Wave Patterns and Slow Motions in Inviscid and Viscous Hyperbolic Equations with Stiff Reaction Terms

Haitao Fan,[1] Shi Jin[2]

Abstract

We study the behavior of solutions to the inviscid ($A = 0$) and the viscous ($A > 0$) hyperbolic conservations laws with stiff source terms

$$u_t + f(u)_x = -\frac{1}{\epsilon} W'(u) + \epsilon A u_{xx} \, ,$$

with $W(u)$ being the double-well potential. The initial value problem of this equation gives, to the leading order, piecewise constants solutions connected by shock layers and rarefaction layers. In this paper we establish the layer motion for the inviscid case at the next order, which moves exponentially slowly. In the viscous case we study the patterns of the traveling wave solutions and structures of the internal layers.

[1]Department of Mathematics, Georgetown University, Washington, DC 20057, USA. Email address: fan@gumath1.math.georgetown.edu. Research supported by NSF grant No. DMS 9705732

[2]Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA. Email address: jin@math.wisc.edu. Research was partly supported by NSF grants No. DMS-9704957 and No. DMS-0072374.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Introduction

In this paper we study the inviscid hyperbolic equations with stiff source term

$$(1.1)\qquad \begin{aligned} u_t + f(u)_x &= \frac{1}{\epsilon} u(1-u^2), \\ u(x,0) &= u_0(x); \end{aligned}$$

and the viscous conservation laws with stiff source term

$$(1.2)\qquad \begin{aligned} u_t + f(u)_x &= \frac{1}{\epsilon} u(1-u^2) + \epsilon A u_{xx} \\ u(x,0) &= u_0(x)\,, \end{aligned}$$

In these problems, $A\epsilon > 0$ is the viscosity coefficient, and $\epsilon > 0$ is the reaction time. These are the simplest models for reacting flows, where the source term, being the derivative of the typical double well potential, accounts for the chemical reaction.

Since most equations governing reacting flows or dynamics of phase transitions are combinations of inhomogeneous fluid dynamics equations and reaction-diffusion equations [AK, VK], equations (1.1) and (1.2) can serve as prototype models to study issues involved in reacting flows. In a reacting flow, the typical scale of the reaction time ϵ is much smaller than the characteristic time scale of the fluid, which makes the source terms in (1.1) and (1.2) stiff (in the so-called fast reaction regime).

First notice that the source terms in (1.1) and (1.2) admit three local equilibria, namely, 0 and ± 1, with 0 being linearly unstable, while ± 1 linearly stable. It was justified in our earlier work [FJT] that as $\epsilon \to 0$, the solution of (1.1) tends to the two stable local equilibria ± 1. When $f(u)$ is convex, the limiting piecewise constant solutions are connected either by a shock which connects 1 from the left to -1 from the right and travels with speed $\frac{1}{2} f(1) - f(-1)$ as determined by the Rankine-Hugoniot jump condition, or by a rarefaction discontinuity that connects -1 from the left to 1 from the right and travels with characteristic speed at the unstable equilibrium, $f'(0)$. These results were extended to the case of the non-convex flux function $f(u)$ in [Mas1].

While the $\epsilon \to 0$ limit reveals the leading order behavior of the solution, it is often interesting to study the behavior at the **next order**, which is secondary in importance and affects the long-time dynamics and the steady state solution. Such questions have been addressed in the context of reaction-diffusion equations [CP, FH, RSK, WR], or in complex fields described by the nonlinear heat equation and nonlinear

Schrödinger equation [Neu, E]. In these examples, the next order solution exhibits exponentially slow motion, since the Ginzburg-Landau type potential, examplified by the source term in (1.1), has exponentially small eigenvalues. Slow motions in viscous conservation laws were studied in [KK, KKP, LO1, RW]. Slow motions in the boundary value problem of a single shock layer was discussed in [LO2]. The more general case of slow motions due to the interactions of neighboring shock and rarefaction layers in the convection-reaction equations or convection-reaction-diffusion equations remains to be explored.

For $\epsilon > 0$, the shock remains a discontinuity but the rarefaction discontinuity in the $\epsilon = 0$ case is a sharp layer of width $O(\epsilon)$, which will be refered to as the *rarefaction layer*.

In order to focus our study on the behavior of the solution at the next order, we assume that, at the leading order, the solution does not move. Namely, we assume that $f(-1) - f(1) = 0$ and $f'(0) = 0$. This happens in, for example, the Burger's flux $f(u) = u^2/2$. When this occurs, the two types of layers do not interact at the $\epsilon \to 0+$ limit, thus it is quite tempting to believe that such a solution will presist for all time. However, easy calculation shows that this solution is not the steady state solution to (1.1). For $\epsilon > 0$, there is a thin layer of width ϵ near the discontinuities, within which the solution decays to ± 1 exponentially. The interactions between two neighboring layers, though exponentially weak, do have a long (exponentially long!) effect on the solution, and our analysis shows that it takes an exponentially-long time for such an interaction to have an $O(1)$ effect on the solution and ultimately the steady state, since the layers move with exponentially small speeds. One of the goals of this paper is to determine such speeds.

Consider the initial data of the form:

$$u(x,0) = u_0(x) = \begin{cases} 1 + O(1)\exp\left(-\frac{M}{\epsilon}|x - a_{2j}|\right) & \text{for} \quad a_{2j+1} > x > a_{2j}\,, \\ -1 + O(1)\exp\left(-\frac{M}{\epsilon}|x - a_{2j}|\right) & \text{for} \quad a_{2j-1} < x < a_{2j}\,, \end{cases} \tag{1.3}$$
$$j = 1, 2, ...2n\,,$$

where $-\infty = a_0 < a_1 < a_2 < \cdots < a_{2n} < a_{2n+1} = \infty$, being constants with $O(1)$ differences, are the locations of $u = 0$ initially. The constant M satisfies

$$0 < M \leq \min_{u \in [-1,1]} \frac{2}{f''(u)}. \tag{1.4}$$

The initial data (1.3) are depicted in Fig.1.1, where a_{2j-1} corresponds to the shock location, while a_{2j} corresponds to the locations of the rarefaction layers.

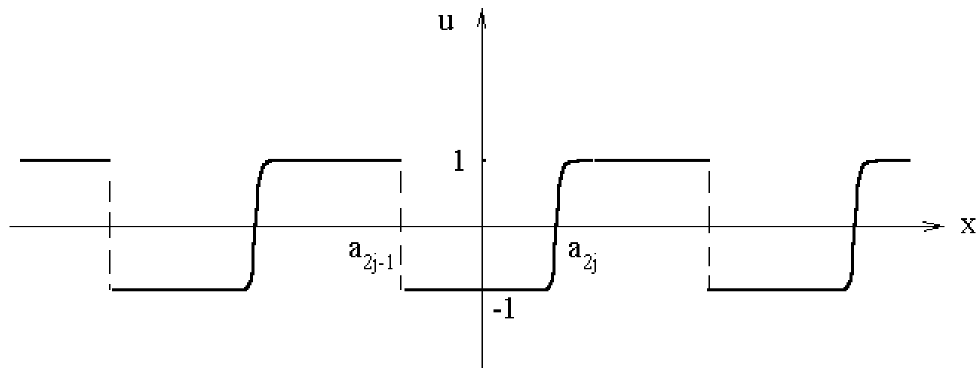

Figure 1.1

It is easy to show by an asymptotic analysis (see section 2.4) that, beyond the initial layer of length $O(\epsilon)$, the solution asymptotically tends to the form (1.3). Thus we do not lose any generality to start with the initial data (1.3).

One of our main results is the dynamics of the slow motion, given by the following theorem.

