A}_1)$*–*$(\mathcal{A}_3)$*.*

(3) *In the proof of Theorem 5.1, the essential parts in the derivation of the error estimate is the contracting property* $(\mathcal{A}_1)$*, network connectivity* $(\mathcal{A}_2)$ *and the boundedness of the states, the RBM states, and the derivatives of* $\Gamma$*.* $(\mathcal{A}_3)$ *is used to deduce these boundedness conditions.*

(4) *It is worthy to mention that the RBM error estimation can be extended to the multi-dimensional setting, where* $q_i$ *and* $\nu_i$ *are multi-dimensional vectors. In this case, one technical difficulty comes from the boundedness of the RBM-approximate states, which needs one more assumption on* $\Gamma$*:*

$(\mathcal{A}_0)$*: (Monotonicity to the equilibrium).*

*For any bounded vectors* $Q$ *with* $\max_i |q_i| < r_0$*, the interaction kernel* $\Gamma$ *satisfies*

$$a_{kj}(\Gamma(q_j - q_k) - \Gamma(\phi_j - \phi_k)) \cdot (q_k - \phi_k) \le 0, \quad j = 1, \dots, N,$$

*where* $v \cdot w$ *is the standard inner product between vectors* $v$ *and* $w$ *in* $\mathbb{R}^N$*, and* $k$ *is any index satisfying* $|q_k - \phi_k| = \max_i |q_i - \phi_i|$*.*

*Under assumptions* $(\mathcal{A}_0)$ *and* $(\mathcal{A}_1)$*–*$(\mathcal{A}_3)$*, Theorem 5.1 holds even for multi-dimensional case, as we later check in Section 6.3.*

(5) *The novelty of Theorem 5.1 and the decomposition of natural frequencies* (3.5) *lie in the uniform-in-time estimate. As noted in Remark 3.1 of* [30]*, if the contraction property* $(\mathcal{A}_1)$ *does not hold, the error estimate* (5.1) *can be formulated with a finite time interval* $[0,T]$*:*

$$\sup_{0\le t<T} \left[ \frac{1}{N}\sum_{i=1}^N \mathbb{E}|q_i^R(t) - q_i(t)|^2 \right] \le C(T) \left[ \frac{N}{\lambda_2}\tau \left( \frac{1}{P-1} - \frac{1}{N-1} \right) + \left( \frac{N}{\lambda_2}\tau \right)^2 \right],$$

*where the constant* $C(T)$ *grows exponentially on* $T$*. The proof of the time-dependent estimate is similar to Theorem 5.1.*

5.2. **Comparison with the previous work.** In this subsection, we briefly discuss the approach and result in [30] for the comparison with our result. Let $X = (X_1, \dots, X_N)$ with $X_i \in \mathbb{R}^d$, $i = 1, \dots, N$ be a state vector whose temporal evolution is governed by the stochastic dynamics with a Brownian motion $\mathbf{B} = (B^1, \dots, B^N)$:

$$\begin{cases} dX_i = -\nabla V(X_i)dt + \dfrac{1}{N-1}\displaystyle\sum_{j\neq i} \Gamma(X_i - X_j)dt + \sigma dB_t^i, \quad i = 1, \dots, N, \quad t > 0, \\ X_i(0) \sim \nu \text{ i.i.d., where } \nu \text{ is a given random variable,} \quad i = 1, \dots, N. \end{cases} \tag{5.2}$$

Then, we consider the RBM-approximation of (5.2):

$$\begin{cases} dX_i^R = -\nabla V(X_i^R)dt + \dfrac{1}{P-1}\displaystyle\sum_{j\in[i]_m} \Gamma(X_i^R - X_j^R)dt + \sigma dB_t^i, \quad t \in [\tau_m, \tau_{m+1}), \\ X_i(0) \sim \nu \text{ i.i.d., where } \nu \text{ is a given random variable,} \quad i = 1, \dots, N, \ m = 0, \cdots, n \end{cases}$$

In [30], the potential $V$ is assumed to be a $C^2$ function with a polynomial growth such that $V(x) - r|x|^2/2$ is convex. In contrast, the interaction function $\Gamma$ is also assumed to be a bounded and Lipschitz continuous $C^2$ function with a Lipschitz constant $L < r/2$. When the coupling is weak enough compared to the drift from $V(x)$, the homogeneous system

$$dX_i^R = -\nabla V(X_i^R)dt + \sigma dB_t^i,$$

is dominant and the RBM approximates system (1.3) with a uniform error over time:

$$\sup_{t>0} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}|X_i^R - X_i|^2 \leq C\frac{\tau}{P-1} + C\tau^2.$$

The uniform error analysis is based on the law of large numbers, where the convergence to the original dynamics is guaranteed as $\tau$ tends to zero, where $T/\tau$ is the large number for the time duration $t \in [0, T]$. A remarkable point is the independence on the final time $T$. From this property, the RBM is considered as a good approximation to the long-time behavior.

Though the uniform-in-time analysis (1.7) is restricted to a specific case, the RBM has shown good performances in the numerical simulations, starting from the examples in [30]: nonlinear opinion dynamics, quantum dynamics [18] and Poisson-Boltzmann equation [36]. Especially, the consensus-based optimization method [9, 24] and the collective behavior models [6, 34] show a quite accurate long-time behavior even if the interaction function $\Gamma$ is dominant in these systems.

## 6. Uniform error estimate

In this section, we provide a proof of Theorem 5.1 on the uniform error estimate for the discrepancy between the full system and the RBM approximate system:

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\kappa}{N-1} \displaystyle\sum_{j=1}^{N} \Big(\bar{\nu}_{ij} + a_{ij}\Gamma(q_j - q_i)\Big), \quad t > 0, \\ \dfrac{dq_i^R}{dt} = \dfrac{\kappa}{P-1} \displaystyle\sum_{j\in[i]_m} \Big(\bar{\nu}_{ij} + a_{ij}\Gamma(q_j^R - q_i^R)\Big), \quad t \in [\tau_m, \tau_{m+1}), \quad m = 0, 1, 2, \cdots. \end{cases} \tag{6.1}$$

Now, we introduce the discrepancies between the full solution and the approximate solution:

$$z_i := q_i^R - q_i, \quad i = 1, \cdots, N.$$

Then, it follows from (6.1) that $z_i$ satisfies

$$\begin{aligned} \frac{dz_i}{dt} &= \frac{\kappa}{P-1} \sum_{j\in[i]_m} \Big(\bar{\nu}_{ij} + a_{ij}\Gamma(q_j^R - q_i^R)\Big) - \frac{\kappa}{N-1} \sum_{j\neq i} \Big(\bar{\nu}_{ij} + a_{ij}\Gamma(q_j - q_i)\Big) \\ &=: \frac{\kappa}{N-1} \sum_{j\neq i} a_{ij} \Big(\Gamma(q_j^R - q_i^R) - \Gamma(q_j - q_i)\Big) + \chi_{m,i}(Q^R, \bar{\nu}), \end{aligned} \tag{6.2}$$

where the random variable $\chi_{m,i}(Q^R, \bar{\nu})$ is given by the following relation:

$$\chi_{m,i}(Q^R, \bar{\nu}) := \frac{\kappa}{P-1} \sum_{j\in[i]_m} \Big(\bar{\nu}_{ij} + a_{ij}\Gamma(q_j^R - q_i^R)\Big) - \frac{\kappa}{N-1} \sum_{j\neq i} \Big(\bar{\nu}_{ij} + a_{ij}\Gamma(q_j^R - q_i^R)\Big). \tag{6.3}$$

The first term in the R.H.S. of (6.2) will be treated by the dissipative assumption $(\mathcal{A}_1)$ (see Lemma 6.1). To apply the condition $(\mathcal{A}_1)$, we need to make sure that there exists $r_0$ such that

$$\sup_{0\le t<\infty}\max_{i,j}|q_j^R(t)-q_i^R(t)|\le r_0 \quad \text{and} \quad \sup_{0\le t<\infty}\max_{i,j}|q_j(t)-q_i(t)|\le r_0. \tag{6.4}$$

For the estimate of the second term, we will estimate the zeroth and first moment estimates of $\chi_{m,i}(Q^R,\bar{\nu})$ (see Lemma 6.2): for a bounded constant vector $\mathcal{P}=(p_1,\dots,p_N)$,

$$\mathbb{E}(\chi_{m,i}(\mathcal{P},\bar{\nu}))=0 \quad \text{and} \quad \mathrm{Var}(\chi_{m,i}(\mathcal{P},\bar{\nu}))\lesssim\left(\frac{1}{P-1}-\frac{1}{N-1}\right). \tag{6.5}$$

In the following subsection, we provide estimates for (6.4) and (6.5).

