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EFFICIENT ASYMPTOTIC-PRESERVING (AP) SCHEMES FOR SOME MULTISCALE KINETIC EQUATIONS

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Abstract. Many kinetic models of the Boltzmann equation have a diffusive scaling that leads to the Navier-Stokes type parabolic equations as the small scaling parameter approaches zero. In practical applications, it is desirable to develop a class of numerical schemes that can work uniformly with respect to this relaxation parameter, from the rarefied kinetic regimes to the hydrodynamic diffusive regimes. An earlier approach in [11] reformulates such systems into the common hyperbolic relaxation system by Jin and Xin for hyperbolic conservation laws used to construct the relaxation schemes, and then use a multistep time splitting method to solve the relaxation system. Here we observe that the combination of the two time-split steps may yield a hyperbolic-parabolic systems that are more advantageous, in both stability and efficiency, for numerical computations. We show that such an approach yields a class of asymptotic-preserving (AP) schemes which are suitable for the computation of multiscale kinetic problems. We use the Goldstein-Taylor and the Carleman models to illustrate this approach.

Key words. Kinetic equations, multiscale, asymptotic-preserving schemes, diffusion limit

AMS subject classifications. 82A40, 35L65, 65M05

1. Introduction.

Many kinetic models of the Boltzmann equation have a diffusive scaling that leads to the Navier-Stokes type parabolic equations as the small parameter goes to zero. In such problems the diffusive relaxation parameter may differ in several orders of magnitude from the rarefied regimes to the hydrodynamic (diffusive) regimes. In real applications, it is desirable to develop a class of numerical schemes that can work uniformly with respect to this relaxation parameter, from the rarefied kinetic regimes to the hydrodynamic diffusive regimes.

A numerical solution for such problems is called *underresolved* if the numerical discretization does not resolve the kinetic scaling characterized by the mean free path ϵ, i.e., if $\Delta t, \Delta x >> \epsilon^2$, where Δt and Δx are the time step and grid size respectively. Such an underresolved discretization is highly desirable for practical applications when the mean free path becomes small. Two numerical issues naturally arise here. The first is the numerical stiffness, and the second, and the more challenging one, is to make sure that the underresolved numerical solutions do capture the macroscopic, diffusive behavior of the solution. Past experience shows that if, as $\epsilon \to 0$, the scheme becomes a consistent, stable and high order accurate discretization of the parabolic equations derived in the same limit from the continuous kinetic equations, then this scheme does capture the macroscopic behavior even if the numerical solution is underresolved. We call such schemes *asymptotic-preserving* (AP) schemes. A robust AP scheme should allow an implicity discretization for a better numerical stability, yet can be solved *explicitly*. In [11] a systematic approach to devise robust AP schemes for multiscale kinetic equations is introduced, which is applicable to a large class of discrete-velocity kinetic models. It is the basis for our approach here.

Developing robust AP schemes for general hyperbolic systems with stiff relaxation terms have been an active area of research in recent years [1, 3, 4, 7, 8, 20]. In these problems the asymptotic limit is a hyperbolic (or Euler type) system. For parabolic

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limit, past successful development of asymptotic-preserving schemes has been mainly focused on the steday-state problem of linear transport equations. See [6, 9, 10, 16,17]. More recently, schemes capable of handling small mean free path problems in kinetic modeling of semiconductor device were developed in [13, 14].

To illustrate the problem and our approach, we first review the study of Jin, Pareschi and Toscani [11]. Consider, as an illuminating example, the one dimensional Goldstein-Taylor model [5, 23] (also called the telegrapher equations) under the diffusive scaling

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{\epsilon}\frac{\partial u}{\partial x} &= \frac{1}{\epsilon^2}(v-u)\,, \\ \frac{\partial v}{\partial t} - \frac{1}{\epsilon}\frac{\partial v}{\partial x} &= \frac{1}{\epsilon^2}(u-v)\,, \end{aligned} \tag{1.1}$$

where $(x,t) \in R \times R^+$, with nonnegative initial data

$$u(x,0) = u_0(x)\,; \quad v(x,0) = v_0(x)\,.$$

In (1.1) ϵ is the positive relaxation parameter. The macroscopic (fluid) variables for this model are the mass density ρ and the flux j, defined by

$$\rho = u + v\,, \qquad j = \epsilon^{-1}(u-v)\,. \tag{1.2}$$

Since u and v can be expressed in terms of ρ and j, system (1.1) is equivalent to the following system for the mass density and the flux

