A steady-state capturing method for hyperbolic systems with geometrical source terms

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Abstract

We propose a simple numerical method for capturing the steady state solution of hyperbolic systems with geometrical source terms. We use the interface value, rather than the cell-averages, for the source terms that balance the nonlinear convection at the cell interface, allowing the numerical capturing of the steady state with a formal high order accuracy. This method applies to Godunov or Roe type upwind methods but requires no modification of the Riemann solver. Numerical experiments on scalar conservation laws and the one dimensional shallow water equations show much better resolution of the steady state than the conventional method, with almost no new numerical complexity.

1 Introduction

Hyperbolic systems with geometric source terms arise in various applications ranging from shallow water with bottom tomography to nozzle flows. In such applications the source terms may have low regularity or possibly concentrations, which cause numerical difficulty when one wants to obtain the steady state solutions [4]. The numerical viscosity, an inevitable feature in modern shock capturing schemes, is blamed for the discrepancy of the steady state solutions. Two possible fixes are available in the literature to handle this numerical difficulty. One is to approximate the steady state balance numerically using an upwind source term, due to an early idea of Roe for linear scalar equations [8]. The nonlinear version of this idea was carried out in [1, 9] using characteristic decomposition also on the source term. Another approach is to incorporate the steady state balance into the numerical flux by modifying the Riemann solver, so the steady state solution is obtained exactly by the cell averages, see [6, 2].

In this paper we propose a very simple method to handle this numerical difficulty. The designing principle is to numerically preserving the steady state at the cell *interface*. To achieve this one needs to use the interface values, instead of the cell-averages, for the source term. In the linear case our method reduces to that of Roe, but the nonliner version is strikingly simple and differs from the more delicate extension proposed in [1].

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In the sequel, we let $x_{j+1/2}$ be a grid point, $\Delta x = x_{j+1/2} - x_{j-1/2}$ be the uniform mesh size, and $t^n = n\Delta t$ be the discrete time with time step Δt. Let u_j^n be the cell average of u at t^n over the cell $[x_{j-1/2}, x_{j+1/2}]$,

$$u_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^n)\,dx\,,$$

and $u_{j+1/2}^n = u(x_{j+1/2}, t^n)$ be the approximate value of u at the cell interface $x_{j+1/2}$.

The paper is organized as follows. In Section 2 we first introduce the method for one-dimensional scalar problem and carry out some numerical experiments. In Section 3 the idea is extended to the shallow-water equations, along with two numerical experiments.

2 One-dimensional scalar problems

Consider the scalar conservation law

$$\partial_t u + \partial_x f(u) = -z'(x)b(u)\,. \tag{2.1}$$

Assume

$$a(u) = f'(u)\,, \qquad D(u) = \int_0^u \frac{a(u)}{b(u)}\,ds\,. \tag{2.2}$$

The steady state solution is given by

$$\partial_x f(u) + z'(x)b(u) = 0\,, \tag{2.3}$$

which can be integrated out to give the algebraic expression

$$D(u) + z(x) = \text{constant}\,. \tag{2.4}$$

Due to the first order numerical viscosity used in shock capturing methods, a standard discretization (split or unsplit) over (2.1) can not preserve the steady state solution (2.4) exactly at the cell average. Rather, a first order numerical discretization error contributes to the numerical discrepancy, especially when the source terms lack regularity [2].

2.1 Numerical methods

We propose a simple method that preserves the steady state solution (2.4) –for smooth solutions – approximately with at least a formally second order accuracy at the cell *interface*. Applying this principle makes the designing of the scheme extremely simple. In fact one only needs to use the interface value of u, which is already available in the numerical flux of the Godunov or Roe type, for the source term. Consequently, one also obtains a formally high order approximation to (2.4) at not only at the cell interface but also for the cell averages.

The numerical method for (2.1) that we propose is the following (will be called the *interface method* thereafter):

$$\partial_t u_j + \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} = -\frac{z_{j+1/2} - z_{j-1/2}}{\Delta x}\,\frac{b_{j+1/2} + b_{j-1/2}}{2}\,, \tag{2.5}$$

where $b_{j+1/2} = b(u_{j+1/2})$. Here $f_{j+1/2}$ is the numerical flux of $f(u)$ defined at the cell interface $x_{j+1/2}$, and different definitions of the flux yield different shock capturing methods. For example, in the Godunov [3] or Roe [7] method, $f_{j+1/2} = f(u_{j+1/2})$ where $u_{j+1/2}$ is the solution of the Riemann or approximate Riemann problem for $u_t + f(u)_x = 0$ with initial data u_j for $x < 0$ and u_{j+1} for $x > 0$. This provides a second order discretization of the source term. Although for generic flux $f(u)$ it is impossible to derive a explicit form for $D(u)$, however, the examples and applications in the sequel shows that this discretization yields formally second order approximations for smooth solutions to the steady state solution at the interfaces.

