Relaxation and the Chapman-Enskog Expansion

Shi Jin*
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332
email: jin@math.gatech.edu

Marshall Slemrod**

Center for the Mathematical Sciences
University of Wisconsin-Madison
1308 W. Dayton St.
Madison, WI 53715-1149
email: slemrod@math.wisc.edu

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0. Introduction

The purpose of this note is to present a short discussion of our recent investigation of the interrelation between relaxation and the Chapman-Enskog expansion for the Boltzmann equation. A more detailed presentation appears in our paper [1].

1. A model problem

As a simple motivation to the relaxation procedure for regularizing the Chapman-Enskog expansion for the Boltzmann equation consider the following model problem

$$u_t = \epsilon q_x, \tag{1.1}$$

$$q_t = -\frac{(q - u_x)}{\epsilon}, \tag{1.2}$$

where $\epsilon > 0$.

System (1.1)-(1.2) of course can be rewritten as

$$u_t - \epsilon q_x = 0 \tag{1.3}$$

$$\epsilon q_t - u_x = -q \tag{1.4}$$

and is hyperbolic. Moreover multiplication of (1.3) by u and (1.4) by ϵq yield

$$\left(\frac{u^2}{2} + \frac{\epsilon^2 q^2}{2}\right)_t - (\epsilon q u)_x = -\epsilon q^2, \tag{1.5}$$

which immediately implies an entropy inequality for a convex entropy. Therefore the rest state $u = 0$, $q = 0$ is stable. Of course, stability can also be proven by direct analysis of (1.3), (1.4) via spectral decomposition.

Now consider the Chapman-Enskog expansion for (1.1), (1.2):

$$\begin{aligned} q - u_x &= -\epsilon q_t \\ q - u_x &= -\epsilon u_{xt} + \text{ higher order terms in } \epsilon. \end{aligned} \tag{1.6}$$

If we substitute (1.6) into the balance law (1.3) we find "Navier-Stokes" order ϵ

$$u_t = \epsilon u_{xx} \tag{1.7}$$

and $u = 0$ is stable. However, if we expand to "Burnett" order ϵ^2 we see

$$u_t = \epsilon u_{xx} - \epsilon^2 u_{xxt}. \tag{1.8}$$

If we set $u(x,t) = e^{\lambda t} e^{\omega x}$ and substitute into (1.8) we find

$$\lambda = \frac{-\epsilon\omega^2}{1 - \epsilon^2\omega^2}.$$

Hence for $w > \frac{1}{\epsilon}$ we have $\lambda > 0$, i.e. high wave number instability. Since the original system (1.1), (1.2) definitely has $u = 0$ stable there is an apparent paradox. This phenomenon has been noted for the case of incompressible viscoelastic fluids by Coleman, Duffin, and Mizel [2] and for compressible gas dynamics by Bobylev [3], [4], see also the papers of Rosenau [5], [6].

Of course, the mistake is that (1.8) is not exact and possible large quantities are omitted in the truncation provided by the "Burnett" equation (1.8). At this point there are (at least) two approaches that may be taken. The first was suggested by Rosenau [5]: simply sum the Chapman-Enskog expansion to recover the exact "constitutive relation" for q as a function u_x. In fact, for a linearized system of equations derived from Grad's 13 moment system of gas dynamics this has been done by Gorban and Karlin [7]. As yet no exact summation has succeeded in non-linear cases. A second approach is perhaps naive but a little thought points to its scientific merit. Specifically if truncation of the Chapman-Enskog expansion beyond Navier-Stokes order leads to unphysical stability results perhaps system (1.1)-(1.2) which has stable equilibria and is well posed is indeed a better model. But in practice this is not so easily done. Recall that in rarefied gas dynamics we are not given initially a system of partial differential equation but only the Boltzmann equation or some near relative of it. So that in reality (1.2) represents the Boltzmann equation and (1.1) represents the conservation laws of momentum and energy. So reversion back to the fundamental Boltzmann equation is not desirable in the actual problem of gas dynamics. Instead we suggest forming a new system of partial differential equations similar to (1.1)-(1.2), i.e. a relaxation system which when expansed on the Knudsen number ϵ (i) captures the Chapman-Enskog expansion to Burnett order, (ii) is hyperbolic (at least locally), (iii) possesses a globally defined "entropy" inequality.