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \frac{1}{\epsilon^2}\frac{\partial \rho}{\partial x} &= -\frac{2}{\epsilon^2} j\,. \end{aligned} \tag{1.3}$$

In the zero relaxation (or diffusion) limit ($\epsilon \to 0^+$), system (1.3) can be approximated to leading order by

$$j = -\frac{1}{2}\frac{\partial \rho}{\partial x}\,, \tag{1.4a}$$

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\frac{1}{2}\frac{\partial \rho}{\partial x} = 0\,. \tag{1.4b}$$

The state satisfying (1.4a) will be called the *local equilibrium,* while (1.4b) is the equation of continuum mechanics generated by the kinetic model, that in this case is the heat equation.

A key idea to attack this problem, as introduced in [11], is to reformulate (1.3) as a (nonstiff) linear hyperbolic system with stiff relaxation term (hereafter called *the diffusive relaxation system*)

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} &= -\frac{2}{\epsilon^2}\left[j + \frac{1-\epsilon^2}{2}\frac{\partial \rho}{\partial x}\right]\,. \end{aligned} \tag{1.5}$$

Without loss of generality we always assume that $\epsilon \le 1$ throughout the paper. The system (1.5) has the form of the Jin-Xin relaxation system used to construct relaxation schemes for conservation laws (cfr. [12]). A discretization for (1.5) is based on a splitting technique (or the higher order extension), which has an implicit scheme for the "collision" step,

$$\begin{aligned} &\frac{\partial \rho}{\partial t} = 0\,, \\ &\frac{\partial j}{\partial t} = -\frac{2}{\epsilon^2}\left[j + \frac{1-\epsilon^2}{2}\frac{\partial \rho}{\partial x}\right]\,; \end{aligned} \tag{1.6a}$$

followed by an explicit scheme for the convection step:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0\,, \\ &\frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} = 0\,. \end{aligned} \tag{1.6b}$$

Appropriate spatial discretizations are taken in both steps. The schemes so constructed, besides its second order accuracy, has a numerical stability independent of ϵ, and are AP schemes.

In this paper we derive a different discretization procedure, based on the same splitting (1.6). Our idea is again to use an implicit time discretization for (1.6a) and explicit time discretization for (1.6b), but keep the spatial derivative continuous. We combine these two steps into one, which turns out to be an explicit time discretization of a well-behaved hyperbolic-parabolic system. This new system will then be solved by standard second order shock capturing methods. The advantage of this new approach is two folds. First, this scheme has a better stability property since the underlying system is a well-behaved hyperbolic-parabolic system in conservative form endowed with a free energy (entropy). In addition, such an approach is numerically more robust and efficient , and a unified spatial discretization can be taken. The disadvantage, however, is the lack of generality since such a formulation does not apply to every kinetic equations, since for some kinetic equations the hyperbolic-parabolic approximations are not in conservative form.

For numerical discretization, we choose the spatial grid points $x_{i+1/2}$ with a uniform mesh width $\Delta x = x_{i+1/2} - x_{i-1/2}$. The discrete time level t^n are also spaced uniformly with time step $\Delta t = t^{n+1} - t^n$ for $n = 0, 1, 2, \ldots$. Here the assumption of a uniform grid is only for simplicity. As usual we denote by U_i^n the cell averages of U in the cell $[x_{i-1/2}, x_{i+1/2}]$ at time t^n

$$U_j^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x, t^n)\,dx,$$

and by $U_{i+1/2}^n = U(x_{i+1/2}, t^n)$ its nodal values.

The rest of the paper is organized as follows. In the next section we introduce the kinetic models under consideration and their diffusion approximations. In section 3 we formulate the numerical scheme for the linear Goldstein-Taylor model, while in section 4 this idea is extended to the nonlinear Carleman model which gives the porous medium equation in the diffusion limit. Some numerical experiments are presented in section 5. We make some concluding remarks in section 6.