The time marching method may be the standard forward Euler method, since for steady state solution the order of time discretization is not important.

The method can easily be extended to systems of hyperbolic equations with source terms, as will be done for the shallow water equations in Section 3.

For numerical comparison we will also use the conventional method, where the source term is defined by the cell average of u:

$$\partial_t u_j + \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} = -\frac{z_{j+1/2} - z_{j-1/2}}{\Delta x}\, b(u_j)\,. \tag{2.6}$$

This is referred to as the *cell-average method.*

If $D(u)$ is monotone, the following method for (2.1) is more accurate:

$$\partial_t u_j + \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} = -\frac{z_{j+1/2} - z_{j-1/2}}{\Delta x}\, \frac{f_{j+1/2} - f_{j-1/2}}{D_{j+1/2} - D_{j-1/2}} \tag{2.7}$$

The discretization of the source term is clearly second order.

For steady state solution, one drops the time derivative so (2.7) becomes

$$\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} + \frac{z_{j+1/2} - z_{j-1/2}}{\Delta x}\, \frac{f_{j+1/2} - f_{j-1/2}}{D_{j+1/2} - D_{j-1/2}} = 0\,, \tag{2.8}$$

or simply

$$D_{j+1/2} - D_{j-1/2} + z_{j+1/2} - z_{j-1/2} = 0\,. \tag{2.9}$$

Thus one gets

$$D_{j+1/2} + z_{j+1/2} \equiv \text{constant} \qquad \text{for all} \quad j\,. \tag{2.10}$$

Therefore, the correct steady state solution (2.4) is preserved *exactly* at the cell interface $x_{j+1/2}$.

2.2 Numerical examples

We first use the Burgers equation with source term describing bathimetry in the SVS model:

$$\partial_t u + \partial_x \frac{u^2}{2} + z'(x) u = 0\,, \quad 0 < x < 10\,, \tag{2.11}$$

$$u(x,0) = 0 \quad \text{for} \quad x > 0\,, \qquad u(0,t) = 2 \quad \text{for} \quad t > 0\,. \tag{2.12}$$

The steady state solution of this problem is given by the simple algebraic expression

$$u + z = 2\,. \tag{2.13}$$

For a continuous bottom the function $z(x)$ is chosen as

$$z(x) = \begin{cases} \cos(\pi x)\,, & 4.5 \le x \le 5.5\,; \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.14}$$

For a discontinous bottom we choose

$$z(x) = \begin{cases} \cos(\pi x)\,, & 5 \le x \le 6\,; \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.15}$$

These are the examples used in [2]. Note that for $f(u) = u^2/2$, method (2.8) is identical to method (2.5). We use the Godunov flux to define $f_{j+1/2}$ and the corresponding $u_{j+1/2}$ for the source term. The time discretization is the forward Euler method.

We use 100 grid points and time step $\Delta t = 0.025$. For the continuous bottom, the cell average value of the exact steady state solution ($u = 2 - z$) is compared with the numerical cell average by the cell-average method (2.6) in Figure 2.1a, while the comparison with the cell average of the numerical solution by the interface method (2.5) is shown in Figure 2.1b. One can see the numerical error beyond the bump for the cell-average method does not appear in the interface method. The L^1 and L^∞ errors of these two methods for steady state solution are given in Table 2.1. We did not normalize the length of the domain (which is 10) in the calculation of the L^1 norm. Note that both methods converge linearly, in L^∞ and L^1 norms, to the exact solution, and the L^∞ errors are of comparable size, but the L^1 errors of the interface method are about four times smaller.

Similar comparisons are made for the case of the discontinuous bottom, and the corresponding results are displayed in Figure 2.2a and 2.2b. The difference between the interface and the cell-average methods is more striking for this case. As shown by Table 2.2, the L^1 error of the cell-average method (2.6) does not decay with shrinking grid sizes, while the interface method (2.5) converges linearly to the steady state and the errors are about the same size as in the case of continuous bottom. We do not compare the L^∞ error since it is not a good measure for discontinuous solutions.