Theorem 1.1. *Let $a_j(t)$ be the location of $u = 0$. There exists an $T > 0$, such that for $t < T$, the speed of the rarefaction layer is given by*

$$\frac{da_{2j}(t)}{dt} = 0, \quad j = 1, 2, ..., n\,, \tag{1.5}$$

while the speed of the shock is

$$\frac{da_{2j-1}(t)}{dt} = O(1)\exp\left(-\frac{M}{\epsilon}\min(|a_{2j-1}(t) - a_{2j}(t)|, |a_{2j-1}(t) - a_{2j-2}(t)|)\right), \tag{1.6}$$
$$j = 1, 2, ..., n.$$

This result shows that the rarefaction layer will never move, while the shock moves with an exponentially small speed. Therefore one should expect the strong interactions among the neighboring layers after an exponentially long time. This motion is determined by using the generalized characteristics method [Daf], a purely hyperbolic technique, while previous works on the slow motion problems in reaction-diffusion equations or the nonlinear Schrödinger equation use the matched asymptotic expansions or spectral analysis [RSK, CP, Neu, E, RW, WR, JO1, JO2] to determine the dynamics of the slow motion.

For the viscous model (1.2), we prove the existence of traveling wave profiles for both the shock layer and the rarefaction layer. We also construct the asymptotic structures of the shock and rarefaction layers. We expect slow motion of these layers in the viscous case, similar to the results stated in Theorem 1.1. However, this topic is left for future investigation.

We point out that there have been increasing activities in the study of problems (1.1) and (1.2) in recent years. The long-time behavior and attractors of hyperbolic conservation laws with source term that admits multiple equilibria similar to (1.2), but in the non-stiff regime $\epsilon = O(1)$, have been studied by several authors [FH1, FH2, Har, Lyb, Mas1, MS, Sin1, Sin2, Sin3]. As mentioned earlier, the zero reaction time limit $\epsilon \to 0+$ of (1.1) were studied in [FJT] when the source term is of bistable type. This result was generalized to the monostable type of source term in [F], while in [FHa], the degenerate bistable type of source terms were considered. Both large time behavior and zero reaction time limit was investigated. It turns out the zero reaction time limit in the degenerate case is similar to the non-degenerate case, although the convergence rate is of algebraic. The large time behavior, however, is different from the non-degenerate case. A novel numerical method, called the random projection method, for problem (1.1) which allows the reaction time ϵ not to be numerically resolved, was developed in [BJ].

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 on the slow motion of waves for the inviscid case. In Section 3, we study the viscous case, (1.2). We establish the existence of wave profiles for both the shock and rarefaction layers and construct their asymptotic structures.

2. Slow Motion in the Inviscid Case

2.1. Basic waves for inviscid conservation laws with source terms

It is clear from the usual entropy condition for conservation laws with convex flux function f, entropy solutions $u(x,t)$ of (1.1) admits jump discontinuities at $x = s(t)$

that satisfy $u(s(t)-,t) > u(s(t)+,t)$. The speed of the shock at $x = s(t)$ is determined by the Rankine-Hugoniot jump condition

$$\frac{ds}{dt} = \frac{f(u(s(t)+,t) - f(u(s(t)-,t)}{u(s(t)+,t) - u(s(t)-,t)}. \tag{2.1.1}$$

Among all entropy shocks, only the one with $u(s(t)\pm,t) = \mp 1$ has a constant speed, which is $(f(1) - f(-1))/2$.

There is a traveling wave $\phi(x - ct)$ of (1.1) with $\phi(\pm\infty) = \pm 1$. It is the solution of

$$\begin{aligned} -c\phi' + f(\phi)' &= \frac{1}{\epsilon} g(\phi), \\ \phi(\pm\infty) &= \pm 1\,, \end{aligned} \tag{2.1.2}$$

where "$'$" denotes $\frac{d}{d\xi}$ and $\xi = x - ct$.

Theorem 2.1.1. *Equation (2.1.2) has a solution if and only if $c = f'(0)$ where 0 is the unstable equilibrium point of (1.1). Furthermore, solutions ϕ satisfy*

$$\phi(\xi) = 1 + O(1)\exp\left(\frac{2}{\epsilon(f'(0) - f'(1))}\xi\right), \text{ as } \xi \to \infty\,, \tag{2.1.3a}$$

and

$$\phi(\xi) = -1 + O(1)\exp\left(\frac{2}{\epsilon(f'(0) - f'(-1))}\xi\right), \text{ as } \xi \to -\infty\,. \tag{2.1.3b}$$

Proof. If $c \neq f'(0)$, then $\phi'(\xi_0) = 0$ when $\phi(\xi_0) = 0$ and hence $\phi \equiv \phi(\xi_0)$ by the uniqueness of initial value problems of ordinary differential equations. Thus, boundary conditions in (2.1.1) cannot be satisfied when $c \neq f'(0)$. If $c = f'(0)$, then $\phi' > 0$ for all $\phi \in (-1, 1)$ and hence (2.1.2) has a solution. The first statement of the theorem then follows.

Estimates (2.1.3) follows immediately from the fact that the eigenvalues of (2.1.2) at $u = \pm 1$ are

$$\frac{2}{\epsilon(f'(0) - f'(\pm 1))}\,.$$

∎

Clearly, the traveling wave in Theorem 2.1.1 corresponds to the rarefaction layer. There are other continuous traveling waves connecting 0 to 1 or -1. Although we do not use them in this paper, for completeness, we list the results on the existence of these waves.

Theorem 2.1.2. *Traveling wave equation* $(2.1.2)_1$ *has a solution if one the following holds.*

(i) $\phi(-\infty) = 0$, $\phi(\infty) = 1$ *and* $f'(0) > c$.

(ii) $\phi(-\infty) = 0$, $\phi(\infty) = -1$ *and* $f'(-1) > c$.

(iii) $\phi(-\infty) = -1$, $\phi(\infty) = 0$ *and* $f'(0) < c$.

(iv) $\phi(-\infty) = 1$, $\phi(\infty) = 0$ *and* $f'(1) < c$.

Proof. Omitted. ∎

2.2. Generalized Characteristics of (1.1)

In the next section, we shall use the extremal backward characteristics of (1.1) in our study of the slow motion in some wave patterns. Here, we recall some properties of extremal backward characteristics.

A Lipschitzian curve $x = \xi(t)$ defined on an interval $[a, b]$ is called a characteristic curve associated to the solution $u(x, t)$ of (1.1) if, for almost all $t \in [a, b]$,

$$\frac{d\xi}{dt} \in [f'(u(\xi(t)+, t)), f'(u(\xi(t)-, t))]. \tag{2.2.1}$$

¿From [Fil], for any $(\bar{x}, \bar{t}) \in \mathbb{R}\times(0, \infty)$, there exists at least one backward characteristic $\xi(t; \bar{x}, \bar{t})$ defined on a maximal interval $(s, \bar{t}]$, $s \geq 0$, with $\xi(\bar{t}; \bar{x}, \bar{t}) = \bar{x}$. The set of all backward characteristics through $(\bar{x}, \bar{t})$ form a funnel confined between the minimal and the maximal backward characteristics through $(\bar{x}, \bar{t})$. We denote the minimal and maximal backward characteristics by $\xi_-(t; \bar{x}, \bar{t})$ and $\xi_+(t; \bar{x}, \bar{t})$ respectively. The following Lemmas 2.2.1, 2.2.2 and 2.2.3 are from [Daf], while the rest results are from [FJT].

Lemma 2.2.1. *The extremal backward characteristic* $\xi_\pm(t; \bar{x}, \bar{t})$ *associated with the solution* $u(x, t)$ *of (1.1) satisfies, for* $t \in (s, \bar{t}]$,

$$\begin{aligned} \frac{d\xi}{dt} &= f'(v(t)), \\ \frac{dv}{dt} &= \frac{1}{\epsilon} v(1 - v^2)\,, \end{aligned} \tag{2.2.2}$$

with initial conditions

$$(\xi_-(\bar{t}; \bar{x}, \bar{t}), v(\bar{t})) = (\bar{x}, u(\bar{x}-, \bar{t})) \tag{2.2.3-}$$

and

$$(\xi_+(\bar{t};\bar{x},\bar{t}),v(\bar{t}))=(\bar{x},u(\bar{x}+,\bar{t})) \tag{2.2.3+}$$

respectively. Furthermore, for both $\xi(t):=\xi_-(t)$ and $\xi(t):=\xi_+(t)$, equations

$$v(t)=u(\xi(t)-,t)=u(\xi(t)+,t) \tag{2.2.4}$$

hold for almost all $t\in(s,\bar{t}]$.

