Regularization of the Burnett Equations for Fast Granular Flows via Relaxation

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Nomenclature

$\mathbf{b}$	body force
$\mathcal{B}$	subset of Euclidean space
div	divergence
e	internal energy density
grad	gradient
$\mathbf{I}$	unit tensor
$\mathbf{L}$	velocity gradient ($\mathbf{L} = (\text{grad } \mathbf{u})^T$)
M	Maxwell number
$\mathbf{n}$	unit exterior normal
$\mathbf{P}$	pressure deviator
p	mean normal pressure
$\mathbf{q}$	energy flux vector
R	gas constant
$\mathbf{S}$	distortion tensor ($\mathbf{S} = \frac{1}{2}(\text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^T - \frac{2}{3} \text{ div } \mathbf{u}\, \mathbf{I})$)
$\mathbf{T}$	stress tensor ($\mathbf{T} = -p\mathbf{I} - \mathbf{P}$)
t	time
tr	trace
$\mathbf{u}$	macroscopic velocity
$\mathbf{x}$	cartesian coordinate ($\mathbf{x} = (x, y, z)$)
μ	viscosity
ρ	mass density
η	specific entropy
ψ	Helmhotz free energy, $\psi = \varepsilon - \theta\eta$
θ	temperature
θ_i	i^{th} coefficient of the Chapman-Enskog expansion for $\mathbf{q}$
ω_i	i^{th} coefficient of the Chapman-Enskog expansion for $\mathbf{P}$
$(\dot{\ })$	material derivative of $(\quad)$, i.e. $(\dot{\ }) = \frac{\partial}{\partial t}(\quad) + \mathbf{u} \cdot \text{ grad } (\quad)$
$\otimes$	dyadic product, i.e. $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j$
$\cdot$	inner product, i.e. $\mathbf{u} \cdot \mathbf{v} = u_i v_i$ for vectors $\mathbf{u}, \mathbf{v}$; $\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{A}\mathbf{B})$ for tensors $\mathbf{A}, \mathbf{B}$.

Abstract

In [21] Sela and Goldhirsch have used the Chapman-Enskog expansion to derive constitutive relations for the pressure deviator $\mathbf{P}$, heat flux $\mathbf{q}$, and rate of energy loss Γ for rapid flows of smooth inelastic spheres. Unfortunately as in the classical Chapman-Enskog expansion for elastic spheres any truncation of the expansion beyond Navier-Stokes order ($n = 1$) will possess unphysical instabilities.

In this paper we propose a visco-elastic relaxation approximation that eliminates the instability paradox, and provide a system of local equations allowing robust numerical approximations of gas dynamics valid to the Burnett order. This system is linearly hyperbolic, and is endowed with a generalized entropy inequality in the case of purely elastic collisions.

0. Introduction

In a recent paper [21] Sela and Goldhirsch have used the Chapman-Enskog expansion to derive constitutive relations for flows of inelastically colliding spheres in three dimensions to Burnett order. This was done by using the relevant Boltzmann equation to recover a double expansion in both Knudsen number and degree of inelasticity. Their work extended earlier derivations of constitutive equations by Lun et al. [18], Lun [19], Jenkins and Richman [8-10], Boyle and Massondi [2], Goldshtein and Shapiro [6]. However to use this expansion to solve initial-boundary value problems will inevitably present difficulties. This is because built into the Sela-Goldhirsch expansion just as in the classical Chapman-Enskog expansion for elastic collisions are truncation instabilities which have no physical relevance [1].

Following our earlier paper for the classical case of elastic collision [11], we provide a relaxation system for rapid granular flow which (i) agrees with the classical Chapman-Enskog expansion up to Burnett order, (ii) is linear hyperbolic, and (iii) yet in the special case of elastic collisions produces a generalized Clausius-Duhem (entropy) inequality.

Relaxation as a stabilizing effect occurs naturally in physics of viscoelastic fluids, where the stress obeys a rate time equation which allows a finite relaxation time for the stress to relax into the equilibrium [13]. It is known that the relaxation provides a subtle stabilizing viscous mechanism, which can be seen from the Chapman-Enskog expansion [3]. In recent years, relaxation approximations have been used as an effective tool to design numerical methods – known as the relaxation schemes. In [12] a generic way to relax a general system of hyperbolic conservation laws was introduced by Jin and Xin, and based on which a class of relaxation schemes free of Riemann solver and local characteristic decomposition were developed for an inviscid gas dynamics. A physically natural pressure relaxation method was developed by Coquel and Perthame for inviscid general gas [4].

Our method differs from the classical approaches of Grad [7] and Levermore [14], [15]. Grad's 13 moment equations, based on setting higher moments to zero, describe the evolution of nonphysical quantities in higher moments. More recently, a procedure was developed by Levermore [14] that generates a hierarchy of moment closure systems, each of which possesses an entropy and is symmetric hyperbolic. The simplest member of this hierarchy is the Euler system. It was remarked in [15] that there is an absence of heat flux in the Gaussian closure at the Navier-Stokes order. Our relaxation system also differs from the rational extended thermodynamics, since the later does not recover the Burnett equations in $O(\mu^2)$ [20].