2. Kinetic Models and Their Diffusion Approximations. The simplest one-dimensional discrete velocity models of the Boltzmann equation are those with two velocities. These models are supposed to describe the evolution of the velocity distribution of a fictitious gas composed of two kinds of particles that move parallel to the x-axis with constant and equal speeds, either in the positive x-direction with a density u, or in the negative x-direction with a density v. The corresponding kinetic equations are given by

$$\begin{aligned} \epsilon\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= \frac{1}{\epsilon}J_\epsilon(u,v)\,, \\ \epsilon\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} &= -\frac{1}{\epsilon}J_\epsilon(u,v)\,, \end{aligned} \tag{2.1}$$

where c is the velocity of the molecules, and the *collision operator* J_ϵ has the form

$$J_\epsilon(u,v) = k(u,v)\left\{-\alpha_\epsilon u + \beta_\epsilon v + \gamma_\epsilon uv\right\}\,, \tag{2.2}$$

where $k(u,v), \alpha_\epsilon, \beta_\epsilon$ and γ_ϵ are nonnegative real constants. In (2.2), three collision rules are allowed, which can be described by $+- \to ++$ with a rate $k\gamma_\epsilon$, $+ \to -$ with a rate $k\alpha_\epsilon$, and $- \to +$ with a rate $k\beta_\epsilon$. When $\gamma_\epsilon > 0$, the presence of the process $+- \to ++$, together with the absence of $++ \to +-$, means the violation of the detailed balance, and time reversal invariance. This phenomenon is obviously reflected in the limit equation.

The macroscopic variables of model (2.1) are the local mass density ρ and the flux j defined by

$$\rho = u + v\,, \quad j = c(u-v)/\epsilon\,. \tag{2.3}$$

Adding and subtracting the two equations in (2.1) one obtains the following equivalent system for ρ and j

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \epsilon^2\frac{\partial j}{\partial t} + c^2\frac{\partial \rho}{\partial x} &= -2\widehat{J}_\epsilon(\rho,j)\,, \end{aligned} \tag{2.4}$$

where

$$\widehat{J}_\epsilon(\rho,j) = -\frac{c}{\epsilon}J_\epsilon(u,v). \tag{2.5}$$

As $\epsilon \to 0$, (2.4b) is approximated by

$$c^2\frac{\partial \rho}{\partial x} = -2\widehat{J}_0(\rho,j)\,, \tag{2.6}$$

If (2.6) can be solved for j in terms of ρ to give

$$j = F\left(\rho, \frac{\partial \rho}{\partial x}\right)\,, \tag{2.7}$$

then the equation (2.4a) will relax towards the second-order partial differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}F\left(\rho, \frac{\partial \rho}{\partial x}\right) = 0\,. \tag{2.8}$$

There are numerous kinetic models that have such a property (see [11] for a review of many different models). In this paper we will concentrate on two models, one linear and one nonlinear. From now on, let us set $c = 1$. We obtain the first model by taking

$$k(u,v) = k_0\,; \quad \alpha_\epsilon = \beta_\epsilon = 1\,; \quad \gamma_\epsilon = 0 \tag{2.9}$$

in the collision operator (2.2). This kinetic model is the Goldstein-Taylor model [5, 23] given by (1.1) with the right hand side multiplied by k_0, which represents the forward equation for the density of a molecule moving with constant speed 1 along the x-axis, subject to spontaneous reversals of direction, at the jump times of a standard Poisson process of constant rate k_0. The corresponding equations for ρ and j are

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \epsilon^2 \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} &= -2k_0 j\,. \end{aligned} \tag{2.10}$$

It has the free energy $\rho^2 + (\epsilon j)^2$ that decays in time:

$$\partial_t \int \rho^2 + (\epsilon j)^2 \, dx \leq 0\,. \tag{2.11}$$

The local equilibrium is given by

$$j = -\frac{1}{2k_0} \frac{\partial \rho}{\partial x}\,, \tag{2.12}$$

while (2.8) becomes the classical heat equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2k_0} \cdot \frac{\partial^2 \rho}{\partial x^2}\,, \tag{2.13}$$

System (2.10), or (1.1), is important in itself. It is often called *hyperbolic heat equations*. It was Maxwell who first introduced the concept of a relaxation time, and for this reason these equations are sometimes referred to in the literature as the Maxwell-Cattaneo equations. Several authors have applied this concept to the problem of heat conduction (see [21] and its references) and there has been a resurgence of interest in these equations in the last five years (see [20, 21] and references therein). Characteristic based numerical schemes to solve (2.4) that are able to handle the small relaxation parameter ϵ were given in [20]. Recently this problem has been studied in [18] as a particular case (the linear one) of a more general situation leading to the porous medium equation. Due to the linearity, uniform bounds on the kinetic densities were obtained in $W^{m,p}$, for $m \geq 0, 1 \leq p \leq \infty$, which implies the convergence in the same spaces when the initial data satisfy a suitable decay at infinity.