Lemma 2.2.2. *Any two extremal backward characteristics do not intersect.*

Lemma 2.2.3. *If the solution $|u(x,t)|\le C$ for some constant $C>0$, then backward characteristics associated with $u(x,t)$ are defined on $[0,\bar{t}]$.*

Lemma 2.2.4. *If $f\in C^1(\mathbb{R};\mathbb{R})$ and $1\ge u_0(x)\ge 0$, then the solution of (1.1) satisfies*

$$1\ge u(x,t)\ge 0 \qquad \text{for} \quad t>0\,. \tag{2.2.5}$$

Corollary 2.2.5. *Backward characteristics through the point $(\bar{x},\bar{t})$ are defined on $[0,\bar{t}]$.*

We say that a curve $x=\zeta(t)$ crosses the curve $x=\xi(t)$ from the left (right) as t decreases if there is a t_0 in the domains of definition of ζ and ξ such that $\zeta(t_0)=\xi(t_0)$ and $\zeta(t_0+\delta)<(>)\xi(t_0+\delta)$, $\zeta(t_0-\delta)>(<)\xi(t_0-\delta)$ for sufficiently small $\delta>0$.

Lemma 2.2.6. *¿From any point $(\bar{x},\bar{t})$, $\bar{t}>0$, there is a unique forward generalized characteristics $\zeta(t;\bar{x},\bar{t})$ of (1.1) defined as*

$$\frac{d\zeta}{dt}=\begin{cases} f'(u(\zeta(t),t)), & \text{if } u(\zeta(t)-,t)=u(\zeta(t)+,t),\\ \frac{f(u(\zeta(t)-,t))-f(u(\zeta(t)+,t))}{u(\zeta(t)-,t)-u(\zeta(t)+,t)}, & \text{if } u(\zeta(t)-,t)>u(\zeta(t)+,t),\end{cases} \tag{2.2.7}$$
$$\zeta(\bar{t})=\bar{x},t>\bar{t}.$$

Furthermore a minimal backward characteristics $x=\xi_-(t)$ of (1.1) cannot cross $x=\zeta(t)$ from the left as t decreases. Similarly, a maximal backward characteristics $x=\xi_+(t)$ of (1.1) cannot cross $x=\zeta(t)$ from the right as t decreases.

2.3. Slow motions in inviscid conservation laws with source terms

¿From last section, we see that there are two types of traveling waves of (1.1) that connect stable equilibrium points. One of them is the ordinary shock wave at $x = s(t)$ with $u(s(t)-, t) = 1 = -u(s(t)+, t)$. The other type of the wave is the continuous traveling wave of (1.1) provided by Theorem 2.1.1, corresponding to the rarefaction layer. Since the goal is to understand the behavior of these waves at the next order, we set $f'(0) = f(1) - f(-1) = 0$ so that the speed of both waves, at the leading order, are 0. We shall investigate the interaction between two types of basic waves of (1.1). For this purpose, we consider the following initial value problem of (1.1):

(2.3.1)

$$u_t + f(u)_x = \frac{1}{\epsilon}u(1-u^2),$$

$$u(x,0) = u_0(x) = \begin{cases} 1 + O(1)\exp\left(-\frac{M}{\epsilon}|x-a_{2j}|\right), & \text{for } a_{2j+1} > x > a_{2j}, \\ -1 + O(1)\exp\left(-\frac{M}{\epsilon}|x-a_{2j}|\right), & \text{for } a_{2j-1} < x < a_{2j}, \end{cases}$$

$$j = 1, 2, ...2n,$$

where $-\infty = a_0 < a_1 < a_2 < \cdots < a_{2n} < a_{2n+1} = \infty$ are constants, and the constant

$$0 < M \leq \min_{u\in[-1,1]} \frac{2}{f''(u)}. \tag{2.3.2}$$

Here, we only present the case where there are even number of a_j's for the simplicity of presentation. The cases when there are odd number of a_j's or when $u_0(x) = -1 + O(1)e^{-M|x-a_1|/\epsilon}$ for $x < a_1$ can be handled similarly to get similar conclusions. We also leave the constants M and the coefficient in front of the exponential term unspecified. A simple matched asymptotic analysis, as presented in section 2.4, can determine these constants.

We first recall the following results on the structure of the solution of (2.3.1) obtained in [FJT]:

Lemma 2.3.1.. *There are Lipschitz continuous curves $x = a_j(t)$, $j = 1, 2, ..., 2n$, defined on $[0, T_j]$ satisfying the following:*

(i) $a_j(0) = a_j$.

(ii) Curves $x = a_j(t)$, $j = 1, 2, ..., 2n$, do not intersect each other except at $t = T_j$.

(iii) Two curves $x = a_j(t)$ and $x = a_{j'}$ may intersect only at the end points of their domain of definition, $T_j = T_{j'}$ where the curve $x = a_{j'}(t)$ is one of the curves among $x = a_k(t)$, $k = 1, 2, ..., 2n$ adjacent to $x = a_j(t)$ for t close to T_j.

(iv) The sign of $u(x\pm, t)$, the solution of (2.3.1), is fixed for x between adjacent points among $a_j(t)$.

(v) The curves $x = a_j(t)$, $j = 1, 2, ...2n$, are forward characteristics of (2.3.1).

We now determine the behavior of u near the shock and rarefaction fronts.

Lemma 2.3.2. *Let $a_j(t)$ be functions defined in Lemma 2.3.1 and the constant $M > 0$ be that in (2.3.2). If $t < \min_{1\leq j\leq 2n} T_j$, then the following hold:*
(i)

$$a_{2j}(t) \equiv a_{2j}. \tag{2.3.3}$$

(ii) If $a_{2j-1}(t) < a_{2j}$, then

$$u(a_{2j-1}(t)-, t) = -1 + O(1)\exp\left(-\frac{M}{\epsilon}|a_{2j-1}(t) - a_{2j}|\right). \tag{2.3.4}$$

(iii) If $a_{2j-1}(t) > a_{2j-2}$, then

$$u(a_{2j-1}(t)+, t) = 1 + O(1)\exp\left(-\frac{M}{\epsilon}|a_{2j-1}(t) - a_{2j-2}|\right). \tag{2.3.5}$$

Proof: (i) By the definition of $a_j(t)$, one has

$$\lim_{x\to a_{2j}(t)-} \mathrm{sign}(u(x\pm, t)) = -1, \tag{2.3.6}$$

and

$$\lim_{x\to a_{2j}(t)+} \mathrm{sign}(u(x\pm, t)) = 1. \tag{2.3.7}$$

Let $\bar{x} < a_{2j}(\bar{t})$ and close to $a_{2j}(\bar{t})$ and consider the minimal backward characteristics issued from $(\bar{x}, \bar{t})$, $\xi_-(t; \bar{x}, \bar{t})$. Since $u(\bar{x}, \bar{t}) < 0$ and the sign of u does not change along extremal backward characteristics, as seen from (2.2), one has $\frac{d\xi_-(t;\bar{x},\bar{t})}{dt} < 0$ and hence

$$\xi_-(0; \bar{x}, \bar{t}) > \xi_-(\bar{t}; \bar{x}, \bar{t}) = \bar{x}. \tag{2.3.8}$$