The paper is divided into three sections after this Introduction. Section 1 reviews the balance laws of mass, momentum, energy for continuum mechanics. Section 2 presents the Chapman-Enskog expansion of Sela and Goldhirsch for pressure deviator $\mathbf{P}$ and heat flux $\mathbf{q}$ up to $n = 2$ Burnett order. Section 3 introduces the relaxation approximation which yields the Burnett equations via the Chapman-Enskog expansion. In Section 4 it is shown that the constitutive relations for $\mathbf{P}$ and $\mathbf{q}$ are consistent with (1) decrease of mechanical energy isothermal motion with no-slip boundary condition, (2) increase of entropy for a rigid heat conductor with insulated boundary conditions. The linear hyperbolicity is shown in Section 5.

1. The field equations of balance

The field equations of balance for continuum fluid dynamics in the absence of heat sources are as follows:

$$\dot{\rho} + \rho \text{ div } \mathbf{u} = 0, \quad \text{(mass conservation)} \tag{1.1}$$

$$\rho\dot{\mathbf{u}} + \text{grad } p + \text{div } \mathbf{P} = \rho\mathbf{b}, \quad \text{(linear momentum conservation)} \tag{1.2}$$

$$\mathbf{P} = \mathbf{P}^T, \quad \text{(rotational momentum conservation)} \tag{1.3}$$

$$\rho\dot{\text{e}} + p \text{ div } \mathbf{u} + \mathbf{P}\cdot\mathbf{S} + \text{div } \mathbf{q} = -\rho\Gamma + \rho r, \quad \text{(energy conservation)} \tag{1.4}$$

where

$$\text{e} = \psi - \theta\frac{\partial\psi}{\partial\theta}, \ \eta = -\frac{\partial\psi}{\partial\theta}, \ p = \rho^2\frac{\partial\psi}{\partial\rho}. \tag{1.5}$$

Differentiation of the expression for the Helmholtz free energy $\psi = \varepsilon - \theta\eta$ yields

$$\rho\theta\dot{\eta} = \rho\dot{\text{e}} - \rho\dot{\rho}\frac{\partial\psi}{\partial\rho}, \tag{1.6}$$

which when combined with (1.1), (1.4), (1.6) yields the entropy production equation

$$\rho\theta\dot{\eta} = -\mathbf{P}\cdot\mathbf{S} - \text{div } \mathbf{q}. \tag{1.7}$$

Division by θ yields the total entropy product rate of a fluid occupying domain $\mathcal{B} \subset \mathbb{R}^3$:

$$\frac{d}{dt}\int_{\mathcal{B}} \rho\eta dV = -\int_{\mathcal{B}} \frac{\mathbf{P}\cdot\mathbf{S}}{\theta} + \frac{\mathbf{q}\cdot\text{grad }\theta}{\theta^2} dV - \int_{\partial\mathcal{B}} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta} dA. \tag{1.8}$$

The Clausius-Duhem inequality is a common albeit not universally accepted form of the second law of thermodynamics. It asserts

$$\frac{d}{dt}\int_{\mathcal{B}} \rho\eta dV + \int_{\partial\mathcal{B}} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta} dA \geq 0 \tag{1.9}$$

which in turn from (1.8) requires $\mathbf{P}$, $\mathbf{q}$ to satisfy

$$\int_{\mathcal{B}} \frac{\mathbf{P}\cdot\mathbf{S}}{\theta} + \frac{\mathbf{q}\cdot \text{grad } \theta}{\theta^2} dV \leq 0 \tag{1.10}$$

for all fluid domains $\mathcal{B}$. However the classical Clausius-Duhem inequality is inconsistent with $\mathbf{P}$, $\mathbf{q}$ delivered by the Chapman-Enskog expansion beyond Navier-Stokes order.

2. The Chapman-Enskog expansion

The Chapman-Enskog expansion for a monatomic gas of spherical molecules yields the constitutive relations

$$\mathrm{e} = \frac{1}{2}\theta, \;\; p = \frac{1}{3}\rho\theta, \;\; \mu = \mu(\theta), \tag{2.1}$$

$$\begin{aligned} &\psi = \frac{1}{3}\theta \log \rho - \frac{1}{2}\theta \log \theta + \frac{1}{2}\theta \\ &\mu(\theta) = \tilde{\mu} n l \theta^{1/2}\,, \tilde{\mu} = 0.3249 + 0.057\epsilon + O(\epsilon^2)\,, \\ &l = (\pi\rho d^2)^{-1}\,, \epsilon = 1 - a^2 \end{aligned}$$

where n is the number density. In addition the expansion provides representations for the pressure deviator tensor $\mathbf{P}$ and heat flux vector $\mathbf{q}$ in terms of a series which may be ordered via powers of the viscosity μ in terms of the total number of space plus time derivatives. Following the notation of Ferziger and Kaper [5] we record

$$\mathbf{P} = \mu\mathbf{P}^{(1)} + \mu^2\mathbf{P}^{(2)} + \cdots, \tag{2.2}$$

$$\mathbf{q} = \mu\mathbf{q}^{(1)} + \mu^2\mathbf{q}^{(2)} + \cdots, \tag{2.3}$$

where the expressions for $\mathbf{P}^{(1)}$, $\mathbf{P}^{(2)}$, $\mathbf{q}^{(1)}$, $\mathbf{q}^{(2)}$ are as follows:

$$\mathbf{P}^{(1)} = -2\mathbf{S}, \tag{2.4}$$

$$\mathbf{q}^{(1)} = -\frac{1}{2}M \text{ grad } \theta - \frac{\lambda}{\tilde{\mu}}\theta \text{grad}(\log\rho), \tag{2.5}$$

$$M = \tilde{k}/\tilde{\mu}, \;\; \tilde{k} = 0.401 + 0.107\varepsilon + 0(\varepsilon^2), \;\; \lambda = 0.211\varepsilon + 0(\varepsilon^2),$$