The idea of relaxation approximation for numerical simulation in gas dynamics is of course not new. In [8] a generic method for relaxation of system hyperbolic conservation laws was introduced; a physically natural pressure relation was presented [9]; Grad [10] and later Levermore [11], [12] developed a theory of moment equations. In a similar spirit rational extended thermodynamics [13] by Müller and Ruggeri develops a moment like system of equations except the closure rules are constituted as in continuum mechanics. The main point of our method is that unlike the above mentioned papers it provided item (i) above while not neglecting (ii), (iii), i.e. its Chapman-Enskog expansion exactly matches the classical Chapman-Enskog expansion to Burnett order ϵ^2 and hence provides a natural regularization of the Burnett equations.

2. The relaxation approximation

Denote by

ρ	density
$\mathbf{u}$	velocity
$\mathbf{P}$	pressure deviator
p	mean normal pressure
$\mathbf{b}$	body force
$\mathbf{L}$	(grad $\mathbf{u})^T$
e	internal energy density

$\mathbf{q}$	energy flux vector
θ	temperature
θ_i	i^{th} coefficient of the Chapman-Enskog expansion for $\mathbf{q}$
ω_i	i^{th} coefficient of the Chapman-Enskog expansion for $\mathbf{P}$
M	Maxwell number
μ	viscosity
η	specific entropy

Our relaxation approximation is given by

$$\dot{\rho} + \rho \text{ div } \mathbf{u} = 0, \tag{2.1}$$

$$\rho\dot{\mathbf{u}} + \text{grad } p + \text{div } \mathbf{P} = \rho\mathbf{b}, \tag{2.2}$$

$$\mathbf{P} = \mathbf{P}^T, \tag{2.3}$$

$$\dot{\mathbf{P}} - \mathbf{L}^T\mathbf{P} - \mathbf{P}\mathbf{L} + \frac{2}{3}\text{tr}(\mathbf{P}\mathbf{L})\mathbf{I} = -\frac{2p}{\omega_2\mu}\left(\mathbf{P} - \mathbf{P}^{eq}\right), \tag{2.4}$$

$$\rho\dot{\text{e}} + p \text{ div } \mathbf{u} + \mathbf{P}\cdot\mathbf{S} + \text{div } \mathbf{q} = 0, \tag{2.5}$$

$$\dot{\mathbf{q}} - \mathbf{L}\mathbf{q} = -\frac{3Mp}{2\theta_2\mu}\left(\mathbf{q} - \mathbf{q}^{eq}\right), \tag{2.6}$$

where

$$\mathbf{P}^{eq} = -2\mu\mathbf{S} + \mathbf{P}_2 + \mathbf{P}_3, \tag{2.7}$$

$$\begin{aligned}
\mathbf{P}_2 = &- \mu\frac{\omega_1}{2p}(\text{div}\mathbf{u})\mathbf{P} + \frac{\omega_2\mu'(\theta)\dot{\theta}}{2p}\mathbf{P} \\
&+ \mu^2\frac{\omega_3}{\rho\theta}\left\{-\text{grad}\left(\frac{\mathbf{q}}{\frac{3}{2}\mu MR}\right) + \frac{1}{3}\text{div}\left(\frac{\mathbf{q}}{\frac{3}{2}\mu MR}\right)\mathbf{I}\right\} \\
&+ \mu\frac{\omega_4}{\rho p\theta}\left\{-\frac{1}{2}\text{grad } p \otimes \left(\frac{\mathbf{q}}{\frac{3}{2}MR}\right) - \frac{1}{2}\left(\frac{\mathbf{q}}{\frac{3}{2}MR}\right)\otimes \text{grad } p\right. \\
&\left.+\frac{1}{3}\text{grad } p \cdot \left(\frac{\mathbf{q}}{\frac{3}{2}MR}\right)\mathbf{I}\right\} \\
&- \mu\frac{\omega_5}{\rho\theta^2}\left\{\frac{1}{2}\text{grad}\,\theta \otimes \left(\frac{\mathbf{q}}{\frac{3}{2}MR}\right) + \frac{1}{2}\left(\frac{\mathbf{q}}{\frac{3}{2}MR}\right)\otimes \text{grad}\,\theta\right. \\
&\left.-\frac{1}{3}\text{grad}\,\theta \cdot \left(\frac{\mathbf{q}}{\frac{3}{2}MR}\right)\mathbf{I}\right\} - \mu\frac{\omega_6}{2p}\left\{\frac{1}{2}(\mathbf{S}\mathbf{P} + \mathbf{P}\mathbf{S}) - \frac{1}{3}\text{tr}(\mathbf{P}\mathbf{S})\mathbf{I}\right\},
\end{aligned} \tag{2.8}$$