Also, it holds that

$$\xi_-(t; \bar{x}, \bar{t}) < a_{2j}(t), \tag{2.3.9}$$

for all $t \in (0, T_{2j})$ because if otherwise, u would change sign along $\xi_-(t; \bar{x}, \bar{t})$ which is impossible. Similarly, one can consider $\xi_+(t; \bar{x}_1, \bar{t})$ for $\bar{x}_1 > a_{2j}(\bar{t})$ and close to $a_{2j}(\bar{t})$. By the same reasoning as above, one obtains

$$\frac{d\xi_+(t; \bar{x}_1, \bar{t})}{dt} > 0, \tag{2.3.10}$$

and

$$\xi_+(t;\bar{x}_1,\bar{t}) > a_{2j}(t), \tag{2.3.11}$$

for all $t \in (0, T_{2j})$. In particular,

$$\xi_+(0;\bar{x}_1,\bar{t}) < \xi_+(\bar{t};\bar{x}_1,\bar{t}) = \bar{x}_1. \tag{2.3.12}$$

Now let $\bar{x} \to a_{2j}(\bar{t})-$ and $\bar{x}_1 \to a_{2j}(\bar{t})+$. Then estimates (2.3.8-12) yield

$$\begin{aligned} a_{2j}(\bar{t}) &= \lim_{\bar{x}\to a_{2j}(\bar{t})-} \xi_-(\bar{t};\bar{x},\bar{t}) \le \lim_{\bar{x}\to a_{2j}(\bar{t})-} \xi_-(0;\bar{x},\bar{t}) \le a_{2j}(0) \\ &\le \lim_{\bar{x}_1\to a_{2j}(\bar{t})+} \xi_+(0;\bar{x}_1,\bar{t}) \le \lim_{\bar{x}_1\to a_{2j}(\bar{t})+} \xi_+(\bar{x}_1;\bar{x}_1,\bar{t}) = a_{2j}(\bar{t})\,, \end{aligned}$$

and hence

$$a_{2j}(\bar{t}) \equiv a_{2j}(0). \tag{2.3.13}$$

(ii) We consider the case where $t < \min(T_j; j = 1, 2, ..., 2n)$ and hence $a_{2j-1}(t) < a_{2j}(0)$. Let $\bar{x} > a_{2j-1}(\bar{t})$ and close to $a_{2j-1}(\bar{t})$ enough. Then the condition $t < \min(T_j; j = 1, 2, ..., 2n)$ implies that

$$\xi_-(0;\bar{x},\bar{t}) \le a_{2j}(0), \tag{2.3.14}$$

because if otherwise, $\xi(t;\bar{x},\bar{t})$ would intersect the forward characteristics $x = a_{2j}(t) \equiv a_{2j}(0)$ from the left while $t > 0$ is decreasing, which is impossible. The choice of $\bar{x}$ and (2.3.14) imply that $a_{2j-1}(0) < \bar{x} < \xi(0,\bar{x},\bar{t}) \le a_{2j}(0)$ and hence

$$u(\xi(0,\bar{x},\bar{t}),0) \le 0\,. \tag{2.3.15}$$

Integrating the system (2.2.2) of equations governing the extremal backward characteristics of (1.1), one obtains that

$$\int_{u(\xi(0;\bar{x},\bar{t}),0)}^{u(\bar{x},\bar{t})} \frac{f'(u)-f'(0)}{u(1-u^2)}du = \frac{1}{\epsilon}(\bar{x}-\xi(0,\bar{x},\bar{t})). \tag{2.3.16}$$

There are a function $\theta(u)$ and a number $B > 0$ such that

$$\begin{aligned} &\int_{u(\xi(0;\bar{x},\bar{t}),0)}^{u(\bar{x},\bar{t})} \frac{f''(\theta(u))}{(1-u^2)}du = B\int_{u(\xi(0;\bar{x},\bar{t}),0)}^{u(\bar{x},\bar{t})} \frac{1}{(1-u^2)}du \\ &= \frac{B}{2}\ln\left|\frac{1+u}{1-u}\right| \Bigg|_{u(\xi(0;\bar{x},\bar{t}),0)}^{u(\bar{x},\bar{t})}, \end{aligned} \tag{2.3.17}$$

where $\theta(u)$ is the function satisfying

$$\frac{f'(u) - f'(0)}{u} = f''(\theta(u)) > 0\,, \tag{2.3.18}$$

and the number B is between $\min_{u\in[-1,0)}(f''(u))$ and $\max_{u\in[-1,0)}(f''(u))$ and depends on $u(\bar{x},\bar{t})$ and $u(\xi(0;\bar{x},\bar{t}),0)$ only. Plugging (2.3.17) into (2.3.16) gives

$$u(\bar{x},\bar{t}) = \frac{C_0\exp(2(\bar{x}-\xi(0;\bar{x},\bar{t}))/B) - 1}{C_0\exp(2(\bar{x}-\xi(0;\bar{x},\bar{t}))/B) + 1}, \tag{2.3.19}$$

where

$$C_0 = \frac{1+u(\xi(0;\bar{x},\bar{t}),0)}{1-u(\xi(0;\bar{x},\bar{t}),0)}. \tag{2.3.20}$$

According to (2.3.14), there are two possibilities:

Case 1. $\xi(0;\bar{x},\bar{t}) < a_{2j}(0)$.

The choice of the initial data $u_0(x)$ in (2.3.1) and the equation (2.3.20) yield that

$$C_0 = O(1)(1+u(\xi(0;\bar{x},\bar{t}),0)) = O(1)\exp\left[\frac{M}{\epsilon}(\xi(0;\bar{x},\bar{t}) - a_{2j}(0))\right]. \tag{2.3.21}$$

Combining (2.3.21) and (2.3.19) and (2.3.2), we obtain

$$u(\bar{x},\bar{t}) = -1 + O(1)\exp(-M|\bar{x} - a_{2j}(0)|/\epsilon) \tag{2.3.22}$$

as desired.

Case 2. $\xi(0;\bar{x},\bar{t}) = a_{2j}(0)$.

In this case, estimates (2.3.19) is already the desired (2.3.22) if C_0 is bounded. Indeed, C_0 is bounded due to (2.3.20), (2.3.2) and (2.3.1).

The proof of (iii) is the same of that of (ii). ∎

Now we can determine the speed of both the shock and the rarefaction fronts.

Theorem 2.3.3. *Let $a_j(t)$ be given in Lemma 2.3.1 and $t < \min_{1\le j\le 2n} T_j$. Then, the following holds:*

(i)

$$\frac{da_{2j}(t)}{dt} = 0, \quad j = 1, 2, ..., n. \tag{2.3.23}$$

(ii) If (2.3.1) and (2.3.2) holds, then

(2.3.24)

$$\frac{da_{2j-1}(t)}{dt} = O(1)\exp\left(-\frac{M}{\epsilon}\min(|a_{2j-1}(t)-a_{2j}(t)|, |a_{2j-1}(t)-a_{2j-2}(t)|)\right),$$

$j = 1, 2, ..., n.$

Proof. Statement (i) follows immediately from (i) of Lemma 2.3.2.

To prove (ii), one only needs to see that $x = a_{2j-1}(t)$ is the location of a shock jump discontinuity of (1.1). According to Lemma 2.3.2, the value of u at the two sides of the shock are

$$u(a_{2j-1}(t)-, t) = -1 + O(1)\exp\left(-\frac{M}{\epsilon}|a_{2j-1}(t) - a_{2j}|\right),$$

and

$$u(a_{2j-1}(t)+, t) = 1 + O(1)\exp\left(-\frac{M}{\epsilon}|a_{2j-1}(t) - a_{2j-2}|\right),$$

respectively. Then the Rankine-Hugoniot condition, which must be satisfied by jump discontinuities of (1.1), reads

$$\frac{da_{2j-1}(t)}{dt} = \frac{f(u(a_{2j-1}(t)+,t)) - f(u(a_{2j-1}(t)-,t),t)}{u(a_{2j-1}(t)+,t) - u(a_{2j-1}(t)-,t)}$$
$$= O(1)\exp\left(-\frac{M}{\epsilon}\min(|a_{2j-1}(t)-a_{2j}(t)|, |a_{2j-1}(t)-a_{2j-2}(t)|)\right).$$

∎

Remark 2.3.4: Theorem 2.3.3 states that when $t < \min_{1\le j\le 2n} T_j$ and $a_j(t)$, $j = 1, 2, ..., 2n$, are well separated in the sense that $|a_j(t) - a_{j+1}(t)| >> \epsilon$, then the motion of the center of shock waves $x = a_{2j-1}(t)$ is exponentially slow.