3 Shallow water equations

Consider the one-dimensional Saint-Vernon equations for shallow water:

$$\partial_t h + \partial_x(hu) = 0\,, \tag{3.1}$$

$$\partial_t(hu) + \partial_x(hu^2 + \tfrac{1}{2}gh^2) = -ghB'(x)\,. \tag{3.2}$$

Here h is the depth of the fluid, u is the mean velocity, g is the gravitational constant, and $B(x)$ is the bottom elevation. This system describes flows in one-dimensional channel with the bottom elevation given by $B(x)$. The steady state solution is given by

$$\partial_x(hu) = 0\,, \tag{3.3}$$

$$\partial_x(hu^2 + \tfrac{1}{2}gh^2) = -ghB'(x)\,, \tag{3.4}$$

Number of nodes	Interface L^∞ error	Interface L^1-error	Average L^∞ error	Average L^1-error
101	$1.564 * 10^{-1}$	$1.000 * 10^{-1}$	$1.710 * 10^{-1}$	$5.025 * 10^{-1}$
201	$7.846 * 10^{-2}$	$5.000 * 10^{-2}$	$8.594 * 10^{-2}$	$2.641 * 10^{-1}$
401	$3.926 * 10^{-2}$	$2.500 * 10^{-2}$	$4.296 * 10^{-2}$	$1.354 * 10^{-1}$
801	$1.963 * 10^{-2}$	$1.250 * 10^{-2}$	$2.150 * 10^{-2}$	$6.851 * 10^{-2}$

Table 1: Comparison of errors for the continuous bottom case. Here "interface" stands for the interface method, while "Average" is for the cell-average method.

Number of nodes	Interface L^1-error	Average L^1-error
101	$2.000 * 10^{-1}$	2.983
201	$1.000 * 10^{-1}$	2.811
401	$5.000 * 10^{-2}$	2.769
801	$2.500 * 10^{-2}$	2.654

Table 2: Comparison of errors for the discontinuous bottom case. Here "interface" stands for the interface method, while "Average" is for the cell-average method.

or

$$(hu) = \text{constant} \quad \text{and} \quad \frac{1}{2}u^2 + gh + gB = \text{constant}\,. \tag{3.5}$$

This goal is to design a numerical approximation to (3.1) that captures this steady state solution with a formally high (at least second) order accuracy.

3.1 Numerical methods

We extend the interface method proposed for scalar problem into system (3.1). The scheme takes the following simple form:

$$\partial_t h_j + \frac{(hu)_{j+1/2} - (hu)_{j-1/2}}{\Delta x} = 0\,, \tag{3.6}$$

$$\partial_t (hu)_j + \frac{(hu^2 + \frac{1}{2}gh^2)_{j+1/2} - (hu^2 + \frac{1}{2}gh^2)_{j-1/2}}{\Delta x} = -g\frac{h_{j+1/2} + h_{j-1/2}}{2}\frac{B_{j+1/2} - B_{j-1/2}}{\Delta x}\,. \tag{3.7}$$

where the flux is the Godunov or Roe type upwind method using the Riemann or approximate Riemann solver. This not only provides a second order approximation to the source term, but also yields the desired property for steady state.

A particular steady state solution is

$$u \equiv 0\,, \qquad h + B = \text{constant}\,. \tag{3.8}$$

It is easy to verify that the method (3.6)-(3.7) captures this solution *exactly* at the cell interface. Note that at steady state, the method gives

$$\frac{(hu)_{j+1/2} - (hu)_{j-1/2}}{\Delta x} = 0\,, \tag{3.9}$$

$$\frac{(hu^2 + \frac{1}{2}gh^2)_{j+1/2} - (hu^2 + \frac{1}{2}gh^2)_{j-1/2}}{\Delta x} = -g\frac{h_{j+1/2} + h_{j-1/2}}{2}\frac{B_{j+1/2} - B_{j-1/2}}{\Delta x}\,. \tag{3.10}$$

If $u_{j+1/2} \equiv 0$ for all j, then the first condition in (3.9) is automatically met, while (3.10) gives

$$\frac{1}{2}g(h_{j+1/2}^2 - h_{j-1/2}^2) = -\frac{1}{2}g(h_{j+1/2} + h_{j-1/2})(B_{j+1/2} - B_{j-1/2})\,. \tag{3.11}$$

This implies that

$$h_{j+1/2} + B_{j+1/2} = h_{j-1/2} + B_{j-1/2} \equiv \text{constant}\,. \tag{3.12}$$

Thus the steady state solution is captured exactly at $x_{j+1/2}$ and the scheme satisfies the so-called *exact C-property* defined in [1] at the cell interface.