$$\begin{aligned} \mathbf{P}^{(2)} = {} & \omega_1 \frac{1}{p}(\text{div } \mathbf{u})\mathbf{S} \\ & + \omega_2\frac{1}{p}\{\dot{\mathbf{S}} - \mathbf{L}^T\mathbf{S} - \mathbf{S}\mathbf{L} + \frac{2}{3}tr(\mathbf{S}\mathbf{L})\mathbf{I}\} \\ & + \omega_3\frac{1}{\rho\theta}\{\text{grad}^2\theta - \frac{1}{3}\Delta\theta\mathbf{I}\} \\ & + \omega_4\frac{1}{\rho p\theta}\{\frac{1}{2}\text{grad } p \otimes \text{grad } \theta \\ & + \frac{1}{2}\text{grad } \theta \otimes \text{grad } p - \frac{1}{3}\text{grad } p \cdot \text{grad } \theta\mathbf{I}\} \\ & + \omega_5\frac{1}{\rho\theta^2}\{\text{grad } \theta \otimes \text{grad } \theta - \frac{1}{3}|\text{grad } \theta|^2\mathbf{I}\} \\ & + \omega_6\frac{1}{p}\{\mathbf{S}^2 - \frac{1}{3}tr(\mathbf{S}^2)\mathbf{I}\}, \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
\mathbf{q}^{(2)} = {} & \theta_1 \frac{1}{\rho\theta}(\mathrm{div}\ \mathbf{u})\mathrm{grad}\ \theta \\
& + \theta_2 \frac{1}{\rho\theta}((\mathrm{grad}\ \theta)^{\bullet} - \mathbf{L}\ \mathrm{grad}\ \theta) \\
& + \theta_3 \frac{1}{p\rho}(\mathbf{S}\ \mathrm{grad}\ p) \\
& + \theta_4 \frac{1}{\rho}\ \mathrm{div}\mathbf{S} + \theta_5 \frac{1}{\rho\theta}\mathbf{S}\ \mathrm{grad}\ \theta \\
\Gamma = {} & \frac{\tilde{\delta}}{\ell}\theta^{3/2} + \tilde{\rho}_1 \varepsilon\ell\theta^{1/2} tr(\mathbf{L}^2) + \tilde{\rho}_2 \frac{\varepsilon\ell}{\theta^{1/2}} |\ \mathrm{grad}\ \theta|^2 \\
& + \frac{\tilde{\rho}_3 \varepsilon\ell}{\rho\theta^{1/2}}\ \mathrm{grad}\ (\rho\theta) \cdot\ \mathrm{grad}\ \theta + \tilde{\rho}_4 \varepsilon\ell\theta^{1/2}\Delta\theta. \qquad (2.8)
\end{aligned}
$$

Also we have followed the representation of Ferziger and Kaper [5] and not changed time derivatives to space derivatives in (2.6), (2.7) even though this was done by Sela and Goldhirsch.

The coefficients $\mu_1, \ldots, \omega_6, \theta_1, \ldots, \theta_5$ are constants but not independent and are given below (see [21]):

$$
\begin{aligned}
&\omega_1 \approx 1.2845 && \theta_1 \approx 1.2291 \\
&\omega_2 \approx 0.6422 && \theta_2 \approx 0.6146 \\
&\omega_3 \approx 0.2552 && \theta_3 \approx 0.3262 \\
&\omega_4 \approx 0.0719 && \theta_4 \approx 0.2552 \\
&\omega_5 \approx 0.0231 && \theta_5 \approx 2.6555 \\
&\omega_6 \approx 2.3510. && \qquad (2.9)
\end{aligned}
$$

Notice the relations are a special case of the universal formulas derived by Truesdell and Muncaster [24]:

$$
\begin{aligned}
\omega_3 &= \theta_4, \\
\theta_1 &= \frac{2}{3}\left(\frac{7}{2} - \frac{\mu'(\theta)}{\mu(\theta)}\theta\right)\theta_2 - \frac{1}{3}\theta\frac{\partial\theta_2}{\partial\theta}, \\
\omega_1 &= \frac{2}{3}\left(\frac{7}{2} - \frac{\mu'(\theta)}{\mu(\theta)}\theta\right)\omega_2 - \frac{1}{3}\theta\frac{\partial\omega_2}{\partial\theta}.
\end{aligned}
$$

Hence since $\frac{\mu'(\theta)\theta}{\mu(\theta)} = \frac{1}{2}$ we have

$$
\begin{aligned}
\omega_3 &= \theta_4, \\
\theta_1 &= 2\theta_2, \\
\omega_1 &= 2\omega_2, \qquad (2.10)
\end{aligned}
$$

which is consistent with (2.9).

Also we recall from [21] that

$$\tilde{\delta} = \left(\frac{16}{27\pi}\right)\varepsilon - 0.0112\varepsilon^2, \quad \tilde{\rho}_1 = 0.1338, \tag{2.11}$$

$$\tilde{\rho}_2 = 0.2444, \quad \tilde{\rho}_3 = .0834, \quad \tilde{\rho}_4 = .0692.$$

In this paper we shall assume that in addition to (2.10) the following relations hold

$$\theta_3 + \omega_3 + \omega_4 = 0, \tag{2.12}$$

$$\theta_5 = \bar{\theta}_5 + \frac{\mu'(\theta)\theta}{\mu(\theta)}\omega_3,$$

$$\omega_3 > 0, \quad \theta_2 > 0$$

$$\bar{\theta}_5 > 0, \bar{\theta}_5 \text{ a constant.}$$

Again all these relations are consistent with (2.9). Also since the relation $\frac{\mu'(\theta)\theta}{\mu(\theta)} = \frac{1}{2}$ is not crucial to the results derived here I have left the general term $\frac{\mu'(\theta)\theta}{\mu(\theta)}$ in all calculations.