Nonlinear degenerate parabolic equations can be obtained as the diffusion limit of (2.4) by choosing $k(u,v)$ varying with u and v. If one sets

$$k(u,v) = \rho\,; \quad \alpha_\epsilon = \beta_\epsilon = 1\,; \quad \gamma_\epsilon = 0\,, \tag{2.14}$$

then (2.4) becomes

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \epsilon^2 \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} &= -2\rho j\,. \end{aligned} \tag{2.15}$$

This model is known as the Carleman model [2].

When $\epsilon \to 0$, the local equilibrium is given by

$$j = -\frac{1}{2\rho} \frac{\partial \rho}{\partial x} = -\frac{1}{2} \frac{\partial \log \rho}{\partial x}\,. \tag{2.16}$$

Thus, the system relaxes to the nonlinear porous medium equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\log \rho)\,. \tag{2.17}$$

This asymptotic problem was first investigated by Kurtz [15] and McKean[19] by means of the theory of nonlinear semigroups. They proved that, for initial data $u_0(x) = v_0(x) \in L^1(R)$ the mass density $\rho_\epsilon(x,t)$ satisfying (2.15) converges in L^1_x for all $t \geq 0$ to $\rho(x,t)$ satisfying the nonlinear diffusion equation (2.17).

In the next two sections we devise a new class of AP schemes for these kinetic equations.

3. The Goldstein-Taylor Model. Let $k_0 = 1$. A natural discretization to (2.10) is an implicit (backward Euler) time discretization for the "collision" step

$$\begin{aligned} \rho^* - \rho^n &= 0\,, \\ \frac{j^* - j^n}{\Delta t} &= -\frac{2}{\epsilon^2}\left[j^* + \frac{1-\epsilon^2}{2}\rho^*_x\right]\,; \end{aligned} \tag{3.1a}$$

followed by an explicit scheme for the convection step:

$$\begin{aligned} \frac{\rho^{n+1} - \rho^n}{\Delta t} + j^*_x &= 0\,, \\ \frac{j^{n+1} - j^*}{\Delta t} + \rho^*_x &= 0\,. \end{aligned} \tag{3.1b}$$

In [11], spatial discretizations are taken independently for the above two steps. Here, we first keep the spatial derivative continuous, and combine the above two steps into:

$$\begin{aligned} \frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{1 + 2\frac{\Delta t}{\epsilon^2}}\, j^n_x &= (1-\epsilon^2)\frac{\frac{\Delta t}{\epsilon^2}}{1 + 2\frac{\Delta t}{\epsilon^2}}\,\rho^n_{xx}\,, \\ \frac{j^{n+1} - j^n}{\Delta t} + \left[1 + \frac{(1-\epsilon^2)\frac{1}{\epsilon^2}}{1 + 2\frac{\Delta t}{\epsilon^2}}\right]\rho^n_x &= -\frac{\frac{2}{\epsilon^2}}{1 + 2\frac{\Delta t}{\epsilon^2}} j^n\,. \end{aligned} \tag{3.2}$$

This is done by first solving ρ^*, j^* in terms of ρ^n, j^n in (3.1a), and then substitute these values in (3.1b). We introduce the following notations (all positive):

$$\begin{aligned} \alpha &= \frac{1}{1 + 2\frac{\Delta t}{\epsilon^2}}\,, \quad \beta = 1 + \frac{(1-\epsilon^2)\frac{1}{\epsilon^2}}{1 + 2\frac{\Delta t}{\epsilon^2}}\,, \\ \gamma &= (1-\epsilon^2)\frac{\frac{\Delta t}{\epsilon^2}}{1 + 2\frac{\Delta t}{\epsilon^2}}\,, \quad \delta = \frac{\frac{2}{\epsilon^2}}{1 + 2\frac{\Delta t}{\epsilon^2}}\,. \end{aligned} \tag{3.3}$$

One can view (3.2) as the forward Euler discretization of the following hyperbolic-parabolic system

$$\begin{aligned} \rho_t + \alpha j_x &= \gamma \rho_{xx}\,, \\ j_t + \beta \rho_x &= -\delta j\,. \end{aligned} \tag{3.4}$$

This is clearly a hyperbolic system with two distinct characteristic speeds $\pm\sqrt{\alpha/\beta}$.