6.1. **Preparatory estimates.** In this subsection, we study the boundedness of the relative states in (6.4). This can be proved using condtions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ as follows.

**Lemma 6.1.** *For $d=1$, suppose conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ hold. Moreover, the initial data $Q^{in}$ and the equilibrium $\Phi$ satisfy*

$$\max_{1\le i,j\le N}|(q_i^{in}-\phi_i)-(q_j^{in}-\phi_j)|+\max_{1\le i,j\le N}|\phi_i-\phi_j|<r_0. \tag{6.6}$$

*Let $Q$ and $Q^R$ be a solution to the system* (6.1)*. Then, one has*

$$\sup_{0\le t<\infty}\max_{1\le i,j\le N}|(q_i(t)-\phi_i)-(q_j(t)-\phi_j)|+\max_{1\le i,j\le N}|\phi_i-\phi_j|<r_0,$$
$$\sup_{0\le t<\infty}\max_{1\le i,j\le N}|(q_i^R(t)-\phi_i)-(q_j^R(t)-\phi_j)|+\max_{1\le i,j\le N}|\phi_i-\phi_j|<r_0.$$

*Proof.* The solution $Q$ to $(6.1)_1$ may be seen as a solution $Q^R$ to $(6.1)_2$ with $P=N$. Hence, we only prove the second estimate and omit the first.

For the second estimate, we use the continuity argument. More precisely, we introduce a set $\mathcal{T}$ consisting of times in which the second estimate is valid, and $T_*$ is the supremum of the set $\mathcal{T}$:

$$\mathcal{T}:=\left\{T>0\;\middle|\;\begin{aligned}&\max_{1\le i,j\le N}|(q_i^R(t)-\phi_i)-(q_j^R(t)-\phi_j)|+\max_{1\le i,j\le N}|\phi_i-\phi_j|<r_0\\&\text{holds for } t\in[0,T)\end{aligned}\right\},\quad T_*:=\sup\mathcal{T}.$$

By assumption (6.6) and the continuity of $\max_{1\le i,j\le N}|(q_i^R(t)-\phi_i)-(q_j^R(t)-\phi_j)|$, there exists $\delta>0$ such that

$$\max_{1\le i,j\le N}|(q_i^R(t)-\phi_i)-(q_j^R(t)-\phi_j)|+\max_{1\le i,j\le N}|\phi_i-\phi_j|<r_0,\quad \forall\, t\in[0,\delta).$$

Hence, $\delta\in\mathcal{T}$ and $T_*>0$.

Next, we claim

$$T_*=\infty.$$

Suppose this is not true, i.e., $T_*<\infty$. Then, one has

$$\max_{1\le i,j\le N}|(q_i^R(t)-\phi_i)-(q_j^R(t)-\phi_j)|+\max_{1\le i,j\le N}|\phi_i-\phi_j|<r_0,\quad \forall\, t\in[0,T_*).$$

For each $t\ge0$, we choose time-dependent indices $i_t$ and $j_t$ to satisfy

$$q_{i_t}^R(t)-\phi_{i_t}=\max_{1\le i\le N}(q_i^R(t)-\phi_i) \quad \text{and} \quad q_{j_t}^R(t)-\phi_{j_t}=\min_{1\le i\le N}(q_i^R(t)-\phi_i).$$

This implies

$$(q_k^R - q_{i_t}^R) - (\phi_k - \phi_{i_t}) = (q_k^R - \phi_k) - (q_{i_t}^R - \phi_{i_t}) \le 0, \quad k \in \{1, \cdots, N\}. \tag{6.7}$$

On the other hand, $q_{i_t}^R$ satisfies

$$\begin{aligned} \frac{dq_{i_t}^R}{dt} &= \frac{\kappa}{P-1} \sum_{k\in[i_t]_m} \Big(\bar{\nu}_{i_t k} + a_{i_t k}\Gamma(q_k^R - q_{i_t}^R)\Big) \\ &= \frac{\kappa}{P-1} \sum_{k\in[i_t]_m} a_{i_t k}\Big(\Gamma(q_k^R - q_{i_t}^R) - \Gamma(\phi_k - \phi_{i_t})\Big). \end{aligned} \tag{6.8}$$

By assumption $(\mathcal{A}_1)$ on $\Gamma$, (6.7) and the maximality (6.8) of $T_*$, we have

$$\frac{d}{dt}(q_{i_t}^R - \phi_{i_t}) \le \frac{L_1\kappa}{P-1} \sum_{k\in[i_t]_m} a_{i_t k}[(q_k^R - q_{i_t}^R) - (\phi_k - \phi_{i_t})] \le 0, \quad t \in [0, T_*).$$

Similarly, one has

$$\frac{d}{dt}(q_{j_t}^R - \phi_{j_t}) \ge \frac{L_1\kappa}{P-1} \sum_{k\in[j_t]_m} a_{j_t k}[(q_k^R - q_{j_t}^R) - (\phi_k - \phi_{j_t})] \ge 0, \quad t \in [0, T_*).$$

These two inequalities induce that

$$\frac{d}{dt}\big((q_{i_t}^R(t) - \phi_{i_t}) - (q_{j_t}^R(t) - \phi_{j_t})\big) \le 0.$$

Therefore, we have

$$\begin{aligned} &\max_{1\le i,j\le N} |(q_i^R(t) - \phi_{i_t}) - (q_j^R(t) - \phi_{j_t})| + \max_{1\le i,j\le N} |\phi_i - \phi_j| \\ &\quad \le \max_{1\le i,j\le N} |(q_i^{in} - \phi_i) - (q_j^{in} - \phi_j)| + \max_{1\le i,j\le N} |\phi_i - \phi_j| < r_0, \quad t \in [0, T_*). \end{aligned}$$

Letting $t \to T_*-$, one has

$$\max_{1\le i,j\le N} |(q_i^R(T_*) - \phi_{i_t}) - (q_j^R(T_*) - \phi_{j_t})| + \max_{1\le i,j\le N} |\phi_i - \phi_j| < r_0.$$

This contradicts to the maximality of $T_*$. Hence, we get $T_* = \infty$ and the desired estimate. □

Next, we study the moment estimates in (6.5). For any $m \ge 0$, $i = 1, \ldots, N$ and a bounded constant vector $\mathcal{P} = (p_1, \ldots, p_N)$, we set

$$\Lambda_i(\mathcal{P}, \bar{\nu}) := \frac{\kappa^2}{N-2} \sum_{j:j\ne i} \left| (\bar{\nu}_{ij} + a_{ij}\Gamma(p_j - p_i)) - \frac{1}{N-1} \sum_{k:k\ne i} (\bar{\nu}_{ik} + a_{ik}\Gamma(p_k - p_i)) \right|^2.$$

**Lemma 6.2.** *Let $\mathcal{P} = (p_1, \ldots, p_N)$ be a bounded constant vector. Then, the random functionals $\chi_{m,i}(\mathcal{P}, \bar{\nu})$ and $\Lambda_i(\mathcal{P}, \bar{\nu})$ satisfy*

$$\mathbb{E}\Big[\chi_{m,i}(\mathcal{P}, \bar{\nu})\Big] = 0, \quad Var\Big[\chi_{m,i}(\mathcal{P}, \bar{\nu})\Big] = \left(\frac{1}{P-1} - \frac{1}{N-1}\right)\Lambda_i(\mathcal{P}, \bar{\nu}).$$

*Proof.* Although the proof can be followed from the same argument of Lemma 3.1 in [30], for reader's convenience, we provide its detailed proof in Appendix A. □

**Remark 6.1.** *From Lemma 6.2, the variance of $\chi_{m,i}(\mathcal{P}, \bar{\nu})$ is bounded for a bounded vector $\mathcal{P}$: in order to bound the right-hand side of (2.1), note that $\Gamma$ is bounded in a bounded domain. Later in the proof of Theorem 5.1, we need to assume that $\Gamma$ is Lipschitz continuous additionally.*

6.2. **Proof of Theorem 5.1.** In this subsection, we provide a proof of our main result.