3. A relaxation approximation

Since it is the material derivative terms on the right hand side of (2.6) and (2.7), and the inelastic term on the right hand side of (2.5), that introduce the linear instability [22], we seek a relaxation approximation that regularizes $\mathbf{P}$ and $\mathbf{q}$. Specifically, we write

$$\dot{\rho} + \rho \text{ div } \mathbf{u} = 0, \tag{3.1}$$

$$\rho\dot{\mathbf{u}} + \text{grad } p + \text{div } \mathbf{P} = \rho\mathbf{b}, \tag{3.2}$$

$$\mathbf{P} = \mathbf{P}^T, \tag{3.3}$$

$$\dot{\mathbf{P}} - \mathbf{L}^T\mathbf{P} - \mathbf{P}\mathbf{L} + \frac{2}{3}\text{tr}(\mathbf{P}\mathbf{L})\mathbf{I} = -\frac{2p}{\omega_2\mu}\left(\mathbf{P} - \mathbf{P}^{eq}\right), \tag{3.4}$$

$$\rho\dot{\text{e}} + p \text{ div } \mathbf{u} + \mathbf{P}\cdot\mathbf{S} + \text{div } (\mathbf{q}) = 0, \tag{3.5}$$

$$\dot{\mathbf{q}} - \mathbf{L}\mathbf{q} = -\frac{3Mp}{2\theta_2\mu}\left(\mathbf{q} - \mathbf{q}^{eq}\right), \tag{3.6}$$

where

$$\mathbf{P}^{eq} = -2\mu\mathbf{S} + \mathbf{P}_2 + \mathbf{P}_3, \tag{3.7}$$

$$
\begin{aligned}
\mathbf{P}_2 = & - \mu\frac{\omega_1}{2p}(\mathrm{div}\mathbf{u})\mathbf{P} + \frac{\omega_2\mu'(\theta)\dot{\theta}}{2p}\mathbf{P} \\
& + \mu^2\frac{\omega_3}{\rho\theta}\left\{-\mathrm{grad}\left(\frac{\mathbf{q}}{\frac{1}{2}\mu M}\right) + \frac{1}{3}\mathrm{div}\left(\frac{\mathbf{q}}{\frac{1}{2}\mu M}\right)\mathbf{I}\right\} \\
& + \mu\frac{\omega_4}{\rho p\theta}\left\{-\frac{1}{2}\mathrm{grad}\ p\otimes\left(\frac{\mathbf{q}}{\frac{1}{2}M}\right) - \frac{1}{2}\left(\frac{\mathbf{q}}{\frac{1}{2}M}\right)\otimes\mathrm{grad}\ p\right. \\
& \left.+\frac{1}{3}\mathrm{grad}\ p\cdot\left(\frac{\mathbf{q}}{\frac{1}{2}M}\right)\mathbf{I}\right\} \\
& - \mu\frac{\omega_5}{\rho\theta^2}\left\{\frac{1}{2}\mathrm{grad}\,\theta\otimes\left(\frac{\mathbf{q}}{\frac{1}{2}M}\right) + \frac{1}{2}\left(\frac{\mathbf{q}}{\frac{1}{2}M}\right)\otimes\mathrm{grad}\,\theta\right. \\
& \left.-\frac{1}{3}\mathrm{grad}\,\theta\cdot\left(\frac{\mathbf{q}}{\frac{1}{2}M}\right)\mathbf{I}\right\} - \mu\frac{\omega_6}{2p}\left\{\frac{1}{2}(\mathbf{SP}+\mathbf{PS}) - \frac{1}{3}\mathrm{tr}(\mathbf{PS})\mathbf{I}\right\},
\end{aligned} \tag{3.8}
$$

$$
\mathbf{P}_3 = \mu^2\left[\frac{\hat{\omega}_2}{p^2}\mathrm{tr}\mathbf{S}^2 + 3\hat{\omega}_3\frac{|\mathrm{grad}\,\theta|^2}{\rho^2\theta^3}\right]\mathbf{P} + \mu\frac{\hat{\gamma}_1}{p\theta}\left(\dot{\theta} + \frac{2}{3}\theta\mathrm{div}\mathbf{u}\right)\mathbf{P}, \tag{3.9}
$$

$$
\mathbf{q}^{eq} = -\frac{1}{2}\mu M\,\mathrm{grad}\,\theta - \mu\frac{\lambda}{\tilde{\mu}}\theta\mathrm{grad}(\log\rho) + \mathbf{q}_2 + \mathbf{q}_3, \tag{3.10}
$$

$$
\begin{aligned}
\mathbf{q}_2 = & -2\mu\frac{\theta_1}{M\rho\theta}(\mathrm{div}\ \mathbf{u})\mathbf{q} + \frac{2\theta_2\dot{\theta}\mu'(\theta)}{M\rho\theta}\mathbf{q} \\
& - \mu\frac{\theta_3}{2p\rho}\mathbf{P}\ \mathrm{grad}\ p - \mu^2\frac{\theta_4}{2\rho}\ \mathrm{div}\left(\frac{\mathbf{P}}{\mu}\right) - \mu\frac{\theta_5}{2\rho\theta}\mathbf{P}\ \mathrm{grad}\ \theta,
\end{aligned} \tag{3.11}
$$

$$
\mathbf{q}_3 = \mu^2\left[\frac{\hat{\theta}_2}{p^2}\mathrm{tr}\mathbf{S}^2 + 3\hat{\theta}_3\frac{|\mathrm{grad}\theta|^2}{\rho^2\theta^3}\right]\mathbf{q} + \mu\frac{\hat{\lambda}_1}{\rho\theta^2}\left(\dot{\theta} + \frac{2}{3}\theta\mathrm{div}\mathbf{u}\right)\left(\frac{\mathbf{q}}{\frac{1}{2}M}\right). \tag{3.12}
$$