We now show that even for nonzero velocity, this method can still capture the steady state with a formally second order accuracy. Assume $u \neq 0$. Equation (3.9) gives

$$(hu)_{j+1/2} = C\,, \qquad \text{or} \quad h_{j+1/2} = \frac{C}{u_{j+1/2}} \tag{3.13}$$

where C is a constant. Applying this in (3.10), one gets

$$C(u_{j+1/2} - u_{j-1/2}) + \frac{1}{2}g(h_{j+1/2}^2 - h_{j-1/2}^2) = -\frac{g}{2}(h_{j+1/2} + h_{j-1/2})(B_{j+1/2} - B_{j-1/2})\,, \tag{3.14}$$

or

$$2C\frac{u_{j+1.2} - u_{j-1/2}}{h_{j+1/2} + h_{j-1/2}} + g(h_{j+1/2} - h_{j-1/2}) = -g(B_{j+1/2} - B_{j-1/2})\,. \tag{3.15}$$

Using (3.13) one obtains

$$2\frac{(u_{j+1/2} - u_{j-1/2})u_{j+1/2}u_{j-1/2}}{u_{j+1/2} + u_{j-1/2}} + g(h_{j+1/2} - h_{j-1/2}) = -g(B_{j+1/2} - B_{j-1/2})\,. \tag{3.16}$$

Since

$$2\frac{(u_{j+1/2}-u_{j-1/2})u_{j+1/2}u_{j-1/2}}{u_{j+1/2}+u_{j-1/2}} - \left(\tfrac{1}{2}u_{j+1/2}^2 - \tfrac{1}{2}u_{j-1/2}^2\right) \tag{3.17}$$

$$= -\frac{1}{2}\frac{(u_{j+1/2}-u_{j-1/2})^3}{u_{j+1/2}+u_{j-1/2}} = O((\Delta x)^3)\,, \tag{3.18}$$

Equation (3.16) gives

$$\left(\frac{1}{2}u_{j+1/2}^2 - \frac{1}{2}u_{j-1/2}^2\right) + g(h_{j+1/2} - h_{j-1/2}) + g(B_{j+1/2} - B_{j-1/2}) = O((\Delta x)^3)\,. \tag{3.19}$$

Thus for finite domain this implies

$$\left(\frac{1}{2}u^2 + gh + gB\right)_{j+1/2} = \text{constant} + O((\Delta x)^2)\,, \qquad \text{for any} \quad j\,. \tag{3.20}$$

Therefore the steady state solution (3.5) is captured at the cell interface with a second order accuracy. Thus this method has the *approximate C-property*, defined in [1], at the cell interface even to the *general* steady state solution with a nonzero velocity.

To our knowledge this is the only method that has been shown to sastify the approximate C-property in the case of nonzero velocity for the shallow water equations.

3.2 Numerical examples

In this section we present two numerical examples. For comparison we introduce the cell-average method that uses the cell average of h in the source term:

$$\partial_t h_j + \frac{(hu)_{j+1/2}-(hu)_{j-1/2}}{\Delta x} = 0\,, \tag{3.21}$$

$$\partial_t (hu)_j + \frac{(hu^2+\frac{1}{2}gh^2)_{j+1/2}-(hu^2+\frac{1}{2}gh^2)_{j-1/2}}{\Delta x} = -gh_j\frac{B_{j+1/2}-B_{j-1/2}}{\Delta x}\,. \tag{3.22}$$

We first solve the shallow water equation over domain $[0, 1]$ with the bottom elevation

$$B(x) = \begin{cases} 0.35 * [\cos(\pi(x - .5)/.1) + 1.]\,, & 0.3 < x < 0.5\,; \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.23}$$

The initial data are $h_i = 1-\frac{1}{2}(B_{i-1/2}+B_{i+1/2})$ and $u_i \equiv 0$. The zero-th order extrapolation is used at the boundary for ghost cells. The Godunov flux is used to define the flux for the convection and the corresponding value of h at $x_{j+1/2}$ is used for the source term. The time discaretization is the forward Euler method. We choose $\Delta x = 0.01$ and $\Delta t = 0.005$. The steady state cell-average solution of $h + B$ for both the interface and cell-average methods are depicted in Figure 3, along with $B(x)$. The exact solution is $h + B = 1$. The discrepancy of the cell-average method appears in the domain where $B(x) \neq 0$, while in the interface method such an error is not visible by eye.