Recall that our purpose is to derive the following uniform error estimate:

$$\sup_{0\le t<\infty}\left[\frac{1}{N}\sum_{i=1}^N \mathbb{E}|q_i^R(t)-q_i(t)|^2\right] \lesssim \left[\frac{N\tau}{\lambda_2}\left(\frac{1}{P-1}-\frac{1}{N-1}\right)+\left(\frac{N\tau}{\lambda_2}\right)^2\right]. \tag{6.9}$$

Let $Q$ and $Q^R$ be solutions to (1.1) and (1.7), respectively. We set

$$z_i := q_i^R - q_i, \qquad 1 \le i \le N.$$

Introduce a functional $U$:

$$U(t) := \frac{1}{N}\mathbb{E}\Big(\sum_{i=1}^N |z_i|^2\Big) = \frac{1}{N}\sum_{i=1}^N \mathbb{E}|z_i|^2.$$

6.2.1. *Step A.* (Derivation of Grönwall's inequality for $U$): In what follows, we derive the following differential inequality for $U$:

$$\frac{dU}{dt} \le -\frac{C\lambda_2}{N}U + C\tau\sqrt{U} + C\tau\left(\frac{1}{P-1}-\frac{1}{N-1}\right),$$

where $C$ is a generic positive constant. From the governing equations for $q_i$ and $q_i^R$, it is easy to see that $z_i = q_i^R - q_i$ satisfies

$$\begin{aligned}\frac{dz_i}{dt} &= \frac{\kappa}{P-1}\sum_{j\in[i]_m}[\bar{\nu}_{ij}+a_{ij}\Gamma(q_j^R-q_i^R)] - \frac{\kappa}{N-1}\sum_{j\ne i}[\bar{\nu}_{ij}+a_{ij}\Gamma(q_j-q_i)]\\ &= \frac{\kappa}{N-1}\sum_{j\ne i} a_{ij}\left[\Gamma(q_j^R-q_i^R)-\Gamma(q_j-q_i)\right] + \chi_{m,i}(Q^R,\bar{\nu}).\end{aligned}$$

Here, the remainder term $\chi_{m,i}(Q^R,\bar{\nu})$ was defined in (6.3), which has been analyzed in Lemma 6.2 for a deterministic state $\mathcal{P}$. We multiply $2z_i$ on the derivative of $z_i$ and sum it over $i$ and take an expectation to get

$$\begin{aligned}\frac{d}{dt}\frac{1}{N}\mathbb{E}\sum_{i=1}^N|z_i|^2 &= \frac{2\kappa}{N(N-1)}\mathbb{E}\sum_{i=1}^N\sum_{j\ne i}a_{ij}\left[\Gamma(q_j^R-q_i^R)-\Gamma(q_j-q_i)\right](q_i^R-q_i)\\ &\quad + \frac{2}{N}\mathbb{E}\sum_{i=1}^N z_i\cdot\chi_{m,i}(Q^R,\tilde{V})\\ &=: \mathcal{S}(t) + \frac{2}{N}\sum_{i=1}^N\mathcal{R}_i(t), \quad \text{a.e., } t>0.\end{aligned} \tag{6.10}$$

In the following lemmas, we estimate $\mathcal{S}$ and $\mathcal{R}_i$.

**Lemma 6.3.** *Suppose assumptions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ *hold, and let* $Q$ *and* $Q^R$ *be solutions to* (1.1) *and* (1.7) *satisfying the boundedness property* (6.4). *Then,* $\mathcal{S}$ *in* (6.10) *satisfies*

$$\mathcal{S}(t) \le -\frac{C\lambda_2}{N^2}\mathbb{E}\sum_{i=1}^N |z_i|^2.$$

*Proof.* In (6.10), we interchange indices $i \leftrightarrow j$ and use (1.2) and (1.3) to get

$$\begin{aligned}\mathcal{S}(t) &= \frac{2\kappa}{N(N-1)}\mathbb{E}\sum_{j=1}^N\sum_{i\neq j} a_{ji}\left[\Gamma(q_i^R - q_j^R) - \Gamma(q_i - q_j)\right](q_j^R - q_j)\\ &= -\frac{2\kappa}{N(N-1)}\mathbb{E}\sum_{i=1}^N\sum_{j\neq i} a_{ij}\left[\Gamma(q_j^R - q_i^R) - \Gamma(q_j - q_i)\right](q_j^R - q_j).\end{aligned} \tag{6.11}$$

We use (6.11) and $(\mathcal{A}_2)$ to obtain

$$\begin{aligned} S(t) &= -\frac{\kappa}{N(N-1)}\mathbb{E}\sum_{i=1}^N\sum_{j\neq i} a_{ij}\left[\Gamma(q_j^R - q_i^R) - \Gamma(q_j - q_i)\right]\big((q_j^R - q_j) - (q_i^R - q_i)\big)\\ &\le -\frac{\kappa C}{N(N-1)}\mathbb{E}\sum_{i=1}^N\sum_{j\neq i} a_{ij}|(q_j^R - q_i^R) - (q_j - q_i)|^2\\ &= -\frac{\kappa C}{N(N-1)}\mathbb{E}\sum_{i=1}^N\sum_{j\neq i} a_{ij}|z_j - z_i|^2\\ &= -\frac{2\kappa C}{N(N-1)}\mathbb{E}(\mathbf{z}^\top L\mathbf{z}) \le -\frac{2\kappa C\lambda_2}{N(N-1)}\mathbb{E}\|\mathbf{z}\|^2 \le -\frac{C\lambda_2}{N^2}\mathbb{E}\sum_{i=1}^N |z_i|^2,\end{aligned}$$

where we followed the approach in [40] in the last line, and used the notation $\mathbf{z} := (z_1 \dots z_N)^\top$ and the conservation $\sum_{i=1}^N z_i \equiv 0$. ☐

**Lemma 6.4.** *Suppose assumptions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ *hold, and let* $Q$ *and* $Q^R$ *be solutions to* (1.1) *and* (1.7) *satisfying the boundedness property* (6.4). *Then, the functional* $\mathcal{R}_i$ *in* (6.10) *satisfies*

$$\mathcal{R}_i(t) \le C\tau\left(\frac{1}{N-1}\sum_{j\neq i}\mathbb{E}|z_j(t)| + \mathbb{E}|z_i(t)| + \left(\frac{1}{P-1} - \frac{1}{N-1}\right)\right) + C\tau^2.$$

*Proof.* Since the proof is tedious and very lengthy, we leave it in Appendix B. ☐

6.2.2. *Step B (Derivation of uniform boundedness).* From (6.10), we collect the results of Lemma 6.3, Lemma 6.4 and the Cauchy-Schwarz' inequality to find

$$\begin{aligned}
\dot{U}(t) &= \mathcal{S}(t) + \frac{2}{N}\sum_{i=1}^{N}\mathcal{R}_i(t) \\
&\le -\frac{C\lambda_2}{N}U(t) + C\tau\sqrt{U(t)} + C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\tau^2 \\
&\le -\frac{C\lambda_2}{2N}U(t) + C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + \frac{CN}{\lambda_2}\tau^2 \quad \text{by Young's inequality.}
\end{aligned}$$

Then, we use comparison principle for ODEs and the explicit formula of the following ODE,

$$y' = -\frac{C\lambda_2}{2N}y + C\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + \frac{CN}{\lambda_2}\tau^2,$$

to find the desired uniform error estimate:

$$U(t) \le C\frac{N\tau}{\lambda_2}\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + C\left(\frac{N\tau}{\lambda_2}\right)^2.$$

6.3. **Extension to the multi-dimensional setting.** As mentioned in Introduction, we used the boundedness of the RBM-approximation (1.7) in the proof of Theorem 5.1. Unfortunately, Lemma 6.1 cannot be used directly, when system (1.3) describes multi-dimensional quantities $q_i, \nu_i \in \mathbb{R}^d$. The following Lemma is an alternative version of Lemma 6.1 if we assume $(\mathcal{A}_0)$.