Remark: 2.3.5. When t increases to near to $t_1 := \min_{1\le j\le 2n} T_j$, the motion of one of $x = a_{2j-1}(t)$, $j = 1, 2, ..., 2n$ accelerates and collide with one of its adjacent curves among $x = a_{2j}(t)$, $j = 1, 2, ..., 2n$. After the collision, $t > t_1$, the collided curves cease to exist. The solution $u(x, t)$ right after t_1 satisfies the same condition (2.3.1) and (2.3.2) again only with n decreased by two. Thus, slow motion of $x = a_j(t)$'s resumes until near the next moment of collision of two curves among $x = a_j(t)$, $j = 1, 2, ..., 2n$.

2.4. Asymptotic layer profiles

In this subsection we use a mtached asymptotic analysis to determine the internal layer profile near a_j. This analysis allows us to determine all the constants unspecified in the previous section. We assume that $\min_j\{a_{j+1}-a_j\} >> \epsilon$. Namely, the distance between two adjacent fronts are much bigger compared to ϵ.

First, the initial layer analysis shows that [FJT] the initial data will be driven to the two linearly stable local equilibria ± 1 exponentially fast, with the positive part of the initial data goes to 1 and the negative part to -1. The structure of the solution around the point which connects from 1 to -1 can be understood asymptotically using the matched asymptotic analysis.

First we consider the profile around $x = a_{2j}$. Introducing the stretched variable

$$\xi = \frac{x - a_{2j}}{\epsilon}\,. \tag{2.4.1}$$

First consider the profile to the right of a_{2j}, namely, for $\xi > 0$. Applying the ansatz

$$u(x,t) = 1 + u_0(\xi,t) + \cdots \tag{2.4.2}$$

where $u_0(x,t)$ is monotonely increasing and satisfies the boundary and exponentially growing conditions

$$u_0(0,t) = -1\,, \qquad u_0(\xi,t) \to 0\,, \quad \text{as} \quad \xi \to +\infty \tag{2.4.3}$$

into (1.1), one gets

$$\begin{aligned} &[1 + u_0(\xi,t) + \cdots]_t + f(1 + u_0(\xi,t) + \cdots)_x \\ =&\tfrac{1}{\epsilon} g(1 + u_0(\xi,t) + \cdots)\,. \end{aligned} \tag{2.4.4}$$

Here we define $g(u) = u(1-u^2)$. By the Taylor expansion, and using $g(1) = 0$, one gets

$$\begin{aligned} &(u_0(\xi,t) + \cdots)_t + f'(1)(\tfrac{1}{\epsilon}\partial_\xi u_0(\xi,t) + \cdots) \\ =&\tfrac{1}{\epsilon} g'(1)(u_0(\xi,t) + \cdots)\,. \end{aligned} \tag{2.4.5}$$

The leading order balance of (2.4.5) gives

$$f'(1)\partial_\xi u_0 = g'(1) u_0\,.$$

This, combined with the conditions (2.4.3), give the desired solution

$$u_0 = -e^{\frac{g'(1)}{f'(1)}\xi} \tag{2.4.6}$$

since $\frac{g'(1)}{f'(1)} < 0$. Thus the asymptotic ansatz for $a_{2j} < x < a_{2j+1}$ is

$$u(x,t) = 1 - e^{\frac{g'(1)}{f'(1)}\xi} + \text{higher order terms}\,. \tag{2.4.7}$$

Now we consider the profile to the left of $x = a_{2j}$, namely, for $\xi < 0$. Applying the ansatz

$$u(x,t) = -1 + u_0(\xi,t) + \cdots \tag{2.4.8}$$

where $u_0(x,t)$ is monotonely decreasing and satisfies the boundary and exponential decaying conditions

$$u_0(0,t) = 1\,, \qquad u_0(\xi,t) \to 0 \quad \text{as} \quad \xi \to -\infty \tag{2.4.9}$$

into (1.1), one gets

$$\begin{aligned}&(-1 + u_0(\xi,t) + \cdots)_t + f(-1 + u_0(\xi,t) + \cdots)_x \\ &= \tfrac{1}{\epsilon} g(-1 + u_0(\xi,t) + \cdots)\,.\end{aligned} \tag{2.4.10}$$

By the Taylor expansion, and using $g(-1) = 0$, one gets

$$\begin{aligned}&(u_0(\xi,t) + \cdots)_t + f'(-1)(\tfrac{1}{\epsilon}\partial_\xi u_0(\xi,t) + \cdots) \\ &= \tfrac{1}{\epsilon} g'(-1)(u_0(\xi,t) + \cdots)\,.\end{aligned} \tag{2.4.11}$$

The leading order balance of (2.4.11) gives

$$f'(-1)\partial_\xi u_0 = g'(-1)u_0$$

This, combined with the conditions (2.4.9), give the desired solution

$$u_0 = e^{\frac{g'(-1)}{f'(-1)}\xi} \tag{2.4.12}$$

since $\frac{g'(-1)}{f'(-1)} > 0$. Thus the asymptotic ansatz for $a_{2j-1} < x < a_{2j}$ is

$$u(x,t) = -1 + e^{\frac{g'(-1)}{f'(-1)}\xi} + \text{higher order terms}\,. \tag{2.4.13}$$

In summary, we get

$$u(x,t) = \begin{cases} 1 - \exp\left(-\frac{g'(1)}{f'(0)-f'(1)}\frac{|x-a_{2j}|}{\epsilon}\right)\,, & \text{for } a_{2j+1} > x > a_{2j}, \\ -1 + \exp\left(\frac{g'(-1)}{f'(-1)-f'(0)}\frac{|x-a_{2j}|}{\epsilon}\right)\,, & \text{for } a_{2j-1} < x < a_{2j}. \end{cases} \tag{2.4.14}$$

$j = 1, 2, \ldots 2n.$

One can apply similar ansatz (2.4.2) or (2.4.8) around the shock discontinuity $x = a_{2j+1}$. It is easy to check that the only solution that satisfies the matching conditions (2.4.3) or (2.4.9) is the trivial solution $u_0 \equiv 0$. Thus around the shock discontinuity $x = a_{2j+1}$ one can not construct an asymptotic solution like (2.4.7) or (2.4.13), namely, there is no layer around the shock so the shock remains a discontinuity.

§3. Existence of Wave Profiles in the Viscous Case

Traveling waves of the viscous equation (1.2) are solutions of

$$-cu' + f(u)' = Au'' + g(u) \tag{3.1}$$

with boundary conditions

$$u(-\infty) = -1,\ u(\infty) = 1 \tag{3.2}$$

or

$$u(-\infty) = 1,\ u(\infty) = -1, \tag{3.3}$$

where "$'$" is $\frac{d}{d\xi}$ and $\xi = (x - ct)/\epsilon$. The condition (3.2) yields the rarefaction layer and (3.3) is for the shock layer connecting equilibrium points -1 to 1. We shall see that under the condition $f'(0) = f(1) - f(-1) = 0$, the speeds of both types of waves are 0.

§3.1. Existence of the viscous profiles for the rarefaction layer

In §2.1, we already proved that when $A = 0$, equation (3.1) with (3.2) has continuous solutions. We rewrite (3.1) and (3.2) as

$$\begin{aligned} &Ap\frac{dp}{du} = (-c + f'(u))p + u(u^2 - 1), \\ &p(u = -1) = p(u = 1) = 0\,, \end{aligned} \tag{3.1.1}$$

where

$$p = u'. \tag{3.1.2}$$

We are interested in monotone solutions of (3.1), (3.2). It is clear that (3.1), (3.2) has a monotone solution if and only if (3.1.1) has a solution $p(u) \geq 0$. To prove the existence of such solutions of (3.1) and (3.2), we consider the unstable manifold in the upper (u, p)-plane.