From the PDE point of view, the initial value problem of (3.4) is well-posed, and is endowed with a free energy (entropy) $\rho^2 + \frac{\alpha}{\beta} j^2$ that decays in time:

$$\partial_t \int \rho^2 + \frac{\alpha}{\beta} j^2 \leq 0\,. \tag{3.5}$$

This energy estimate is consistent (as $\Delta t \to 0$) with the energy for its continuous counterpart (2.11). It is based on (3.4) that we will construct the numerical scheme.

Remark: If one first discretizes spatially the two steps in (3.1), and then combines the fully discrete steps, as was done in [11], one will obtain a system that involves both second and third order derivatives, whose stability is a much more subtle issue than the hyperbolic-parabolic system (3.4), or (3.2). This is one of the advantages of the new approach. In addition, in [11], two independent spatial discretizations need to be taken in (3.1a) and (3.1b) respectively, while now we just need one spatial discretization for (3.2).

We now discuss the spatial discretization of (3.2). We use the second order upwind scheme of van Leer [24] that utilizes a slope limiter for the convection terms, and the center difference on the dissipation term:

$$\begin{aligned}\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} &= -\frac{\alpha}{2\Delta x}(j_{i+1}^n - j_{i-1}^n) + \frac{\sqrt{\alpha\beta}}{2\Delta x}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n) \\ &\quad - \frac{\sqrt{\alpha\beta}}{4}(\sigma_{i+1}^- + \sigma_i^+ - \sigma_i^- - \sigma_{i-1}^+) + \frac{\gamma}{\Delta x^2}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n)\,, \\ \frac{j_i^{n+1}-j^n}{\Delta t} &= -\frac{\beta}{2\Delta x}(\rho_{i+1}^n - \rho_{i-1}^n) + \frac{\sqrt{\alpha\beta}}{2\Delta x}(j_{i+1}^n - 2j^n + j_{i-1}^n) \\ &\quad + \frac{\beta}{4}(\sigma_{i+1}^- - \sigma_i^+ - \sigma_i^- + \sigma_{i-1}^+) - \delta j_i^n\,,\end{aligned} \tag{3.15}$$

where $\sigma_i^\pm$, the slope of the Riemann invariants $\rho \pm \sqrt{\frac{\alpha}{\beta}}j$ on the i-th cell, is given by

$$\begin{aligned}\sigma_i^\pm &= \frac{1}{\Delta x}\left(\rho_{i+1} \pm \sqrt{\frac{\alpha}{\beta}}j_{i+1} - \rho_i \mp \sqrt{\frac{\alpha}{\beta}}j_i\right)\psi(\theta_i^\pm), \\ \theta_i^\pm &= \left[\frac{\rho_i^n \pm \sqrt{\frac{\alpha}{\beta}}j_i - \rho_{i-1}^n \mp \sqrt{\frac{\alpha}{\beta}}j_{i-1}}{\rho_{i+1}^n \pm \sqrt{\frac{\alpha}{\beta}}j_{i+1} - \rho_i^n \mp \sqrt{\frac{\alpha}{\beta}}j_i}\right],\end{aligned} \tag{3.16}$$

and ψ is the van Leer slope limiter function:

$$\psi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}\,. \tag{3.17}$$

To achieve a higher order accuracy in time, one can use the classical Runge-Kutta method to replace the forward Euler method in (3.15).

With the above scheme (3.15), the numerical stability condition is

$$\sqrt{\frac{\beta}{\alpha}} \le 1\,, \quad \text{and} \quad \gamma\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}\,, \quad \text{and} \quad \Delta t \le \delta\,, \tag{3.18}$$

where $\alpha, \beta, \gamma, \delta$ all depend on ϵ. One can easily show that the following stability condition

$$\frac{\Delta t}{\Delta x^2} \le \frac{1}{4} \tag{3.19}$$

implies (3.18) for all $0 < \epsilon \le 1$. (3.19) is the stability condition for the explicit discretization of the limit heat equation (2.13). Clearly (3.19) allows the choice of Δt independent of ϵ, which is our desired stability condition.

Another important properties of (3.15) is that it preserves the correct diffusion limit. By passing $\epsilon \to 0$, (3.15a) becomes

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} = \frac{1}{2\Delta x^2}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n)\,, \tag{3.20}$$

the centered difference discretization to the limiting heat equation (2.13). Thus this scheme is AP.