**Lemma 6.5.** *For $d \ge 2$, suppose assumptions $(\mathcal{A}_0)$–$(\mathcal{A}_3)$ hold, and assume that the solution $Q$ is a small perturbation of the stable equilibrium $\Phi$:*

$$\max_{1\le i\le N}|q_i^{in} - \phi_i| + \max_{1\le i\le N}|\phi_i| < r_0/2,$$

*and let $Q$ and $Q^R$ be a solution to the system* (1.1) *and* (1.7)*, respectively. Then, one has*

$$\sup_{0\le t<\infty}\max_{1\le i\le N}|q_i(t) - \phi_i| + \max_{1\le i\le N}|\phi_i| < r_0/2,$$

*which implies*

$$\sup_{0\le t<\infty}\max_{1\le i,j\le N}|(q_i^R(t) - \phi_i) - (q_j^R(t) - \phi_j)| + \max_{1\le i,j\le N}|\phi_i - \phi_j| < r_0.$$

*Proof.* For some $t \ge 0$, let $k$ be an index satisfying

$$|q_k^R(t) - \phi_k| = \sup_i |q_i - \phi_i|.$$

Then, the dynamics of $q_k^R$ is given as

$$\frac{dq_k^R}{dt} = \frac{\kappa}{P-1}\sum_{j\in[k]_m}[\bar{\nu}_{kj} + a_{kj}\Gamma(q_j^R - q_k^R)] = \frac{\kappa}{P-1}\sum_{j\in[k]_m}a_{kj}[\Gamma(q_j^R - q_k^R) - \Gamma(\phi_j - \phi_k)].$$

Since we may use $(\mathcal{A}_4)$ at $t = 0$, we have

$$\frac{d}{dt}|q_k^R - \phi_k|^2 = \frac{2\kappa}{P-1}\sum_{j\in[k]_m}a_{kj}[\Gamma(q_j^R - q_k^R) - \Gamma(\phi_j - \phi_k)]\cdot(q_k^R - \phi_k) \le 0, \quad t = 0.$$

From the continuity argument as in Lemma 6.1, we conclude the boundedness of

$$\max_{1\le i\le N} |q_i^R(t)-\phi_i|^2$$

for the whole time interval. We will omit the details. □

With the same arguments in Section 6.2, we conclude the RBM error estimate for the multi-dimensional system.

**Corollary 6.1.** *Suppose that the multi-dimensional system* (1.7) *with* $q_i, \nu_i \in \mathbb{R}^d$ *satisfies assumptions* $(\mathcal{A}_0)$-$(\mathcal{A}_3)$. *Let* $Q=(q_1,\dots,q_N)$ *be the solution of* (1.1) *for a small initial datum near the equilibrium:*

$$\max_{1\le i\le N} |q_i^{in}-\phi_i| + \max_{1\le i\le N} |\phi_i| < r_0/2,$$

*where* $r_0$ *and* $\phi$ *are from* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_3)$. *Then, for the RBM-approximation* $Q^R=(q_1^R,\dots,q_N^R)$ *of* (1.7), *the error can be estimated as*

$$\sup_{t>0} \frac{1}{N}\sum_{i=1}^N \mathbb{E}|q_i^R-q_i|^2 \le C\frac{N\tau}{\lambda_2}\left(\frac{1}{P-1}-\frac{1}{N-1}\right) + C\left(\frac{N\tau}{\lambda_2}\right)^2$$

*for some constant* $C$ *independent of* $t$, $\tau$, $P$, $N$ *and* $\lambda_2$.

## 7. Three first-order consensus models

In this section, we present three first-order consensus models satisfying assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, and derive uniform error estimates for the RBM to such models.

7.1. **The linear consensus model.** For constants $\nu_i \in \mathbb{R}$ $(i=1,\dots,N)$ and $\kappa>0$, we assume the states $q_i, q_i^R \in \mathbb{R}^d$ $(i=1,\dots,N)$ follow the dynamics of the linear consensus model:

$$\begin{cases} \dfrac{dq_i}{dt} = \nu_i + \dfrac{\kappa}{N-1}\displaystyle\sum_{j\ne i} a_{ij}(q_j-q_i), \quad t>0, \\ \dfrac{dq_i^R}{dt} = \dfrac{\kappa}{P-1}\displaystyle\sum_{j\in[i]_m} \left(\bar{\nu}_{ij} + a_{ij}(q_j^R-q_i^R)\right),\ t\in(\tau_m,\tau_{m+1}), \end{cases} \tag{7.1}$$

subject to the same initial data:

$$q_i(0)=q_i^R(0)=q_i^{in}, \quad i=1,\cdots,N. \tag{7.2}$$

Note that each component of $q$ satisfies the same form of equations (7.1). Hence, we may assume $d=1$. In this case, the coupling function $\Gamma_L$ is given as

$$\Gamma_L(q) := q, \quad q\in\mathbb{R}.$$

Condition $(\mathcal{A}_1)$ is satisfied for an arbitrary $r_0>0$ with $L_1=L_2=1$. By Corollary 4.1, Condition $(\mathcal{A}_3)$ holds for $r_0 > \frac{\sqrt{2}(N-1)\|\nu\|}{\kappa L_1\lambda_2}$. Since we can choose such $r_0$ for any given system parameters $N,\nu,\kappa,\lambda_2$, the RBM error estimate (6.9) holds unconditionally.

**Corollary 7.1.** *Let* $Q$ *and* $Q^R$ *be the solutions to linear consensus models* (7.1) - (7.2). *Then, we have*

$$\sup_{t\ge0}\sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{E}|q_i^R-q_i|^2} \le C\left[\frac{N\tau}{\lambda_2}\left(\frac{1}{P-1}-\frac{1}{N-1}\right) + \left(\frac{N\tau}{\lambda_2}\right)^2\right].$$

7.2. **The Kuramoto model.** Let $q_i \in \mathbb{R}$ be the phase of the $i$-th Kuramoto oscillator. Then, $q_i$ and its approximation $q_i^R$ satisfy

$$
\begin{cases} \dfrac{dq_i}{dt} = \nu_i + \dfrac{\kappa}{N-1}\displaystyle\sum_{j\neq i} a_{ij}\sin(q_j - q_i), \quad i = 1, \cdots, N, \quad t > 0, \\ \dfrac{dq_i^R}{dt} = \dfrac{\kappa}{P-1}\displaystyle\sum_{j\in[i]_m}\big(\bar{\nu}_{ij} + a_{ij}\sin(q_j^R - q_i^R)\big), \quad t \in (\tau_m, \tau_{m+1}), \end{cases} \tag{7.3}
$$

subject to the same initial data:

$$
q_i(0) = q_i^R(0) = q_i^{in}, \quad i = 1, \cdots, N. \tag{7.4}
$$

In this case, the interaction function $\Gamma_K$ takes the form of

$$
\Gamma_K(q) := \sin q, \quad q \in \mathbb{R}.
$$

The first condition $(\mathcal{A}_1)$ is satisfied with parameters:

$$
0 < r_0 < \frac{\pi}{2}, \quad L_1 = \cos r_0 \quad \text{and} \quad L_2 = 1.
$$

By Corollary 4.1, if $\frac{\sqrt{2}(N-1)\|\nu\|}{\kappa\lambda_2} < r_0 \cos r_0$, condition $(\mathcal{A}_2)$ holds.

**Corollary 7.2.** *Suppose that the initial data and the coupling strength satisfy*

$$
\max_{1\le i,j\le N} |(q_i^{in} - \phi_i) - (q_j^{in} - \phi_j)| + \max_{1\le i,j\le N} |\phi_i - \phi_j| < r_0 < \frac{\pi}{2}, \quad \frac{\sqrt{2}(N-1)\|\nu\|}{\kappa\lambda_2} < r_0 \cos r_0
$$

*and let $Q$ and $Q^R$ be solutions to* (7.3) - (7.4). *Then, one has*

$$
\sup_{t\ge 0}\sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{E}|q_i^R - q_i|^2} \le C\left[\frac{N\tau}{\lambda_2}\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + \left(\frac{N\tau}{\lambda_2}\right)^2\right].
$$

Note that the contracting property of $\Gamma_K$ was already known in [11, 19]. We have provided an alternative proof using Proposition 4.1.

7.3. **1D Cucker-Smale model.** Let $q_i$ be the position of the $i$-th CS particle with a unit mass on the real line. Then, $q_i$ and its approximation $q_i^R$ satisfy

$$
\begin{cases} \dfrac{dq_i}{dt} = \nu_i + \dfrac{\kappa}{N-1}\displaystyle\sum_{j\neq i} a_{ij}\Gamma_{cs}(q_j - q_i), \quad t > 0, \\ \dfrac{dq_i^R}{dt} = \dfrac{\kappa}{P-1}\displaystyle\sum_{j\in[i]_m}\big(\bar{\nu}_{ij} + a_{ij}\Gamma_{cs}(q_j^R - q_i^R)\big), \quad t \in (\tau_m, \tau_{m+1}), \\ q_i(0) = q_i^R(0) = q_i^{in}, \quad i = 1, \cdots, N, \end{cases} \tag{7.5}
$$

where $\nu_i$ and $\Gamma_{cs}$ are natural velocity of the $i$-th particle and coupling function which is the anti-derivative of the communication weight function $\psi$:

$$
\Gamma_{cs}(q) = \int_0^q \psi(r)dr, \quad \nu_i := v_i^{in} - \frac{\kappa}{N-1}\sum_{j=1}^N a_{ij}\Gamma_{cs}(q_j^{in} - q_i^{in}).
$$

For the flocking estimate of the RBM-approximation $(7.5)_2$, we refer to [15].