Since the energy equation (3.5) implies that

$$
\dot{\theta} + \frac{2}{3}\theta\mathrm{div}\mathbf{u} = \frac{1}{3\rho}(-\mathbf{P}\cdot\mathbf{S} - \mathrm{div}\mathbf{q}), \tag{3.13}
$$

system (3.1)-(3.6) is local (does not contain $\dot{\theta}$ on the right hand side) after using (3.13). Moreover, (3.13) suggests that $\dot{\theta}+\frac{2}{3}\theta\mathrm{div}\mathbf{u} = O(\mu)$, and $\mathbf{P}_3$ and $\mathbf{q}_3$ are $O(\mu^3)$, thus belong to the super Burnett order. It is a trivial observation that (3.1)-(3.6) yield a representation of $\mathbf{P}$, $\mathbf{q}$ in powers of μ, which agrees with the classical Chapman-Enskog expansion (2.2)-(2.7) to Burnett order, i.e., terms of order μ^2, with a correction at $O(\mu^3+\mu^2\varepsilon)$ since $\lambda = O(\varepsilon)$.

4. A Generalized Entropy Inequality

We recall a generalized entropy inequality (Theorem 4.2) for the relaxation systems (3.1)-(3.7) in the case of purely elastic collisions, $a = 1, \varepsilon = 0$ [11].

Lemma 4.1. Let $\mathbf{P}, \mathbf{q}$ be given by (3.1)-(3.12) with

$$\hat{\lambda_1} = -\frac{1}{2}\theta\frac{\partial\theta_2}{\partial\theta} - \theta_2\theta\frac{\mu'(\theta)}{\mu(\theta)} + \frac{3}{2}\theta_2,$$
$$\hat{\gamma_1} = -\frac{1}{2}\theta\frac{\partial\omega_2}{\partial\theta} - \omega_2\theta\frac{\mu'(\theta)}{\mu(\theta)} + \omega_2, \tag{4.1}$$

in (3.9), (3.12) respectively. Then the following equality holds:

$$\begin{aligned}
&\rho\left\{-\eta + \frac{1}{2}\mathrm{tr}\left(\frac{\omega_2\mathbf{P}^2}{4\rho p\theta}\right) + \frac{1}{M}\left(\frac{2\theta_2|\mathbf{q}|^2}{M\rho^2\theta^3}\right)\right\}^{\bullet} + \mathrm{div}\left\{\frac{\mathbf{q}}{\theta} + \frac{\omega_3\mathbf{P}\mathbf{q}}{M\rho\theta^2}\right\}\\
=& -\frac{1}{2}\frac{\mathrm{tr}\mathbf{P}^2}{\mu\theta} - \frac{2}{M}\frac{|\mathbf{q}|^2}{\mu\theta^2} + \frac{2\omega_2-\omega_6}{4p\theta}\mathrm{tr}(\mathbf{S}\mathbf{P}^2)\\
&+ \frac{2}{M}\left(\frac{2\theta_2}{M\rho\theta^3}\right)\mathrm{tr}(\mathbf{S}\mathbf{q}\otimes\mathbf{q}) + \frac{1}{M}(-\overline{\theta_5} - \omega_3 + \omega_3'(\theta)\theta)\mathrm{tr}(\mathbf{P}\mathrm{grad}\theta\otimes\mathbf{q})\\
&+ \left\{\hat{\omega_2}\frac{\mathrm{tr}\mathbf{S}^2}{p^2} + 3\hat{\omega_3}\frac{|\mathrm{grad}\theta|^2}{\rho^2\theta^3}\right\}\frac{3\mu}{\theta}\mathrm{tr}\mathbf{P}^2\\
&+ \left\{\hat{\theta_2}\frac{\mathrm{tr}\mathbf{S}^2}{p^2} + 3\hat{\theta_3}\frac{|\mathrm{grad}\theta|^2}{\rho^2\theta^3}\right\}\mu\frac{|\mathbf{q}|^2}{\theta^2}\left(\frac{2}{M}\right).
\end{aligned} \tag{4.2}$$