$$\mathbf{P}_3 = \mu^2\left[\frac{\hat{\omega}_2}{p^2}\text{tr}\mathbf{S}^2 + \hat{\omega}_3\frac{|\text{grad}\,\theta|^2}{R\rho^2\theta^3}\right]\mathbf{P} + \mu\frac{\hat{\gamma}_1}{p\theta}\left(\dot{\theta} + \frac{2}{3}\theta\text{div}\mathbf{u}\right)\mathbf{P}, \tag{2.9}$$

$$\mathbf{q}^{eq} = -\frac{3}{2}\mu MR\,\text{grad}\,\theta + \mathbf{q}_2 + \mathbf{q}_3, \tag{2.10}$$

$$\mathbf{q}_2 = -2\mu\frac{\theta_1}{3MR\rho\theta}(\text{div }\mathbf{u})\mathbf{q} + \frac{2\theta_2\dot{\theta}\mu'(\theta)}{3MR\rho\theta}\mathbf{q}$$
$$-\mu\frac{\theta_3}{2p\rho}\mathbf{P}\text{ grad }p - \mu^2\frac{\theta_4}{2\rho}\text{ div}\left(\frac{\mathbf{P}}{\mu}\right) - \mu\frac{\theta_5}{2\rho\theta}\mathbf{P}\text{ grad }\theta, \tag{2.11}$$

$$\mathbf{q}_3 = \mu^2\left[\frac{\hat{\theta_2}}{p^2}\text{tr}\mathbf{S}^2 + \hat{\theta}_3\frac{|\text{grad}\theta|^2}{R\rho^2\theta^3}\right]\mathbf{q} + \mu\frac{\hat{\lambda_1}}{\rho\theta^2}\left(\dot{\theta} + \frac{2}{3}\theta\text{div}\mathbf{u}\right)\left(\frac{\mathbf{q}}{\frac{3}{2}MR}\right). \tag{2.12}$$

Since the energy equation (2.5) implies that

$$\dot{\theta} + \frac{2}{3}\theta\text{div}\mathbf{u} = \frac{2}{3\rho R}(-\mathbf{P}\cdot\mathbf{S} - \text{div}\mathbf{q}), \tag{2.13}$$

system (2.1)-(2.6) is local (does not contain $\dot{\theta}$ on the right hand side) after using (2.13). Moreover, (2.13) suggests that $\dot{\theta} + \frac{2}{3}\theta\text{div}\mathbf{u} = O(\mu)$, and $\mathbf{P}_3$ and $\mathbf{q}_3$ are $O(\mu^3)$, thus belong to the super Burnett order.

It is a relatively easy exercise to see that (2.1)-(2.7) when expanded in powers of μ (which is proportion to the Knudsen number ϵ) agrees with the classical Chapman-Enskog expansion for the Boltzmann equation to Burnett order μ^2 as given in [14], [15].

Furthermore, in the case of one dimensional motion $\mathbf{u} = (u(x_1,t),0,0)$, $\rho = \rho(x_1,t)$, $\theta = q(x_1,t)$ linearized about $u = 0$, $\rho = \bar{\rho}$, $\theta = \bar{\theta}$ where $\bar{\rho}$, $\bar{\theta}$ are constants (2.1)-(2.7) is hyperbolic. Finally we note that the following generalized entropy inequality holds.

3. A generalized entropy inequality

Let $\mathbf{P},\mathbf{q}$ be given by (2.1)-(2.12) with

$$\begin{aligned}\hat{\lambda_1} &= -\frac{1}{2}\theta\frac{\partial\theta_2}{\partial\theta} - \theta_2\theta\frac{\mu'(\theta)}{\mu(\theta)} + \frac{3}{2}\theta_2,\\ \hat{\gamma_1} &= -\frac{1}{2}\theta\frac{\partial\omega_2}{\partial\theta} - \omega_2\theta\frac{\mu'(\theta)}{\mu(\theta)} + \omega_2,\end{aligned} \tag{3.1}$$