Lemma 3.1.1. *Let $p_-(u;A)$ be the portion of the unstable manifold of (3.1.1) in $\{p>0\}$ issued from $(u,p)=(-1,0)$ and entering $\{p>0\}$. If $0\le A_1<A_2$, then $p_-(u,A_1)>p_-(u,A_2)$ for $u>-1$.*

Proof: We investigate $\frac{dp}{du}$ near $u=\pm1$. Taking $\frac{d}{du}$ on (3.1.1) and $u=-1$, one gets

$$A\left(\frac{dp}{du}\Big|_{u=-1}\right)^2=(-c+f'(-1))\frac{dp}{du}\Big|_{u=-1}+2. \tag{3.1.3}$$

Solving the above equation for $\frac{dp}{du}$ yields

$$\frac{dp}{du}\Big|_{u=-1}=\frac{1}{2A}\left[-c+f'(-1)\pm\sqrt{(c-f'(-1))^2+8A}\right]. \tag{3.1.4}$$

We are interested in the unstable manifold of (3.1.2) entering the region $p>0$. The slope of this unstable manifold at $u=-1$ is, from (3.1.4),

$$\begin{aligned}&\frac{dp_-}{du}\Big|_{u=-1}=\frac{1}{2A}\left[-c+f'(-1)+\sqrt{(c-f'(-1))^2+8A}\right]\\ &=\frac{4}{c-f'(-1)+\sqrt{(c-f'(-1))^2+8A}}>0.\end{aligned} \tag{3.1.6}$$

It is clear that $\frac{dp_-}{du}\big|_{u=-1}$ decreases as $A\ge0$ increases. Thus, there is a point $u_0>-1$ such that if $0\le A_1<A_2$, then the inequality $p_-(u,A_1)>p_-(u,A_2)$ holds for $u\in(-1,u_0)$. To prove the Lemma, it suffices to prove that $p_-(u,A_1)$ and $p_-(u,A_2)$ do not intersect. To this end, we assume its contrary, i.e. $p_-(u_1,A_1)=p_-(u_1,A_2)>0$. Define

$$u_2:=\inf\{u_1\ge u_0\ :\ p_-(u_1,A_1)=p_-(u_1,A_2)\}. \tag{3.1.7}$$

This means that $p_1:=p_-(u,A_1)>p_2:=p_-(u,A_2)$ for $u\in(-1,u_2)$ and $p_-(u_2,A_1)=p_-(u_2,A_2)$, and hence

$$\frac{dp_1}{du}\Big|_{u=u_2}\le\frac{dp_2}{du}\Big|_{u=u_2}. \tag{3.1.8}$$

The difference of equations (3.1.1) for p_1 and p_2 at $u=u_2$ reads

$$0=p_1\left(A_1\frac{dp_1}{du}-A_2\frac{dp_2}{du}\right). \tag{3.1.9}$$

When

$$\frac{dp_1}{du}\Big|_{u=u_2} \neq 0 \quad \text{or} \quad \frac{dp_2}{du}\Big|_{u=u_2} \neq 0, \tag{3.1.10}$$

the estimate (3.1.8) and $0 \leq A_1 < A_2$ applied to (3.1.9) leads to a contradiction. If

$$\frac{dp_1}{du}\Big|_{u=u_2} = \frac{dp_2}{du}\Big|_{u=u_2} = 0, \tag{3.1.11}$$

then

$$\frac{d^2p_1}{du^2}\Big|_{u=u_2} = \frac{d^2p_2}{du^2}\Big|_{u=u_2} \tag{3.1.12}$$

and

$$\frac{d^3p_1}{du^3}\Big|_{u=u_2} \leq \frac{d^3p_2}{du^3}\Big|_{u=u_2} \tag{3.1.13}$$

hold. Eqautions (3.1.11) and (3.1.12) at $u = u_2$ are

$$\begin{aligned} &(-c + f'(u))p + u(1 - u^2) = 0, \\ &f''(u)p + 1 - 3u^2 = 0. \end{aligned} \tag{3.1.14}$$

Plugging $c = 0$, $f'' > 0$ and $f'(0) = 0$ into (3.1.14), one sees that there is no solution for (3.1.14) and hence (3.1.11) is impossible. These contradictions complete the proof. ∎

Similarly, the following lemma can be established for the stable manifold of (3.1) entering $(u, p) = (1, 0)$ from the upper $(u, p)-$plane:

Lemma 3.1.2. *Let $p_+(u; A)$ be the portion of the stable manifold of (3.1.1) in $\{p > 0\}$ entering $(u, p) = (1, 0)$. If $0 \leq A_1 < A_2$, then $p_+(u, A_1) > p_+(u, A_2)$ for $u < 1$.*

Proof. The proof is almost the same as that of Lemma 3.1.1. ∎

The next theorem establishes the existence of the traveling wave solution for the rarefaction layer.

Theorem 3.1.3. *For any $A > 0$, the system (3.1), (3.2) has a monotone solution when $c = f'(0)$.*

Proof. When $A = 0$ and $f'(0) = 0$, there is a monotone solution of (3.1) and (3.2), as stated in Theorem 2.1.1. Thus, there is a solution $p(u)$ of (3.1.1) with

$p(u) \geq 0$ when $A = 0$ and $c = f'(0)$. It is clear that (3.1.1) has a solution if and only if an unstable manifold $p = p_-(u; A)$ of (3.1.1) leaving $(u,p) = (-1, 0)$ intersects a stable manifold $p = p_+(u; A)$ of (3.1.1) entering $(u, p) = (1, 0)$. When $p_-(u; A)$ and $p_+(u; A)$ intersect, they coincide. Thus, $p_-(1; A = 0) = 0$ and $p_+(-1; A = 0) = 0$. From Lemma 3.1.2, $p_-(u, A) < p_-(u, 0) = p_+(u, 0) > p_+(u, A)$ for $A > 0$. When $A > 0$ is small, $\frac{dp_-}{du}(u = -1; A)$ at $u = -1$ changes little from $\frac{dp_-}{du}(u = -1; 0)$, and hence $p_-(u; A)$ changes little from $p_-(u; 0)$. This implies that $p = p_-(u, A)$ hits the u-axis at some point $(u, p) = (u_0, 0)$ with $u_0 \leq 1$ and close to 1. Similarly, the stable manifold $p_+(u; A)$ intersects the u-axis at some point $u_1 > -1$ and close to -1. Therefore, the manifolds $p_-(u; A)$ and $p_+(u, A)$ must intersect for small $A > 0$. The same argument shows that there is no upper bound for $A > 0$ for which $p_-(u; A)$ and $p_+(u; A)$ intersect. Thus, there are monotone solutions of (3.1), (3.2) for all $A > 0$. ∎

§3.2. Existence of the viscous profiles of shocks

In this section, we prove that the solution of (1.2) with some special initial data converges as $t \to \infty$ to a stationary wave.

For convenience, we can do transformation $(x, t) \mapsto (x/\epsilon, t/\epsilon)$ on (1.2) to eliminate ϵ. Thus, in the rest of this section, we omit ϵ from (1.2).

Lemma 3.2.1. *Let $u(x, t)$ solves (1.2) with initial data $u(x, 0)$. If $u_x(x, 0) < 0$ (> 0), then $u_x(x, t) < 0$ (> 0).*

Proof. Let $\phi = u_x$, then ϕ satisfies

$$\begin{aligned} &\phi_t + f'(u)\phi_x + f''(u)\phi^2 = \phi_{xx} + g'(u)\phi, \\ &\phi(x, 0) = u_x(x, 0) > 0 \quad (< 0). \end{aligned} \tag{3.2.1}$$

The maximum principle on (3.2.1) yields that $\phi(x, t) > 0$ (< 0) if $\phi(x, 0) > 0$ (< 0). ∎

In the rest of this section, we assume the initial data satisfy

$$u_x(x, 0) < 0, \text{ and } -1 \leq u(x, 0) \leq 1. \tag{3.2.2}$$

In this case the solution $u(x, t)$ is decreasing. Then the transformation from (x, t) to

$$\begin{aligned} &w = u(x, t), \\ &s = t \end{aligned} \tag{3.2.3}$$

is one-to-one. Then, for any smooth function $v(x,t)$, the chain rule reads

$$v_t = v_w u_t + v_s, \qquad v_x = v_w u_x. \tag{3.2.4}$$

Let $\phi := u_x$. After changing variables according to (3.2.3) and (3.2.4), the equation (3.2.1) and (3.2.2) become

$$\begin{aligned} &\phi_s = \phi^2 \left[\phi_{ww} + \left(\frac{g(w)}{\phi}\right)_w - f''(w)\right], \quad w \in (-1,1), \\ &\phi(w,0) < 0, \quad \phi(w=\pm 1, 0) = 0. \end{aligned} \tag{3.2.5}$$

¿From Lemma 3.2.1, we know that the solution of (3.2.5) satisfies $\phi(w,s) < 0$ for $s > 0$.