Theorem 4.2: Under the hypothesis of Lemma 4.1, we have

$$\begin{aligned}
&\rho\left\{-\eta + \frac{1}{2}\mathrm{tr}\left(\frac{\omega_2\mathbf{P}^2}{4\rho p\theta}\right) + \frac{1}{M}\left(\frac{2\theta_2|\mathbf{q}|^2}{M\rho^2\theta^3}\right)\right\}^{\bullet} + \mathrm{div}\left\{\frac{\mathbf{q}}{\theta} + \frac{\omega_3\mathbf{P}\mathbf{q}}{M\rho\theta^2}\right\}\\
\leq& -\frac{1}{2}\frac{\mathrm{tr}\mathbf{P}^2}{\mu\theta} - \frac{2}{M}\frac{|\mathbf{q}|^2}{\mu\theta^2} + \frac{|2\omega_2-\omega_6|}{4\sqrt{2}p\theta}\left((\mathrm{tr}\mathbf{S}^2)(\mathrm{tr}\mathbf{P}^2)\right)^{1/2}(\mathrm{tr}\mathbf{P}^2)^{1/2}\\
&+ \frac{2}{M}\left(\frac{2\theta_2}{M\rho\theta^3}\right)\left((\mathrm{tr}\mathbf{S}^2)^{1/2}|\mathbf{q}|\right)|\mathbf{q}|\\
&+ \frac{1}{M\rho\theta^3}|-\overline{\theta_5} - \omega_3 + \omega_3'(\theta)\theta|(\mathrm{tr}\mathbf{P}^2)^{1/2}|\mathrm{grad}\theta||\mathbf{q}|\\
&+ \left\{\hat{\omega_2}\frac{\mathrm{tr}\mathbf{S}^2}{p^2} + 3\hat{\omega_3}\frac{|\mathrm{grad}\theta|^2}{\rho^2\theta^3}\right\}\frac{3\mu}{\theta}\mathrm{tr}\mathbf{P}^2 + \left\{\hat{\theta_2}\frac{\mathrm{tr}\mathbf{S}^2}{p^2} + 3\hat{\theta_3}\frac{|\mathrm{grad}\theta|^2}{\rho^2\theta^3}\right\}\mu\frac{|\mathbf{q}|^2}{\theta^2}\left(\frac{2}{M}\right).
\end{aligned}$$

Furthermore, if we define $\mathbf{z} \in \mathbb{R}^5$ by

$$\mathbf{z} = \left[\left(\frac{tr\mathbf{P}^2}{\theta}\right)^{1/2},\ \sqrt{2}\frac{|\mathbf{q}|}{\sqrt{M}\theta},\ \mu\frac{(tr\mathbf{S}^2tr\mathbf{P}^2)^{1/2}\sqrt{3}}{p(\theta)^{1/2}},\ \mu\sqrt{2}\frac{(tr\mathbf{S}^2)^{1/2}|\mathbf{q}|}{p\theta(M)^{1/2}},\ \sqrt{\frac{2}{3}}\frac{|\mathrm{grad}|\ \theta|\mathbf{q}|}{p\theta^{3/2}}\right], \tag{3.14}$$

then (4.18) maybe written as

$$\begin{aligned}
&\rho\left\{-\eta + \frac{1}{2}\mathrm{tr}\left(\frac{\omega_2\mathbf{P}^2}{4\rho p\theta}\right) + \frac{1}{M}\left(\frac{2\theta_2|\mathbf{q}|^2}{M\rho^2\theta^3}\right)\right\}^{\bullet} + \mathrm{div}\left\{\frac{\mathbf{q}}{\theta} + \frac{\omega_3\mathbf{P}\mathbf{q}}{M\rho\theta^2}\right\}\\
\leq& -\frac{1}{\mu}\mathbf{z}\cdot\mathbf{D}\mathbf{z},
\end{aligned} \tag{4.20}$$

where

$$\mathbf{D} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{|\omega_6 - 2\omega_2|}{8\sqrt{2}} & 0 & \frac{1}{2M}\sqrt{\frac{2}{3}}| - \bar{\theta}_5 - \omega_3 + \theta\omega_3'(\theta)| \\ 0 & 1 & 0 & \frac{1}{3M} & 0 \\ \frac{|\omega_6 - 2\omega_2|}{8\sqrt{2}} & 0 & \hat{\omega}_2 & 0 & 0 \\ 0 & \frac{1}{3M} & 0 & \hat{\theta}_2 & 0 \\ \frac{1}{2M}\sqrt{\frac{2}{3}}| - \bar{\theta}_5 - \omega_3 + \theta\omega_3'(\theta)| & 0 & 0 & 0 & \frac{\hat{\theta}_3}{M} \end{bmatrix}.$$

$\mathbf{D}$ is positive definite if $\hat{\omega}_2 > 0, \hat{\theta}_2 > 0$, $\hat{\theta}_3 > 0$ are sufficiently large, $\hat{\omega}_3 = 0$, and $| - \bar{\theta}_5 - \omega_3 + \theta\omega_3'(\theta)|$, $|\omega_6 - 2\omega_3|$ are bounded.

5. Hyperbolicity

In this section we study the hyperbolicity of the relaxation approximation (3.1)-(3.6). Let $\mathbf{P} = (P^{ij})_{3\times 3}$ and $\mathbf{q} = (q_1, q_2, q_3)$. While it is too difficult to study its hyperbolicity in three space dimension, we assume that all quantities depend on x only,

$$\mathbf{u} = (u(x,t), 0, 0)$$

and look for special solution

$$P^{23} = P^{13} = P^{12} = q_2 = q_3 = 0.$$

It is easy to show that these are exact solutions to (3.4) and (3.6). Furthermore, one can show that

$$P^{22} = P^{33}$$

is also consistent with (3.4) and (3.6). Since $\mathbf{P}$ has zero trace, this implies that

$$P^{22} = -\frac{1}{2}P^{11}.$$

Thus we are left with five independent variables ρ, u, θ, P^{11} and $q_1 = q$, satisfying the system