Next we use the initial data are $h_i = 1 - 0.5 * 0.1^2 - \frac{1}{2}(B_{i-1/2} + B_{i+1/2})$ and $u_i \equiv 0.1$. The steady state solution of $u^2/2 + h + B$, where u and h are cell averages, for both methods are depicted in Figure 3.2, along with $B(x)$. The exact solution is $u^2/2 + h + B = 1$. One can see that the interface method gives a better approximation than the cell-average method, although the result is slightly inferior to the zero-velocity case.

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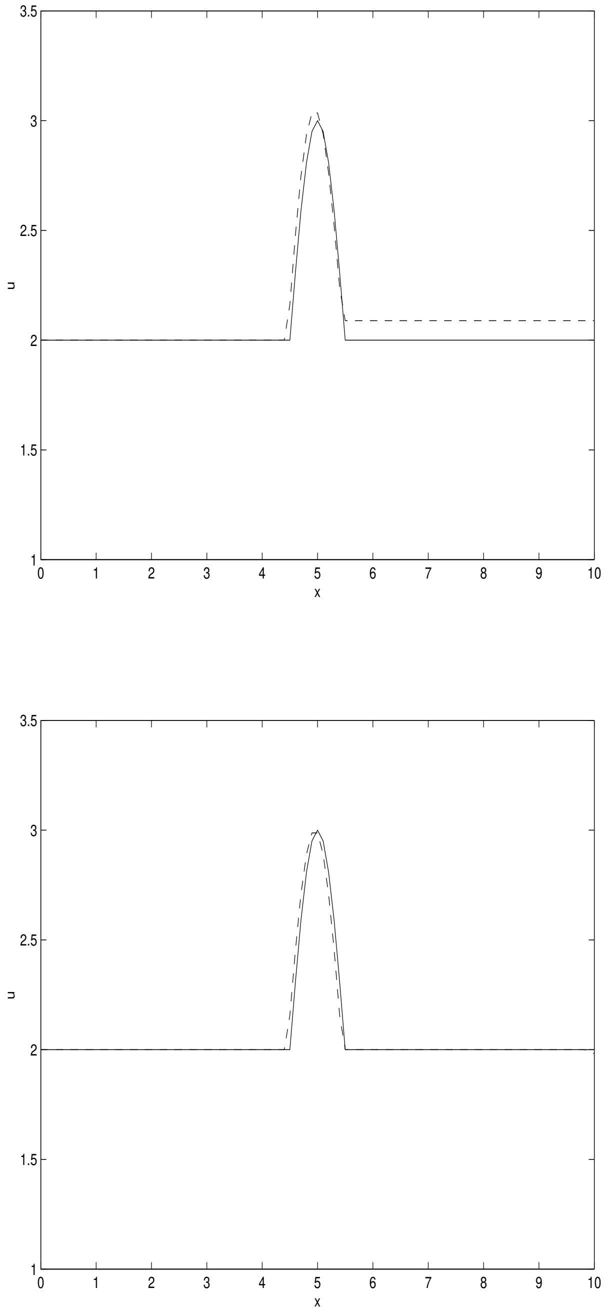

Figure 2.1. Numerical cell average solutions of $h + B$ compared with the exact solution at steady state for a continuous bottom. Solid line: the exact solution; dashed line: the numerical solution. Top: the conventional method (2.6); bottom: the new method (2.5).