Lemma 3.2.2. *If $\phi_s(w,0) > 0 \;\; (< 0)$, then $\phi_s(w,s) > 0 \;\; (< 0)$ for all $s > 0$.*

Proof. Taking $\frac{\partial}{\partial s}$ on (3.2.5) and leting $\psi = \phi_s$ give

$$\psi_s = \phi^2 \psi_{ww} + 2\phi[\phi_{ww} - f''(w)]\psi - g(w)\psi_w + g'(w)\psi. \tag{3.2.6}$$

Again, the maximum principle on (3.2.6) yields that if $\phi_s(w,0) > 0 \;\; (< 0)$, then $\phi_s(w,s) > 0 \;\; (< 0)$ for all $s > 0$. ∎

Lemma 3.2.3. *Let*

$$u(x,0) = -\tanh(x/\delta) \tag{3.2.7}$$

where $\delta > 0$ is a constant and $\phi(w,0)$ be the function $u_x(x,0)$ with variables (w,s) given in (3.2.3). Then

- *(i) when $\delta > 0$ is small enough, the solution ϕ of (3.2.5) satisfies $\phi_s(w,s) > 0$ for all $s > 0$;*
- *(ii) when $\delta > 0$ is large enough, the solution ϕ of (3.2.5) satisfies $\phi_s(w,s) < 0$ for all $s > 0$.*

Proof. ¿From Lemma 3.2.2, it suffices to prove that the initial data given by (3.2.7) satisfy $(3.2.5)_2$ and $\phi_s(w,0) > 0$ for $w \in (-1,1)$. It is easy to see that $\phi(w,0) = u_x(x,0) < 0$ for $u(x,0) = -\tanh(x/\delta)$. Also, the limiting process $w \to \pm 1$

corresponds to $x \to \mp\infty$ and hence $\phi(w \to \pm 1, 0) = 0$. Straightforward computation gives

$$
\begin{aligned}
\phi_s(w,0) &= \phi_t + \phi_x \frac{\partial x}{\partial s} = \phi_t - \phi_x \frac{u_t}{u_x} \\
&= u_{xxx} - u_{xx}^2/u_x - f''(u)u_x^2 - g'(u)u_x - g(u)u_{xx}/u_x \\
&= \operatorname{sech}^4\left(\frac{x}{\delta}\right)\left[\frac{2}{\delta^3} - \frac{f''(u)}{\delta^2} - \frac{1}{\delta}\right].
\end{aligned} \tag{3.2.8}
$$

In the above calculation, we used $g(u) = u(1-u^2)$. ¿From (3.2.8), it is clear that for $\delta > 0$ small enough, $\phi(w,0) > 0$ holds, while for large enough $\delta > 0$, $\phi(w,0) < 0$ holds. The desired statements then follow immediately from Lemma 3.2.2. ∎

Corollary 3.2.4. *Let $\phi(w,s)$ be the solution of (3.2.5) given in Lemma 3.2.3 with $\delta > 0$ small enough. Then $\phi(w,s) \to \theta(w) < 0$ as $s \to \infty$, for $w \in (-1,1)$. Furthermore, the function $\theta(w)$satisfies*

$$
-\infty = \int_0^1 \frac{1}{\theta(w)}dw < \int_0^w \frac{1}{\phi(w,0)}dw < \int_0^{-1} \frac{1}{\phi(w,0)}dw < \int_0^{-1} \frac{1}{\theta(w)} = \infty. \tag{3.2.9}
$$

Proof. Lemma 3.2.3 states that this solution of (3.2.5) with intial data (3.2.7) with $\delta > 0$ sufficiently small satisfies $\phi_s(w,s) > 0$ and hence $\phi(w,s)$ is increasing as s increases. On the other hand, ϕ is also bounded from above by 0. Therefore, the limit $\lim_{s\to\infty} \phi(w,s) =: \theta(w)$ exists for all $w \in [-1,1]$. To prove that $\theta(w) < 0$ for $w \in (-1,1)$, we consider the solution ϕ_1 of (3.2.5) with initial data (3.2.7) with $\delta_1 > 0$ large enough. ¿From Lemma 3.2.3, $0 > \phi_1(w,0) > \phi_1(w,s)$ for all $s > 0$ and $w \in (-1,1)$. Since the comparison principle holds for equation (3.2.5) and $0 > \phi_1(w,0) > \phi(w,0)$, thus,

$$
0 > \phi_1(w,0) > \phi_1(w,s) > \phi(w,s) > \phi(w,0) \tag{3.2.10}
$$

and hence $0 > \phi_1(w,0) \geq \theta(w)$. Furthermore, the equality

$$
\frac{\partial x}{\partial w} = \frac{1}{u_x}
$$

and inequalities (3.2.10), together with the property of (3.2.7) and hence that of $\phi(w,0)$ imply that

$$
\int_0^1 \frac{1}{\theta(w)}dw < \int_0^w \frac{1}{\phi(w,0)}dw = -\infty.
$$

The other half of (3.2.9) can be proved similarly. ∎.

Theorem 3.2.5. *If $f(u) = f(-u)$, then the solution $u(x,t)$ of (1.2) with initial data*

$$u(x,0) = -\tanh(x/\delta), \tag{3.2.11}$$

with constant $\delta > 0$ small enough, converges to a stationary solution of (1.2).

Proof. ¿From the definition (3.2.3) of the transformation $(x,t) \mapsto (u,s)$, one gets

$$\frac{\partial x}{\partial u} = \frac{1}{u_x}. \tag{3.2.12}$$

For the solution given in Corollary 3.2.4, one has

$$\frac{\partial x}{\partial u} = \frac{1}{\theta(u)} \tag{3.2.13}$$

or equivalently

$$\lim_{s\to\infty} (x(u,s) - x(0,s)) = \int_0^u \frac{1}{\theta(u)} du. \tag{3.2.14}$$

The integral in (3.2.14) is regular for all $w \in (-1,1)$ in view of Corollary 3.2.4. When $f(u) = f(-u)$, the solution of (1.2) with initial data (3.2.11) is antisymmetric about the point $x = 0$ and hence $x(0,s) \equiv 0$. Then (3.2.14) yields

$$x = \lim_{s\to\infty} x(u,s) = G(u). \tag{3.2.15}$$

Estimate (3.2.9) and $\theta(u) < 0$ garrantee that for each $x \in \mathbb{R}$, there is a $u(x) \in (-1,1)$ satisfies (3.2.15). By definition (3.2.3), we have

$$u(x) = u(x(u(x),s),s) = u(x,s) + u_x(\eta,s)(x(u(x),s) - x)$$

for some η between x and $x(u(x),s)$. Since u_x is bounded as indicated by (3.2.10), taking $s \to \infty$ gives

$$\lim_{s\to\infty} u(x,s) = u(x)$$

for all $x \in \mathbb{R}$. ∎

§3.3. Asymptotic layer structures

In this subsection we use a mtached asymptotic analysis to determine the shock and rarefaction layer profiles near a_j.