in (2.9), (2.12) respectively. Then the following equality holds:

$$\begin{aligned}&\rho\left\{-\eta + \frac{1}{2}\text{tr}\left(\frac{\omega_2\mathbf{P}^2}{4\rho p\theta}\right) + \frac{1}{3MR}\left(\frac{2\theta_2|\mathbf{q}|^2}{3MR\rho^2\theta^3}\right)\right\}^{\bullet} + \text{div}\left\{\frac{\mathbf{q}}{\theta} + \frac{\omega_3\mathbf{P}\mathbf{q}}{3MR\rho\theta^2}\right\}\\
&= -\frac{1}{2}\frac{\text{tr}\mathbf{P}^2}{\mu\theta} - \frac{2}{3MR}\frac{|\mathbf{q}|^2}{\mu\theta^2} + \frac{2\omega_2 - \omega_6}{4p\theta}\text{tr}(\mathbf{S}\mathbf{P}^2)\\
&\quad + \frac{2}{3MR}\left(\frac{2\theta_2}{3MR\rho\theta^3}\right)\text{tr}(\mathbf{S}\mathbf{q}\otimes\mathbf{q}) + \frac{1}{3MR}(-\overline{\theta_5} - \omega_3 + \omega_3'(\theta)\theta)\text{tr}(\mathbf{P}\text{grad}\theta\otimes\mathbf{q})\\
&\quad + \left\{\hat{\omega_2}\frac{\text{tr}\mathbf{S}^2}{p^2} + \hat{\omega_3}\frac{|\text{grad}\theta|^2}{R\rho^2\theta^3}\right\}\frac{\mu}{R\theta}\text{tr}\mathbf{P}^2\\
&\quad + \left\{\hat{\theta_2}\frac{\text{tr}\mathbf{S}^2}{p^2} + \hat{\theta_3}\frac{|\text{grad}\theta|^2}{R\rho^2\theta^3}\right\}\mu\frac{|\mathbf{q}|^2}{\theta^2}\left(\frac{2}{3MR}\right).\end{aligned} \tag{3.2}$$

It is then a relatively easy matter to show that the right side of (3.2) is less than or equal to $-\frac{1}{\mu}\mathbf{z}\cdot\mathbf{D}\mathbf{z}$ where

$$\mathbf{D}=\begin{bmatrix} \frac{1}{2} & 0 & \frac{|\omega_6-2\omega_2|}{8\sqrt{2}} & 0 & \frac{1}{2M}\sqrt{\frac{2}{3}}|-\bar{\theta}_5-\omega_3+\theta\omega_3'(\theta)| \\ 0 & 1 & 0 & \frac{1}{3M} & 0 \\ \frac{|\omega_6-2\omega_2|}{8\sqrt{2}} & 0 & \hat{\omega}_2 & 0 & 0 \\ 0 & \frac{1}{3M} & 0 & \hat{\theta}_2 & 0 \\ \frac{1}{2M}\sqrt{\frac{2}{3}}|-\bar{\theta}_5-\omega_3+\theta\omega_3'(\theta)| & 0 & 0 & 0 & \frac{\hat{\theta}_3}{M} \end{bmatrix},$$

$$\mathbf{z}=\left[\left(\frac{tr\mathbf{P}^2}{\theta}\right)^{1/2},\ \sqrt{\frac{2}{3}}\frac{|\mathbf{q}|}{\sqrt{MR\theta}},\ \mu\frac{(tr\mathbf{S}^2tr\mathbf{P}^2)^{1/2}}{p(R\theta)^{1/2}},\ \mu\sqrt{\frac{2}{3}}\frac{(tr\mathbf{S}^2)^{1/2}|\mathbf{q}|}{p\theta(MR)^{1/2}},\ \sqrt{\frac{2}{3}}\frac{|\mathrm{grad}|\ \theta|\mathbf{q}|}{p\theta^{3/2}}\right].$$

$\mathbf{D}$ is positive definite if $\hat{\omega}_2>0,\hat{\theta}_2>0$, $\hat{\theta}_3>0$ are sufficiently large, $\hat{\omega}_3=0$, and $|-\bar{\theta}_5-\omega_3+\theta\omega_3'(\theta)|$, $|\omega_6-2\omega_3|$ are bounded.

Hence we have produced a global generalized Clausius-Duhem "entropy inequality" for the relaxation system (2.1)-(2.7).

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