$$\rho_t + u\rho_x + \rho u_x = 0, \tag{5.1}$$

$$u_t + uu_x + \frac{1}{\rho}p_x + \frac{1}{\rho}P^{11}_x = b, \tag{5.2}$$

$$\theta_t + u\theta_x + \frac{2p}{\rho}u_x + \frac{2}{\rho}P^{11}u_x + \frac{2}{\rho}q_x = 0, \tag{5.3}$$

$$P^{11}_t + uP^{11}_x - \frac{4}{3}P^{11}u_x = -\frac{2p}{\omega_2\mu}\left(P^{11} - P^{11}_{eq}\right), \tag{5.4}$$

$$q_t + uq_x - qu_x = -\frac{3Mp}{2\theta_2\mu}\left(q - q_{eq}\right), \tag{5.5}$$

where

$$P_{eq}^{11} = -\frac{4}{3}\mu u_x + P_2^{11} + P_3^{11}, \tag{5.6}$$

$$\begin{aligned} P_2^{11} = & -\mu\frac{\omega_1}{2p}P^{11}u_x + \frac{\omega_2\mu'(\theta)\dot{\theta}}{2p}P^{11} - \mu^2\frac{4\omega_3}{3\rho\theta}\left(\frac{q}{\mu M}\right)_x \\ & -\mu\frac{4\omega_4}{3\rho p\theta}\frac{q}{M}p_x - \mu\frac{4\omega_5}{3\rho\theta^2}\frac{q}{M}\theta_x - \mu\frac{\omega_6}{6p}P^{11}u_x, \end{aligned} \tag{5.7}$$

$$P_3^{11} = \mu^2\left[\frac{2\hat{\omega}_2}{3p^2}{u_x}^2 + 3\hat{\omega}_3\frac{{\theta_x}^2}{\rho^2\theta^3}\right]P^{11} - \mu\frac{\hat{\gamma}_1}{p\theta}\frac{2}{\rho}\left({P^{11}}^2u_x + P^{11}q_x\right), \tag{5.8}$$

$$q_{eq} = -\frac{1}{2}\mu M\,\theta_x - \mu\frac{\lambda}{\tilde{\mu}}\theta(\log\rho)_x + q_2 + q_3 \tag{5.9}$$

$$\begin{aligned} q_2 = & -2\mu\frac{\theta_1}{M\rho\theta}qu_x + \frac{2\theta_2\dot{\theta}\mu'(\theta)}{M\rho\theta}q \\ & -\mu\frac{\theta_3}{2p\rho}P^{11}p_x - \mu^2\frac{\theta_4}{2\rho}\left(\frac{P^{11}}{\mu}\right)_x - \mu\frac{\theta_5}{2\rho\theta}P^{11}\theta_x, \end{aligned} \tag{5.10}$$

$$q_3 = \mu^2\left[\frac{2\hat{\theta}_2}{3p^2}{u_x}^2 + 3\hat{\theta}_3\frac{{\theta_x}^2}{\rho^2\theta^3}\right]q - \mu\frac{\hat{\lambda}_1}{\rho\theta^2}\frac{2}{\rho}\left(P^{11}u_x + q_x\right)\left(\frac{q}{\frac{1}{2}M}\right). \tag{5.11}$$

Upon using $p = \rho\theta/3$, and (5.3) to replace $\dot{\theta}$, one obtains the Jacobi matrix for the relaxation system (5.1)-(5.5):

$$\mathbf{J} = \begin{bmatrix} u & \rho & 0 & 0 & 0 \\ \frac{\theta}{3\rho} & u & \frac{1}{3} & \frac{1}{\rho} & 0 \\ 0 & J_{32} & u & 0 & \frac{2}{\rho} \\ J_{41} & J_{42} & J_{43} & u & J_{46} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{56} \end{bmatrix}, \tag{5.12}$$

where

$$\begin{aligned} J_{32} &= \frac{2}{3}\theta + \frac{2}{\rho}P^{11}, \\ J_{41} &= \frac{8\omega_4 q}{9\omega_2\rho M}, \\ J_{42} &= -\frac{4}{3}P^{11} + \frac{8p}{3\omega_2} + \frac{\omega_1}{\omega_2}P^{11} - 2\frac{\mu'}{\mu}P^{11}\left(-\frac{1}{3}\theta - \frac{1}{\rho}P^{11}\right) \\ &\quad + \frac{\omega_6}{3\omega_2}P^{11} + \frac{4\hat{\gamma}_1}{3\omega_2 p}{P^{11}}^2, \\ J_{43} &= -\frac{8\omega_3 q\mu'}{9\omega_2\mu M} + \frac{8(\omega_4+\omega_5)q}{9\omega_2\theta M}, \end{aligned}$$

$$
\begin{aligned}
J_{46} &= \frac{2\mu' P^{11}}{\mu\rho} + \frac{8\omega_3}{9\omega_2 M} + \frac{4\hat{\gamma}_1 P^{11}}{3\omega_2 p}, \\
J_{51} &= \frac{\theta_3 M\theta}{4\theta_2\rho}P^{11} + \frac{3M\lambda\theta p}{2\theta_2\tilde{\mu}\rho}, \\
J_{52} &= -q + \frac{\theta_1}{\theta_2}q - \frac{\mu'}{\mu}q\left(-\frac{2}{3}\theta - \frac{2}{\rho}P^{11}\right) + \frac{2\hat{\lambda}_1 q}{3\theta_2 p}P^{11}, \\
J_{53} &= \frac{3M^2 p}{4\theta_2} - \frac{3Mp\theta_4\mu'}{4\theta_2\rho\mu}P^{11} + \frac{M(\theta_3+\theta_5)}{4\theta_2}P^{11}, \\
J_{54} &= \frac{3Mp\theta_4}{4\theta_2\rho}, \\
J_{56} &= u + \frac{2\mu' q}{\mu\rho} + \frac{2\hat{\lambda}_1 q}{3\theta_2 p}.
\end{aligned} \tag{5.13}
$$