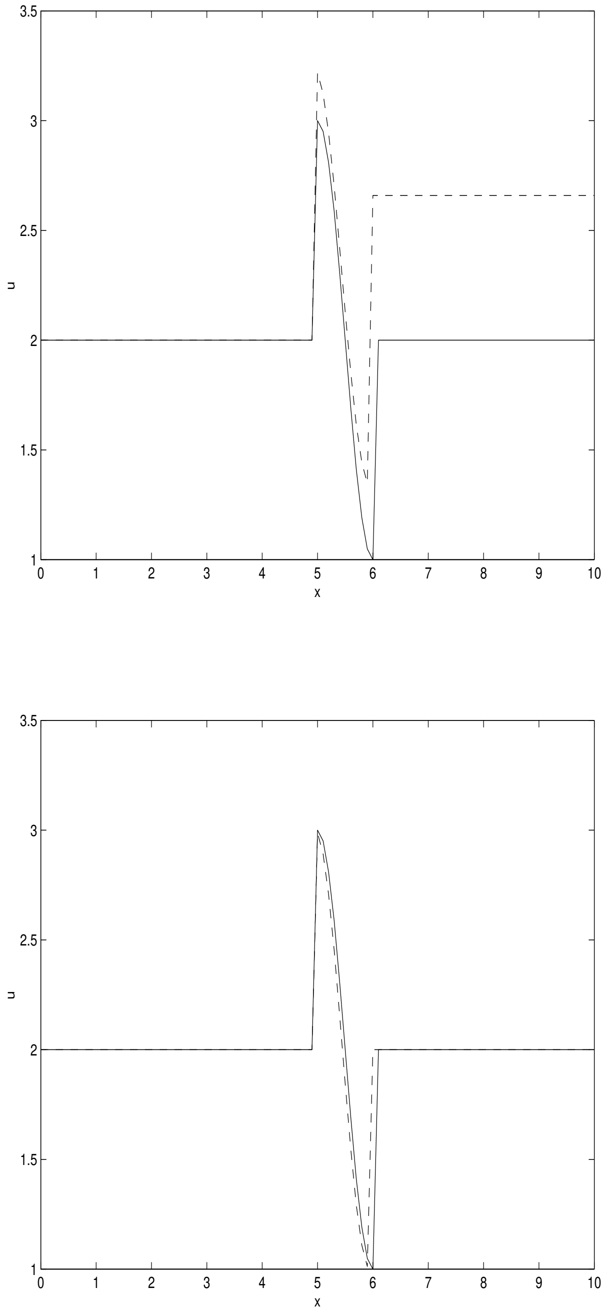

Figure 2.2. Numerical cell average solutions of $h + B$ compared with the exact solution at steady state for a discontinuous bottom. Solid line: the exact solution; dashed line: the numerical solution. Top: the conventional method (2.6); bottom: the new method (2.5).

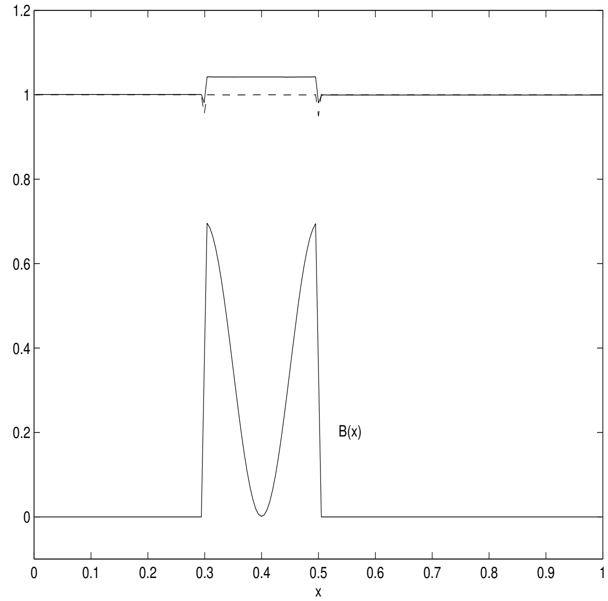

Figure 3. Numerical cell average solutions of $h + B$ compared with the exact solution at steady state for the shallow water equations. The velocity is zero. Solid line: the cell-average method; dashed line: the interface method.

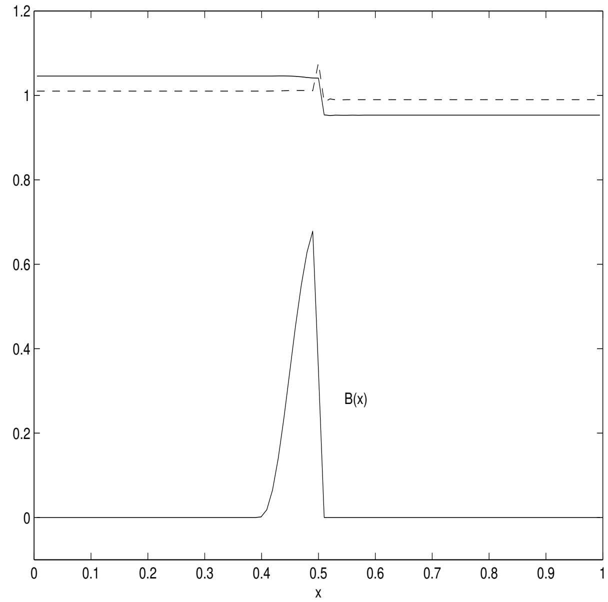

Figure 4. Numerical cell average solutions of $u^2/2 + h + B$ compared with the exact solution at steady state for the shallow water equation. The velocity is nonzero. Solid line: the cell-average method; dashed line: the interface method.