Note that (3.15) is an explicit scheme. If one discretizes the original Goldstein-Taylor equations (2.10), one needs a CFL condition like $\Delta t \sim \epsilon^2$, which is impractically expensive (especially in multi-D problems) when one needs to take $\Delta x >> \epsilon$.

4. The Carleman Model. We now apply the same approach to the nonlinear Carleman model

$$\begin{aligned} \rho_t + j_x =& 0\,, \\ j_t + \frac{1}{\epsilon^2}\rho_x =& -\frac{2}{\epsilon^2}\rho j\,, \end{aligned} \tag{4.1}$$

to device a robust AP scheme with similar numerical advantages. As was done in [11], we rewrite (4.1) as

$$\begin{aligned} \rho_t + j_x =& 0\,, \\ j_t + \rho_x =& -\frac{1}{\epsilon^2}[2\rho j + (1-\epsilon^2)\rho_x]\,, \end{aligned} \tag{4.2}$$

and then split (4.2) into a "collision" step

$$\begin{aligned} \rho_t =& 0\,, \\ j_t =& -\frac{1}{\epsilon^2}[2\rho j + (1-\epsilon^2)\rho_x]\,; \end{aligned} \tag{4.3a}$$

and a convection step

$$\begin{aligned} \rho_t + j_x =& 0\,, \\ j_t + \rho_x =& 0\,. \end{aligned} \tag{4.3b}$$

The "collision" step will be treated implicitly:

$$\begin{aligned} \rho^* - \rho^n =& 0\,, \\ \frac{j^* - j^n}{\Delta t} =& -\frac{1}{\epsilon^2}\left[2\rho^* j^* + (1-\epsilon^2)\rho^*_x\right]\,; \end{aligned} \tag{4.4a}$$

while the convection step is discretized explicitly:

$$\begin{aligned} \frac{\rho^{n+1} - \rho^*}{\Delta t} + j^*_x =& 0\,, \\ \frac{j^{n+1} - j^*}{\Delta t} + \rho^*_x =& 0\,. \end{aligned} \tag{4.4b}$$

As long as one keeps the spatial derivative continuous, one can explicitly solve for ρ^* and j^* in terms of ρ^n, j^n from (4.4a) to get

$$\begin{aligned} \rho^* &= \rho^n\,, \\ j^* &= \frac{1}{1+2\frac{\Delta t}{\epsilon^2}\rho^n} j^n - \frac{(1-\epsilon^2)\frac{\Delta t}{\epsilon^2}}{1+2\frac{\Delta t}{\epsilon^2}\rho^n}\,\rho^n_x\,. \end{aligned} \tag{4.5}$$

Applying (4.5) in (4.4b), we get

$$\begin{aligned} \frac{\rho^{n+1}-\rho^n}{\Delta t} + \left(\frac{1}{1+2\frac{\Delta t}{\epsilon^2}\rho^n} j^n\right)_x &= (1-\epsilon^2)\frac{\Delta t}{\epsilon^2}\left(\frac{\rho^n_x}{1+2\frac{\Delta t}{\epsilon^2}\rho^n}\right)_x\,, \\ \frac{j^{n+1}-j^n}{\Delta t} + \left(1 + \frac{\frac{1-\epsilon^2}{\epsilon^2}}{1+2\frac{\Delta t}{\epsilon^2}\rho^n}\right)\rho^n_x &= -\frac{\frac{2}{\epsilon^2}\rho^n}{1+2\frac{\Delta t}{\epsilon^2}}\,j^n\,. \end{aligned} \tag{4.6}$$

(4.6) can be written in conservative form as

$$\begin{aligned}
&\frac{\rho^{n+1}-\rho^n}{\Delta t}+\left(\frac{1}{1+2\frac{\Delta t}{\epsilon^2}\rho^n}j^n\right)_x=(1-\epsilon^2)\frac{\Delta t}{\epsilon^2}\left(\frac{\rho^n_x}{1+2\frac{\Delta t}{\epsilon^2}\rho^n}\right)_x\,,\\
&\frac{j^{n+1}-j^n}{\Delta t}+\left[\rho^n+\frac{1-\epsilon^2}{2\Delta t}\ln\left(1+2\frac{\Delta t}{\epsilon^2}\rho^n\right)\right]_x=-\frac{\frac{2}{\epsilon^2}\rho^n}{1+2\frac{\Delta t}{\epsilon^2}\rho^n}j^n\,.
\end{aligned}\tag{4.7}$$