It is common to assume that the interaction kernel $\psi$ is symmetric, non-negative, non-increasing, smooth and bounded. For example, we may set

$$\psi(q) = \frac{1}{(1+|q|^2)^{\beta/2}}, \quad \beta > 0.$$

Since $\psi$ is a nonnegative function, the coupling function $\Gamma_{cs}$ is a smooth non-decreasing function. Thus, the situation is similar to the case of linear consensus models. If the interaction kernel $\psi$ is positive in a bounded domain $\{x \mid |x| < r_0\}$, then the contracting property $(\mathcal{A}_1)$ holds with $r_0$ with $L_1 = \psi(r_0)$. By Corollary 4.1, $(\mathcal{A}_2)$ also holds if $\frac{\sqrt{2}(N-1)\|\nu\|}{\kappa\lambda_2} < r_0\psi(r_0)$.

**Corollary 7.3.** *Suppose the interaction kernel $\psi$ and initial data satisfy*

$$\max_{1\le i,j\le N} |(q_i^{in} - \phi_i) - (q_j^{in} - \phi_j)| + \max_{1\le i,j\le N} |\phi_i - \phi_j| < r_0, \quad \frac{\sqrt{2}(N-1)\|\nu\|}{\kappa\lambda_2} < r_0\psi(r_0). \tag{7.6}$$

*Then, there exists $\delta = \delta(\eta) > 0$ such that if $\kappa > \delta\|\nu\|$, one has*

$$\sup_{0\le t<\infty} \sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{E}|q_i^R - q_i|^2} \le C\left[\frac{N\tau}{\lambda_2}\left(\frac{1}{P-1} - \frac{1}{N-1}\right) + \left(\frac{N\tau}{\lambda_2}\right)^2\right].$$

**Remark 7.1.** *Consider the case $1/\psi(r) = O(r^\beta)$. If $\beta < 1$ then the two inequalities in* (7.6) *always hold for a sufficiently large $r_0$, hence, the error estimate holds for any initial data, i.e. the estimate holds unconditionally. But this is not the case for $\beta \ge 1$: the second inequality in* (7.6) *bounds the admissible value of $r_0$, and this makes the first inequality in* (7.6) *hold only for well-arranged initial data. Detailed analysis on the conditional and unconditional convergence of the Cucker-Smale model depending on the value of the exponent $\beta$ is given in* [25].

## 8. Numerical simulations

In this section, we perform several numerical simulations on the RBM-approximation for the first-order models to compare with the analytical result on the RBM error estimate in Theorem 5.1. The $\ell^2$-errors will be evaluated for each RBM-approximation along time-evolution, where the dependence on $P$ and the boundedness over time $t$ are our primary concerns.

In order to compare the errors clearly, we used the forward Euler method with time step $\tau$, where $\tau$ is also the time step for the RBM-approximation. The tested system is the one-dimensional Cucker-Smale model (7.5) with system parameters:

$$N = 64, \quad \tau = 0.1, \quad a_{ij} \equiv 1, \quad \kappa = 2 \quad \text{and} \quad \psi(q) = \frac{1}{\sqrt{1+|q|^2}}.$$

We take initial data randomly and uniformly distributed.

Note that the interaction network is fully connected and the communication weight is always positive, hence, conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. We selected the equilibrium point $\Phi$ a priori. Recall the original system and RBM-approximated system:

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\kappa}{N-1}\displaystyle\sum_{j\ne i}[\sinh^{-1}(q_j - q_i) - \sinh^{-1}(\phi_j - \phi_i)], \quad t \ge 0 \\ \dfrac{dq_i^R}{dt} = \dfrac{\kappa}{P-1}\displaystyle\sum_{j\in[i]_m}[\sinh^{-1}(q_j^R - q_i^R) - \sinh^{-1}(\phi_j - \phi_i)], \quad t \in [\tau_m, \tau_{m+1}), \end{cases}$$

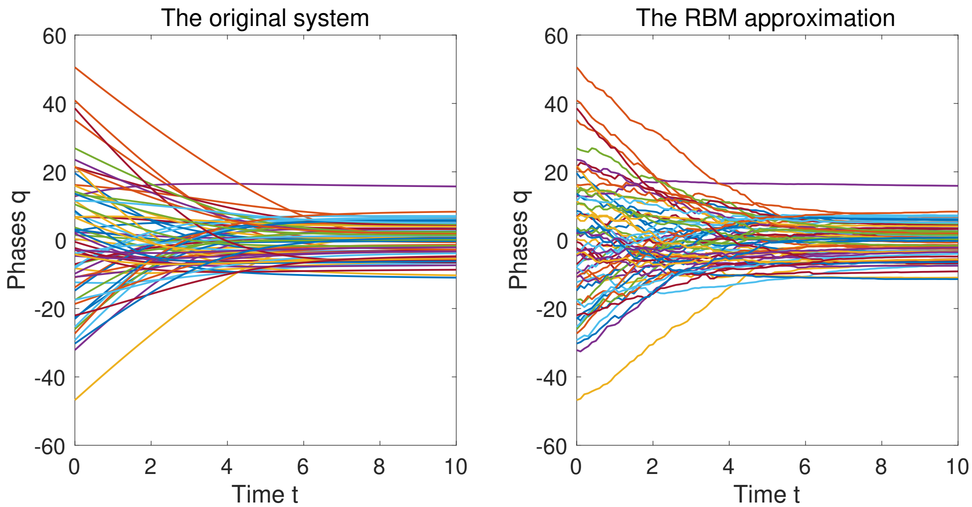


FIGURE 1. A simulation on trajectories along time on the original system (left) and the RBM-approximation ($P = 2$).

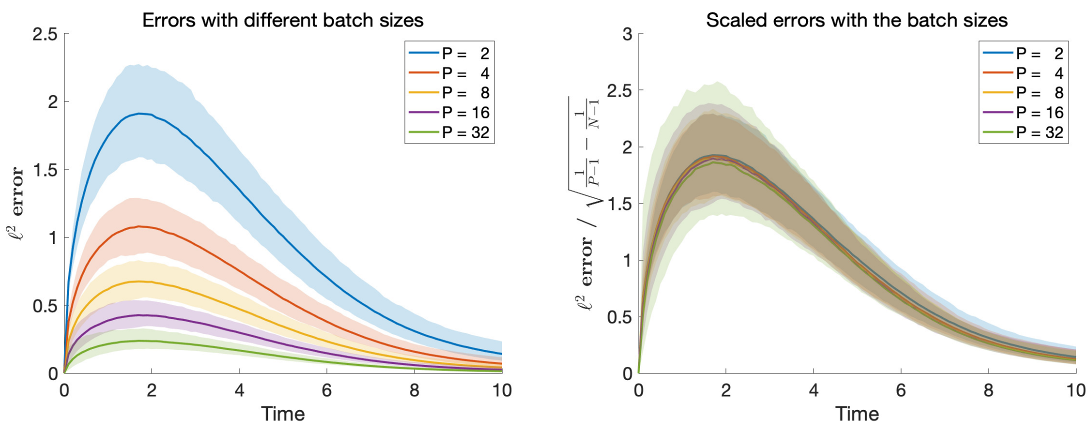


FIGURE 2. Left: The $\ell^2$-errors from 1000 simulations, calculated with different $P$. The colored region is drawn with 95% of the total simulations excluding bad 5% ones. The middle thick colored line is the median trajectories. Right: Scaled error by the term $\sqrt{\frac{1}{P-1} - \frac{1}{N-1}}$. Errors from different $P$ shows similar values.

subject to the same initial data:

$$q_i(0) = q_i^{in}, \quad i = 1, \cdots, N.$$

In Figure 1, we plot the trajectories along time from the full system and the RBM-approximated system with $P = 2$. One can see that both systems converge to the same equilibrium point $\phi$. On the other hand, (1.4) is not stable around $\phi$ from the equations and the example in Section 2.2.