The characteristic polynomial of the Jacobian matrix (5.12) is

$$
\begin{aligned}
&(u-\lambda)^4(J_{56}-\lambda) + (u-\lambda)^3\left[-\frac{2}{\rho}J_{53} - J_{46}J_{54}\right] \\
&\quad + (u-\lambda)^2(J_{56}-\lambda)\left[-\frac{1}{3}J_{32} - \frac{1}{\rho}J_{42} - \frac{1}{3}\theta\right] \\
&\quad + (u-\lambda)^2\left[\frac{2}{\rho}J_{43}J_{54} + \frac{2}{3\rho}J_{52} + \frac{1}{\rho}J_{46}J_{52}\right] + (u-\lambda)(J_{56}-\lambda)\left[\frac{1}{\rho}J_{32}J_{43} + J_{41}\right] \\
&\quad + (u-\lambda)\left[-\frac{2}{3\rho}J_{42}J_{54} + \frac{1}{3}J_{32}J_{46}J_{54} + \frac{2}{\rho^2}J_{42}J_{53} - \frac{2}{\rho^2}J_{43}J_{52}\right. \\
&\qquad\qquad \left. -\frac{1}{\rho}J_{32}J_{46}J_{53} - J_{46}J_{51} + \frac{1}{3}\theta J_{46}J_{54} - \frac{2}{3}J_{51} + \frac{2\theta}{3\rho}J_{53}\right] \\
&\quad + \left[-\frac{2\theta}{3\rho}J_{43}J_{54} - \frac{2}{\rho}J_{41}J_{53} + \frac{2}{\rho}J_{43}J_{51} + \frac{2}{3}J_{41}J_{54}\right].
\end{aligned} \tag{5.14}
$$

While it is very difficult to see whether the characteristic polynomial (5.14) has five real roots, we instead investigate the linearized system (5.1)-(5.5) around

$$(\overline{\rho}, 0, \overline{\theta}, 0, 0),$$

where $\overline{\rho}, \overline{\theta}$ are constants. In this case, the quantities in (5.13) becomes

$$
\begin{aligned}
&J_{32} = \frac{2}{3}\theta\,, \\
&J_{41} = 0\,, \quad J_{42} = \frac{8\overline{p}}{3\omega_2}\,, \quad J_{43} = 0\,, \quad J_{46} = \frac{8\omega_3}{9\omega_2 M}\,, \\
&J_{51} = \frac{3M\lambda\overline{\theta}\overline{p}}{2\theta_2\tilde{\mu}\overline{\rho}}\,, \quad J_{52} = 0\,, \quad J_{53} = \frac{3M^2\overline{p}}{4\theta_2}\,, \quad J_{54} = \frac{3M\overline{p}\theta_4}{4\theta_2\overline{\rho}}\,, \quad J_{56} = 0\,,
\end{aligned} \tag{5.15}
$$

where $\overline{p} = \frac{1}{3}\overline{\theta}\overline{\rho}$. The characteristic polynomial (5.13) reduces to (upon changing $u - \lambda \to -\lambda$):

$$\lambda\left[\lambda^4 + a_1\lambda^2 + a_0\right], \tag{5.16}$$

with the coefficients

$$\begin{aligned} a_0 = &\left[-\frac{4M\theta_4}{3\omega_2\theta_2} + \frac{4\omega_3\theta_4}{9\omega_2\theta_2} + \frac{4M^2}{\omega_2\theta_2} - \frac{4M\omega_3}{3\omega_2\theta_2} - \frac{4\omega_3\lambda}{\omega_2\theta_2\tilde{\mu}}\right.\\ &\left.+\frac{2\omega_3\theta_4}{3\omega_2\theta_2} - \frac{3M\lambda}{\theta_2\tilde{\mu}} + \frac{3M^2}{2\theta_2}\right]\left(\frac{\overline{p}}{\overline{\rho}}\right)^2 \\ a_1 = &-\left[\frac{3M^2}{2\theta_2} + \frac{2\omega_3\theta_4}{3\omega_2\theta_2} + \frac{5}{3} + \frac{8}{3\omega_2}\right]\left(\frac{\overline{p}}{\overline{\rho}}\right) \end{aligned} \tag{5.17}$$

After plugging in all the parameters,

$$\begin{aligned} a_0 =&[17.1919 - 2.2852\varepsilon + O(\varepsilon^2)]\left(\frac{\overline{p}}{\overline{\rho}}\right)^2 \\ a_1 =&[-9.6467 - 0.6795\varepsilon + O(\varepsilon^2)]\left(\frac{\overline{p}}{\overline{\rho}}\right) \end{aligned} \tag{5.18}$$

The characteristic polynomial (5.16) has five distinct real roots

$$0, \quad \pm\sqrt{4.8233 + 0.3397\varepsilon \pm \sqrt{6.0728 + 5.5627\varepsilon}}\sqrt{\frac{\overline{p}}{\overline{\rho}}}$$

after ignoring the $O(\varepsilon^2)$ terms. Thus the linearized relaxation system is hyperbolic.

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