First we consider the profile around $x = a_{2j}$. Introducing the stretched variable

$$\xi = \frac{x - a_{2j}}{\epsilon} . \tag{3.3.1}$$

Applying the ansatz

$$u(x,t) = \pm 1 + u_0(\xi, t) + \cdots \tag{3.3.2}$$

subject to boundary conditions for $u_0(\xi, t)$ into (1.2), one gets

$$\begin{aligned} &(\pm 1 + u_0(\xi,t) + \cdots)_t + f(\pm 1 + u_0(\xi,t) + \cdots)_x \\ =&\tfrac{1}{\epsilon} g(\pm 1 + u_0(\xi,t) + \cdots) + \epsilon(\pm 1 + u_0(\xi,t) + \cdots)_{xx} , \end{aligned} \tag{3.3.3}$$

where $g(u) = u(1 - u^2)$. By the Taylor expansion, and using $g(\pm 1) = 0$, one gets

$$\begin{aligned} &(u_0(\xi,t) + \cdots)_t + f'(\pm 1)(\tfrac{1}{\epsilon}\partial_\xi u_0(\xi,t) + \cdots) \\ =&\tfrac{1}{\epsilon} g'(\pm 1)(u_0(\xi,t) + \cdots) + \frac{1}{\epsilon}(u_0(\xi,t) + \cdots)_{\xi\xi} . \end{aligned} \tag{3.3.4}$$

The leading order balance of (3.3.4) gives

$$\partial_{\xi\xi} u_0 - f'(\pm 1)\partial_\xi u_0 + g'(\pm 1) u_0 = 0 . \tag{3.3.5}$$

This equation has two eigenvalues

$$\begin{aligned} \lambda_\pm^{(1)} &= \tfrac{1}{2}\left[f'(\pm 1) + \sqrt{f'(\pm 1)^2 - 4g'(\pm 1)} \right] , \\ \lambda_\pm^{(2)} &= \tfrac{1}{2}\left[f'(\pm 1) - \sqrt{f'(\pm 1)^2 - 4g'(\pm 1)} \right] . \end{aligned} \tag{3.3.6}$$

First we construct the solution at $x = a_{2j+1}$. If $x > a_{2j+1}$, then $\xi = \frac{x - a_{2j+1}}{\epsilon} > 0$. Then u_0 is monotonely decreasing and satisfies the boundary conditions

$$u_0(0,t) = 1 , \quad u_0 \to 0 \quad \text{as} \quad \xi \to \infty . \tag{3.3.7}$$

The only possible solution of u_0 subject to these conditions is

$$u_0 = e^{-\lambda_-^{(2)} \frac{x - a_{2j+1}}{\epsilon}} . \tag{3.3.8}$$

Similarly, for $x < a_{2j+1}$, one can obtain that

$$u_0 = -e^{\lambda_-^{(1)} \frac{x - a_{2j+1}}{\epsilon}} . \tag{3.3.9}$$

The profile around $x = a_{2j}$ can be similarly constructed and we list them here:

$$u_0 = -e^{-\lambda_+^{(2)} \frac{x-a_{2j}}{\epsilon}} \quad \text{if} \quad x > a_{2j}\,,$$
$$u_0 = -e^{\lambda_+^{(1)} \frac{x-a_{2j}}{\epsilon}} \quad \text{if} \quad x < a_{2j}\,.$$

References

[AK] S. Allen and J. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta Metallurgica 27, 1084-1095, 1979.

[BJ] W.Z. Bao and S. Jin, The random projection method for hyperbolic systems with stiff reaction terms, J. Comp. Phys. 163, 216-248, 2000.

[CP] J. Carr and P.L. Pego, Metastable patterns in solutions of $u_t = \epsilon u_{xx} - f(u)$, Comm. Pure Appl. Math. 42, 523-576, 1989.

[Daf] C. M. Dafermos, Generalized characteristics and structure of solutions of hyperbolic conservation laws, Indiana Univ. Math. J. 26, 1097-1119, 1977.

[E] W. E, Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity. Phys. D 77, 383–404, 1994.

[Fan] H. Fan, Hyperbolic conservation laws with stiff reaction terms of monostable type, to appear in Transaction of AMS.

[FH1] H. Fan and J. K. Hale, Large-time behavior in inhomogeneous conservation laws, Arch. Rational Mech. Anal., 125, 201-216, 1993.

[FH2] H. Fan and J. K. Hale, Attractors in inhomogeneous conservation laws and parabolic regularizations, Trans. Ameri. Math. Soc. 347, 1239-1254, 1995.

[FHa] H. Fan, J. Härterich, Large time behavior and zero reaction time limit of conservation laws with degenerate source, preprint, 2000.

[FJT] H. Fan, S. Jin and Z.-H. Teng, Zero reaction limit for hyperbolic conservation laws with source terms, J. Diff. Eqn., to appear, 2000

[Fil] A. F. Filippov, Differential equations with discontinuous right hand side, Mat. Sb. (N.S.)42, 99-128; English Transl., Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 1960.

[FH] G. Fusco and J.K. Hale, Slow-motion manifolds, dormant instability, and singular perturbations. J. Dyn. Diff. Eqn. 1, 75–94, 1989.

[Har] J. Härterich, Heteroclinic orbits between rotating waves in hyperbolic balance laws, preprint, 1998.

[KK] G. Kreiss and H.-O. Kreiss, Convergence to steady-state of solutions of Burgers' equations, Appl. Numer. Math. 2, 161-179, 1986.

[KK] G. Kreiss, H.-O. Kreiss and N.A. Peterson, On the convergence to steady-state of solutions of nonlinear hyperbolic-parabolic systems, SIAM J. Num. Anal. 31, 1577-1604, 1994.

[Kru] S.N. Kruzhkov, First order quasilinear equations in several independent variables, Math. USSR-Sb. 10, 217-243, 1970.

[LO1] J.G. Laforgue and R.E. O'Malley, Jr., Shock layer movement for Burgers' equation, SIAM J. Appl. Math. 55, 332-347, 1995.

[LO2] J.G. Laforgue and R.E. O'Malley, Jr., Exponential asymptotics, the viscid Burgers' equation, and standing wave solutions for a reaction-advection-diffusion model, Studies Appl. Math. 102, 137-172, 1999.

[Lyb] A. N. Lyberopoulos, A Poincarè-Bendixson theorem for scalar conservation laws, Proc. Roy. Soc. Edinburgh, 124A, 589-607, 1994.

[Mas1] C. Mascia, Traveling wave solutions for a balance law, Proc. Roy. Soc. Edinburgh 127A, 567-593, 1997.

[Mas2] C. Mascia, First order singular perturbations of quasilinear nonconvex type. Preprint, 1999.

[MS] C. Mascia and C. Sinestrari, The perturbed Riemann problem for a balance law, Adv. in Diff. Eqn. 2(5), 779-810, 1997.

[Neu] J.C. Neu, Vortices in complex scalar fields. Phys. D 43, 385–406, 1990.

[RW] L.G. Reyna and M.J. Ward, On the exponentially slow motion of a viscous shock, Comm. Pure Appl. Math. 48, 79-120, 1995.

[RSK] J. Rubinstein, P. Sternberg and J.B. Keller, Fast reaction, slow diffusion, and curve shortening, SIAM J. Appl. Math. 49, 116-133, 1989.

[Sin1] C. Sinestrari, Large time behavior of solutions of balance laws with periodic initial data, Nonl. Diff. Appl. 2, 111-131, 1995.

[Sin2] C. Sinestrari, Asymptotic profile of solutions of conservation laws with source, Differential Integral Equations 9, 499-525, 1996.

[Sin3] C. Sinestrari, Instability of discontinuous traveling waves for hyperbolic balance law, J. Diff. Eqn. 134, 269-285, 1997.

[VK] W.G. Vincenti and C.H. Kruger, Jr., Introduction to Physical Gas Dynamics, Krieger, 1975.

[WR] M.J. Ward and L.G. Reyna, Internal layers, small eigenvalues, and the sensitivity of metastable motion, SIAM J. Appl. Math. 55, 425-445, 1995.