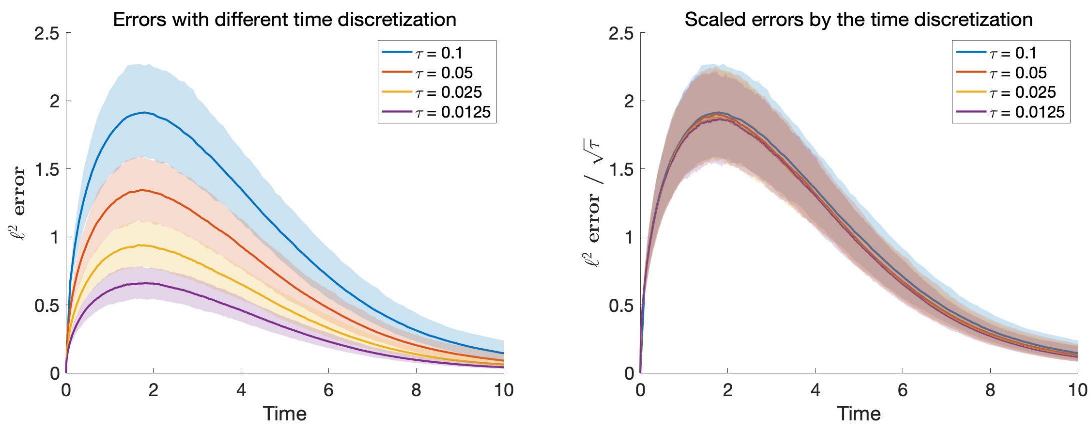


FIGURE 3. Left: The $\ell^2$-errors from 1000 simulations, calculated with different $\tau$. The colored region is drawn with 95% of the total simulations excluding bad 5% ones. The middle thick colored line is the median trajectories. Right: Scaled error by $\sqrt{\tau}$. Errors from different $\tau$ shows similar values.

In Figure 2, we plotted the $\ell^2$-errors from the RBM-approximation, which is defined as

$$\ell^2\text{-error} := \sqrt{\frac{1}{N}\sum_{i=1}^{N}|q_i^R(t) - q_i(t)|^2}.$$

The RBM-approximation is computed with 1000 random simulations for each $P$, where the error is tested with $P = 2, 4, 8, 16, 32$. The thick colored line shows the RBM trajectories with the median error for each $P$, while the colored area is the 95% confidential intervals at each time. This means that we exclude 5% simulations with extraordinary big or small errors for better visibility.

Note that the error grows rapidly in the starting stage, but eventually decreases to zero due to the convergence of the system. This was the main idea to expect uniform-in-time error bounds for the RBM-approximation.

On the other hand, Figure 2 also shows how error decreases along $P$. Note that the $\ell^2$-error depends on $P$ and $\tau$:

$$\ell^2\text{-error} \le C\sqrt{\tau\left(\frac{1}{P-1} - \frac{1}{N-1}\right)} + O(\tau). \tag{8.1}$$

In the right figure, the errors are scaled with the factor of $P$ in the above inequality. One can see that the errors are similar for different $P$, which shows that the estimate (8.1) suggests the sharpest order on $P$.

Figure 3 is the same plot as Figure 2 with $P = 2$ but with different $\tau$: $\tau = 1/10$, $1/20$, $1/40$, $1/80$. As in Figure 2, it shows that the $\ell^2$-error is proportional to $\sqrt{\tau}$.

## 9. CONCLUSION

In this paper, we have analyzed the RBM-approximation for the deterministic first-order model with convolution-type interactions and the interaction network, in the case when the system experiences the $\ell^2$-contraction and convergence to an equilibrium. In the error analysis of the RBM model, contraction property and the boundedness of states are two important ingredients for the uniform-in-time error analysis. In particular, the boundedness of the RBM trajectories is the main obstacle, if there are nonidentical natural drift $\nu_i$ for different $i$. By assuming a priori knowledge on the equilibrium point, the boundedness of the RBM trajectories is guaranteed. To construct a unified framework, our focus was confined to the deterministic consensus model including the linear consensus model and the Kuramoto model as special cases. There are still lots of interesting issues for the RBM-approximation for the first-order consensus model. Some of the issues can be listed below.

- **Stochastic perturbations**: When a system lies in a noisy environment, then the boundedness of the system cannot be achieved in the classical sense, so we need $L^2$-boundedness of the states and interactions. We excluded noisy systems due to the Kuramoto model since the contracting property $(\mathcal{A}_1)$ in such models needs deterministic bounds $r_0$. The RBM was originally considered with additive noise in [30], and the multiplicative noise is also important for the particle swarm optimization as in [9].
- **Non-contracting systems.** Non-convolution systems are also good examples to study the RBM error estimates. Moreover, we expect that non-contracting systems, such as the Keller-Segel aggregation model or Coulomb potential dynamics, may also satisfy uniform-in-time error estimates. We used contraction to nullify the error along time, however, we may bound the error without it if we know the equilibria.
- **Non-unique equilibria.** Another interesting example is the Van der Waals dynamics, where the equilibrium point is not unique due to the symmetry. For example, if three particles want to make their relative distances to a specific value, then the system does not care whether two particles change their positions. In such cases, the RBM is not expected to converge to the exact equilibrium point of the original system, but the final density profile may coincide.
- **The mean-field limit.** It will be interesting to study the mean-field limit $(N \to \infty)$ of the random batch model, and then compare the limit with that of the original model, as was done in [29].

We leave the above interesting problems as future work.

## Appendix A. Proof of Lemma 6.2

In this appendix, we provide proofs for two stochastic estimates for $\chi_{m,i}(\mathcal{P},\bar\nu)$:

$$\mathbb{E}\Big[\chi_{m,i}(\mathcal{P},\bar\nu)\Big]=0,\quad \mathrm{Var}\Big[\chi_{m,i}(\mathcal{P},\bar\nu)\Big]=\left(\frac{1}{P-1}-\frac{1}{N-1}\right)\Lambda_i(\mathcal{P},\bar\nu).$$

(i) Recall that

$$\chi_{m,i}(\mathcal{P},\bar\nu):=\frac{\kappa}{P-1}\sum_{j\in[i]_m}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)-\frac{\kappa}{N-1}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big), \tag{A.1}$$

and we define

$$\begin{aligned} f_{m,i}(\mathcal{P},\bar\nu)&:=\frac{\kappa}{P-1}\sum_{j\in[i]_m}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\\ &=\underbrace{\frac{\kappa}{P-1}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)}_{\text{deterministic part}}\ \underbrace{\mathbf{1}_{\{j\in[i]_m\}}}_{\text{random part}}. \end{aligned}\tag{A.2}$$

Then, since the probability of $\{j\in[i]_m\}$ is $\frac{P-1}{N-1}$, we get

$$\begin{aligned} \mathbb{E}f_{m,i}(\mathcal{P},\bar\nu)&=\frac{\kappa}{P-1}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\mathbb{E}\mathbf{1}_{\{j\in[i]_m\}}\\ &=\frac{\kappa}{P-1}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\frac{P-1}{N-1}\\ &=\frac{\kappa}{N-1}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big). \end{aligned}\tag{A.3}$$

From (A.2) and (A.3), one has

$$\chi_{m,i}(\mathcal{P},\bar\nu)=f_{m,i}(\mathcal{P},\bar\nu)-\mathbb{E}f_{m,i}(\mathcal{P},\bar\nu), \tag{A.4}$$

which trivially implies

$$\mathbb{E}[\chi_{m,i}(\mathcal{P},\bar\nu)]=0.$$

(ii) It follows from (A.4) that

$$\mathrm{Var}\left[\chi_{m,i}(\mathcal{P},\bar\nu)\right]=\mathbb{E}|\chi_{m,i}(\mathcal{P},\bar\nu)|^2=\mathbb{E}\Big(|f_{m,i}(\mathcal{P},\bar\nu)|^2\Big)-\Big(\mathbb{E}f_{m,i}(\mathcal{P},\bar\nu)\Big)^2. \tag{A.5}$$

Next, we compute the term $\mathbb{E}\Big(|f_{m,i}(\mathcal{P},\bar\nu)|^2\Big)$.