It is a straightforward calculation to show that the convective part of (4.7) is of hyperbolic type, with two distinct characteristic speeds

$$\lambda_\pm=-\frac{\frac{\Delta t}{\epsilon^2}j}{2(1+\frac{\Delta t}{\epsilon^2}\rho)^2}\pm\frac{1}{2(1+\frac{\Delta t}{\epsilon^2}\rho)}\sqrt{\left(\frac{\frac{\Delta t}{\epsilon^2}j}{2(1+\frac{\Delta t}{\epsilon^2}\rho)}\right)^2+4\left(1+\frac{\Delta t}{\epsilon^2}\rho+\frac{1-\epsilon^2}{\epsilon^2}\right)}\,.\tag{4.8}$$

Since the convection in (4.7) is *nonlinear*, we use the second order relaxed scheme derived in [12] which is free of Riemann solver and local characteristic decomposition. Denote the first order part of (4.7) by (with continuous time variable)

$$U_t+F(U)_x=0\,,\tag{4.9}$$

where $U=(u^{(1)},u^{(2)})=(\rho,j)^T$ is the vector of the conserved quantity, and $F(U)=(F^{(1)}(U),F^{(2)}(U))$ is the flux vector of (4.7). The relaxed scheme takes the form

$$\begin{aligned}
\frac{u_i^{(p),n+1}-u_i^{(p),n}}{\Delta t}&=-\frac{(F^{(p)}(U_{i+1})-F^{(p)}(U_{i-1}))}{2\Delta x}+a\frac{u_{i+1}^{(p),n}-2u_i^{(p),n}+u_{i-1}^{(p),n}}{2\Delta x}\\
&\quad-\frac{1}{4}(\sigma_i^+-\sigma_{i+1}^--\sigma_{i-1}^++\sigma_i^-)\,,\qquad p=1,2\,,
\end{aligned}\tag{4.10}$$

where the limited slope is defined by

$$\begin{aligned}
\sigma_j^\pm\big|_{v=F^{(p)}(U)}&=\frac{1}{\Delta x}(F^{(p)}(U_{i+1})\pm a\,u_{i+1}-F^{(p)}(U_i)\mp a\,u_i)\,\phi(\theta_i^\pm)\,,\\
\theta_i^\pm&=\frac{F^{(p)}(U_i)\pm a\,u_i-F^{(p)}(U_{i-1})\mp a\,u_{i-1}}{F^{(p)}(U_{i+1})\pm a\,u_{i+1}-F^{(p)}(U_i)\mp a\,u_i}\,,\qquad q=1,2\,,
\end{aligned}\tag{4.11}$$

with ϕ given in (3.17). The simplest choice of a is $a=b(\Delta x/\Delta t)^2$ with $0<b<1$. The dissipation term in (4.7a) is treated by the explicit second order conservative center difference

$$(1-\epsilon^2)\frac{\Delta t}{\epsilon^2\Delta x^2}\left(\frac{\rho^n_{i+1}-\rho^n_i}{1+2\frac{\Delta t}{\epsilon^2}\frac{\rho^n_{i+1}+\rho^n_i}{2}}-\frac{\rho^n_i-\rho^n_{i-1}}{1+2\frac{\Delta t}{\epsilon^2}\frac{\rho^n_i+\rho^n_{i-1}}{2}}\right)\,,\tag{4.12}$$

and the damping term in (4.7b) is treated explicitly.

The stability condition for this scheme is

$$\sup_U|\lambda_\pm|\frac{\Delta t}{\Delta x}\le b\,,\qquad\text{and}\qquad(1-\epsilon^2)\frac{\Delta t}{\epsilon^2}\frac{1}{1+2\frac{\Delta t}{\epsilon^2}\rho}\sim O(1)\,.\tag{4.13}$$

It is easy to show that an CFL condition of the type

$$\frac{1}{\rho}\frac{\Delta t}{(\Delta x)^2}\sim O(1)\tag{4.14}$$

implies (4.13) for all $0<\epsilon\le 1$. (4.14) is the stability condition for the explicit center difference of the porous medium equation (4.16). Moreover, as $\epsilon\to 0$, (4.7a), with