By (A.2), one has

$$\begin{aligned} |f_{m,i}(\mathcal{P},\bar\nu)|^2&=\Big(\frac{\kappa}{P-1}\Big)^2\sum_{j\in[i]_m}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\sum_{k\in[i]_m}\Big(\bar\nu_{ik}+a_{ik}\Gamma(p_k-p_i)\Big)\\ &=\Big(\frac{\kappa}{P-1}\Big)^2\Big\{\sum_{j\in[i]_m}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)^2\\ &\qquad+\sum_{j\in[i]_m}\sum_{\substack{k\in[i]_m\\k\neq j}}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\Big(\bar\nu_{ik}+a_{ik}\Gamma(p_k-p_i)\Big)\Big\} \end{aligned}\tag{A.6}$$

Now we apply expectation to both sides of (A.6) to get

$$
\begin{aligned}
\mathbb{E}\Big(|f_{m,i}(\mathcal{P},\bar\nu)|^2\Big) &= \Big(\frac{\kappa}{P-1}\Big)^2\Big\{\mathbb{E}\sum_{j\in[i]_m}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)^2 \\
&\qquad + \mathbb{E}\sum_{j\in[i]_m}\sum_{\substack{k\in[i]_m\\ k\neq j}}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\Big(\bar\nu_{ik}+a_{ik}\Gamma(p_k-p_i)\Big)\Big\} \\
&=: \Big(\frac{\kappa}{P-1}\Big)^2\Big\{\mathcal{I}_1+\mathcal{I}_2\Big\}.
\end{aligned}
\tag{A.7}
$$

In the sequel, we evaluate the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ one by one.

• (Computation of $\mathcal{I}_1$): Similar to (A.3), one has

$$
\mathcal{I}_1 = \frac{P-1}{N-1}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)^2. \tag{A.8}
$$

• (Computation of $\mathcal{I}_2$): We use

$$
\mathbb{E}\Big(\mathbf{1}_{\{j,k\in[i]_m,\ k\neq j\}}\Big) = \mathbb{P}\{j\in[i]_m\}\mathbb{P}\{k\in[i]_m,\ k\neq j \mid j\in[i]_m\} = \frac{P-1}{N-1}\frac{P-2}{N-2},
$$

to get

$$
\mathcal{I}_2 = \frac{P-1}{N-1}\frac{P-2}{N-2}\sum_{j\neq i}\sum_{k\neq i,j}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\Big(\bar\nu_{ik}+a_{ik}\Gamma(p_k-p_i)\Big). \tag{A.9}
$$

In (A.7), we combine estimates (A.8) and (A.9) to get

$$
\begin{aligned}
&\mathbb{E}\Big(|f_{m,i}(\mathcal{P},\bar\nu)|^2\Big) \\
&\quad = \frac{P-1}{N-1}\cdot\frac{\kappa^2}{(P-1)^2}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)^2 \\
&\quad + \frac{(P-1)(P-2)}{(N-1)(N-2)}\cdot\frac{\kappa^2}{(P-1)^2}\sum_{j\neq i}\sum_{k\neq i,j}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\Big(\bar\nu_{ik}+a_{ik}\Gamma(p_k-p_i)\Big).
\end{aligned}
\tag{A.10}
$$

For the evaluation of $\Big(\mathbb{E}f_{m,i}(\mathcal{P},\bar\nu)\Big)^2$, we use (A.3) and similar argument as for $\mathcal{I}_i$ to derive

$$
\begin{aligned}
\Big(\mathbb{E}f_{m,i}(\mathcal{P},\bar\nu)\Big)^2 &= \frac{\kappa^2}{(N-1)^2}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)^2 \\
&\qquad + \frac{\kappa^2}{(N-1)^2}\sum_{j\neq i}\sum_{k\neq i,j}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\Big(\bar\nu_{ik}+a_{ik}\Gamma(p_k-p_i)\Big).
\end{aligned}
\tag{A.11}
$$

In (A.5), we use (A.10) and (A.11) to find

$$
\begin{aligned}
\mathrm{Var}\left[\chi_{m,i}(\mathcal{P},\bar\nu)\right] &= \left(\frac{1}{P-1}-\frac{1}{N-1}\right)\Bigg(\frac{\kappa^2}{N-1}\sum_{j\neq i}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)^2 \\
&\quad -\frac{\kappa^2}{(N-1)(N-2)}\sum_{j\neq i}\sum_{k\neq i,j}\Big(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i)\Big)\Big(\bar\nu_{ik}+a_{ik}\Gamma(p_k-p_i)\Big)\Bigg) \\
&= \left(\frac{1}{P-1}-\frac{1}{N-1}\right)\frac{\kappa^2}{N-2}\sum_{j\neq i}\left|(\bar\nu_{ij}+a_{ij}\Gamma(p_j-p_i))-\frac{1}{N-1}\sum_{k\neq i}(\bar\nu_{ik}+a_{ik}\Gamma(p_k-p_i))\right|^2 \\
&= \left(\frac{1}{P-1}-\frac{1}{N-1}\right)\Lambda_i(\mathcal{P},\bar\nu).
\end{aligned}
$$

This completes the proof of Lemma 6.2.

## Appendix B. Proof of Lemma 6.4

The main ingredient is the moment estimates of $\chi_{m,i}$, Lemma 6.2. Since $Q^R(\tau_m)$ is independent of the random batches at $[\tau_m,\tau_{m+1})$, we can treat it as a constant vector:

$$
\begin{aligned}
\mathbb{E}\left|\chi_{m,i}(Q^R(\tau_m),\bar\nu)\right|^2 &= \mathbb{E}\left[\mathbb{E}\left[|\chi_{m,i}(Q^R(\tau_m),\bar\nu)|^2 \mid \mathcal{F}_m\right]\right] \\
&= \mathbb{E}\left[\mathrm{Var}\left[\chi_{m,i}(Q^R(\tau_m),\bar\nu)\mid \mathcal{F}_m\right]\right] \\
&\le C\left(\frac{1}{P-1}-\frac{1}{N-1}\right),
\end{aligned}
\tag{B.1}
$$

where $\mathcal{F}_m$ denotes the $\sigma$-algebra generated from the random choices before $t=\tau_m$ and we used the boundedness $|\Lambda_i(Q^R(\tau_m),\bar\nu)|\le C$. In order to use (B.1), we split the terms of $\mathcal{R}_i(t)$ into the differences as follows:

$$
\begin{aligned}
\mathcal{R}_i(t) &= \mathbb{E}[z_i(t)\cdot\chi_{m,i}(Q^R(t),\bar\nu)] \\
&= \mathbb{E}[(z_i(t)-z_i(\tau_m))\cdot\chi_{m,i}(Q^R(\tau_m),\bar\nu)] \\
&+ \mathbb{E}[z_i(\tau_m)\cdot\chi_{m,i}(Q^R(\tau_m),\bar\nu)] \\
&+ \mathbb{E}[z_i(t)\cdot(\chi_{m,i}(Q^R(t),\bar\nu)-\chi_{m,i}(Q^R(\tau_m),\bar\nu)] \\
&=: \mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5.
\end{aligned}
$$

In the sequel, we estimate $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ one by one.

- (Estimate of $\mathcal{I}_3$): Note that

$$
\begin{aligned}
\frac{dz_i}{dt} &= \frac{\kappa}{P-1}\sum_{j\in[i]_m}[\bar\nu_{ij}+a_{ij}\Gamma(q_j^R-q_i^R)]-\frac{\kappa}{N-1}\sum_{j\neq i}[\bar\nu_{ij}+a_{ij}\Gamma(q_j-q_i)] \\
&= \frac{\kappa}{P-1}\sum_{j\in[i]_m}a_{ij}\left[\Gamma(q_j^R-q_i^R)-\Gamma(q_j-q_i)\right]+\chi_{m,i}(Q,\bar\nu),\quad t\in[\tau_m,\tau_{m+1}).
\end{aligned}
\tag{B.2}
$$

Then, we use (B.2) and defining relation of $\mathcal{I}_3$ to get

$$
\begin{aligned}
\mathcal{I}_3 &= \mathbb{E}[(z_i(t) - z_i(\tau_m)) \cdot \chi_{m,i}(Q^R(\tau_m), \bar{\nu})] \\
&= \mathbb{E}\left[\Big(\int_{\tau_m}^t \frac{dz_i}{ds} ds\Big) \cdot \chi_{m,i}(Q^R(\tau_m), \bar{\nu})\right] \\
&\le \mathbb{E}\Big[\Big(\int_{\tau_m}^t \frac{\kappa}{P-1} \sum_{j \in [i]_m} a_{ij} \left[\Gamma(q_j^R - q_i^R) - \Gamma(q_j - q_i)\right] ds\Big) \cdot \chi_{m,i}(Q^R(\tau_m), \bar{\nu})\Big] \\
&\quad + \int_{\tau_m}^t \mathbb{E}\left[\chi_{m,i}(Q(s), \bar{\nu}) \cdot \chi_{m,i}(Q^R(\tau_m), \bar{\nu})\right] ds \\
&=: \mathcal{I}_{31} + \mathcal{I}_{32}.
\end{aligned} \tag{B.3}
$$

Below, we estimate two terms in (B.3) as follows.

$\diamond$ (Estimate of $\mathcal{I}_{31}$): In this case, one has

$$
\mathcal{I}_{31} \le C\mathbb{E}\Big(\int_{\tau_m}^t \Big|\frac{\kappa}{P-1} \sum_{j \in [i]_m} a_{ij} \left[\Gamma(q_j^R - q_i^R) - \Gamma(q_j - q_i)\right] \Big| ds\Big). \tag{B.4}
$$

By Lipschitz continuity of $\Gamma$, we have

$$
\frac{\kappa}{P-1} \sum_{j \in [i]_m} a_{ij} \left[\Gamma(q_j^R - q_i^R) - \Gamma(q_j - q_i)\right] \le \frac{\kappa L_2}{P-1} \sum_{j \in [i]_m} a_{ij}(|z_i| + |z_j|). \tag{B.5}
$$

On the other hand, we use the boundedness of $\frac{dz_i}{dt}$ to see

$$
\begin{aligned}
&|z_i(s)| - |z_i(\tau_m)| \le |z_i(s) - z_i(\tau_m)| = \Big|\int_{\tau_m}^s \frac{dz_i(r)}{dr}\Big| \le C\tau, \quad s \in [\tau_m, \tau_{m+1}), \\
&\text{i.e.,} \quad |z_i(s)| \le |z_i(\tau_m)| + C\tau, \quad s \in [\tau_m, \tau_{m+1}).
\end{aligned} \tag{B.6}
$$

Then, we use (B.5) and (B.6) to get that, for $t \in [\tau_m, \tau_{m+1})$,

$$
\begin{aligned}
&\int_{\tau_m}^t \Big|\frac{\kappa}{P-1} \sum_{j \in [i]_m} a_{ij} \left[\Gamma(q_j^R(s) - q_i^R(s)) - \Gamma(q_j(s) - q_i(s))\right] \Big| ds \\
&\quad \le \frac{\kappa L_2}{P-1} \sum_{j \in [i]_m} a_{ij} \int_{\tau_m}^t (|z_i(s)| + |z_j(s)|) ds \\
&\quad \le \frac{\kappa L_2}{P-1} \sum_{j \in [i]_m} \int_{\tau_m}^t (|z_i(\tau_m)| + |z_j(\tau_m)| + C\tau) ds \\
&\quad \le \frac{C\tau}{P-1} \sum_{j \in [i]_m} (|z_i(\tau_m)| + |z_j(\tau_m)|) + C\tau^2.
\end{aligned} \tag{B.7}
$$

Now, we combine (B.4) and (B.7) to find

$$
\mathcal{I}_{31} \le C\Big(\frac{\tau}{N-1} \sum_{j \ne i} (\mathbb{E}|z_i(\tau_m)| + \mathbb{E}|z_j(\tau_m)|) + \tau^2\Big), \tag{B.8}
$$

where we used

$$
\mathbb{P}(j \in [i]_m) = \frac{P-1}{N-1}.
$$

By the same way as in (B.6), we have

$$|z_i(\tau_m)| \le |z_i(t)| + C\tau. \tag{B.9}$$

We combine (B.8) and (B.9) to deduce

$$\mathcal{I}_{31} \le C\Big(\frac{\tau}{N-1}\sum_{j\neq i}(\mathbb{E}|z_i(t)| + \mathbb{E}|z_j(t)|) + \tau^2\Big). \tag{B.10}$$

$\diamond$ (Estimate of $\mathcal{I}_{32}$): We use the Cauchy-Schwarz inequality and boundedness of $\Lambda_i$ to find

$$\begin{aligned}\mathcal{I}_{32} &= \int_{\tau_m}^t \mathbb{E}\left[\chi_{m,i}(Q(s),\bar\nu)\cdot\chi_{m,i}(Q^R(\tau_m),\bar\nu)\right] ds\\ &\le \int_{\tau_m}^t \sqrt{\mathbb{E}|\chi_{m,i}(Q(s),\bar\nu)|^2}\sqrt{\mathbb{E}|\chi_{m,i}(Q^R(\tau_m),\bar\nu)|^2}ds\\ &\le C\tau\left(\frac{1}{P-1}-\frac{1}{N-1}\right).\end{aligned} \tag{B.11}$$

In (B.3), we combine the estimates (B.10) and (B.11) to get

$$\mathcal{I}_3 \le C\Big(\frac{\tau}{N-1}\sum_{j\neq i}(\mathbb{E}|z_i(t)| + \mathbb{E}|z_j(t)|) + \tau^2\Big) + C\tau\left(\frac{1}{P-1}-\frac{1}{N-1}\right).$$

• (Estimate of $\mathcal{I}_4$): This term is zero since the time is fixed at $\tau_m$, which is independent of the random choice at $\tau_m$:

$$\begin{aligned}\mathcal{I}_4 &= \mathbb{E}[z_i(\tau_m)\cdot\chi_{m,i}(Q^R(\tau_m),\bar\nu)]\\ &= \mathbb{E}\left[z_i(\tau_m)\cdot\mathbb{E}[\chi_{m,i}(Q^R(\tau_m),\bar\nu)\mid\mathcal{F}_m]\right] = 0.\end{aligned}$$

• (Estimate of $\mathcal{I}_5$): By the Cauchy-Schwarz inequality, one has

$$\begin{aligned}\mathcal{I}_5 &= \mathbb{E}[z_i(t)\cdot(\chi_{m,i}(Q^R(t),\bar\nu) - \chi_{m,i}(Q^R(\tau_m),\bar\nu))]\\ &\le \sqrt{\mathbb{E}|z_i(t)|^2}\sqrt{\mathbb{E}|\chi_{m,i}(Q^R(t),\bar\nu) - \chi_{m,i}(Q^R(\tau_m),\bar\nu)|^2}.\end{aligned} \tag{B.12}$$

Now we claim that the second factor in (B.12) satisfies

$$|\chi_{m,i}(Q^R(t),\bar\nu) - \chi_{m,i}(Q^R(\tau_m),\bar\nu)|^2 \le C\tau^2. \tag{B.13}$$

For this, we use defining relation (A.1) of $\chi_{m,i}(Q^R(t),\bar\nu)$, the Lipschitz continuity of $\Gamma$ and $Q^R$ and the uniform boundedness,

$$|q_j^R - q_i^R| + \Big|\frac{d}{dt}(q_j^R - q_i^R)\Big| \le C$$

to see

$$\begin{aligned}&\Big|\frac{d}{dt}\chi_{m,i}(Q^R(t),\bar\nu)\Big|\\ &= \Big|\frac{d}{dt}\Big[\frac{\kappa}{P-1}\sum_{j\in[i]_m}\Big(\bar\nu_{ij} + a_{ij}\Gamma(q_j^R - q_i^R)\Big) - \frac{\kappa}{N-1}\sum_{j\neq i}\Big(\bar\nu_{ij} + a_{ij}\Gamma(q_j^R - q_i^R)\Big)\Big]\Big|\\ &\le C.\end{aligned}$$

This implies

$$\Big|\chi_{m,i}(Q^R(t),\bar{\nu}) - \chi_{m,i}(Q^R(\tau_m),\bar{\nu})\Big|^2 = \Big|\int_{\tau_m}^t \frac{d}{ds}\chi_{m,i}(Q^R(s),\bar{\nu})ds\Big|^2 \le C\tau^2,$$

which verifies (B.13). Now, we combine (B.12) and (B.13) to obtain

$$\mathcal{I}_5 \le C\tau\sqrt{\mathbb{E}|z_i(t)|^2}$$

for some positive constant $C$. Finally, we collect all the estimates for $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ to find

$$\mathcal{R}_i(t) \le C\tau\Big[\frac{1}{N-1}\sum_{j\neq i}\mathbb{E}|z_j(t)| + \mathbb{E}|z_i(t)| + \left(\frac{1}{P-1} - \frac{1}{N-1}\right)\Big] + C\tau^2.$$

This completes the proof of Lemma 6.4.

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