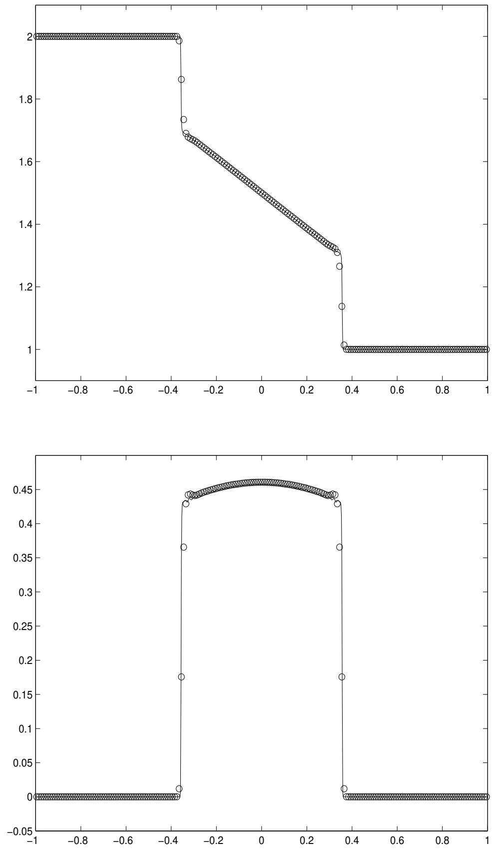

FIG.5.1A,B. *Numerical solution (circles) of the Goldstein-Taylor model with initial data (5.1).* $\epsilon = 0.7, \Delta t = 0.0025, \Delta x = 0.01, t = 0.25$. *The mass density ρ (top) and the flow j (bottom) are compared against the "exact" solution represented by the solid lines.*

(4.12), becomes

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = \frac{1}{2(\Delta x)^2}\left(\frac{\rho_{i+1}^n - \rho_i^n}{\frac{\rho_{i+1}^n + \rho_i^n}{2}} - \frac{\rho_i^n - \rho_{i-1}^n}{\frac{\rho_i^n + \rho_{i-1}^n}{2}}\right), \tag{4.15}$$

which is clearly the second order discretization of the porous medium equation

$$\rho_t = \frac{1}{2}\partial_x\left(\frac{\rho_x}{\rho}\right) = \frac{1}{2}\partial_{xx}\log\rho\,, \tag{4.16}$$

the correct porous medium equation (2.17). Thus our scheme is AP.

In summary, the scheme derived this section has a stability condition independent of ϵ (as indicated in (4.14)), can be solved explicitly, and is asymptotic-preserving. It has all the desired properties of a robust AP scheme.

5. Numerical Results.

In this section we present several numerical results that demonstrate the performance of the schemes constructed in this paper.

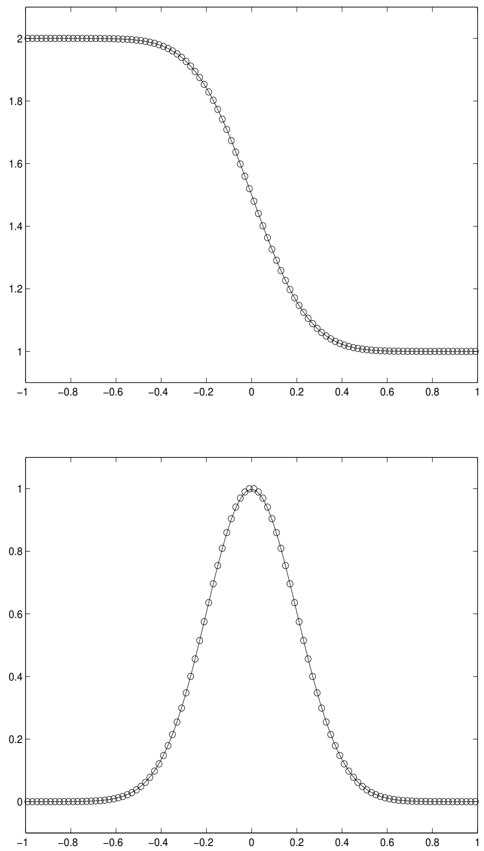

FIG.5.1C,D. *Numerical solution (circles) of the Goldstein-Taylor model with initial data (5.1).* $\epsilon = 10^{-6}, \Delta t = 0.0002, \Delta x = 0.02, t = 0.04$. *The mass density* ρ *(top) and the flow* j *(bottom) are compared against the "exact" solution represented by the solid lines.*

We first solve the Goldstein-Taylor model (2.15) with the following initial data

$$\begin{aligned} \rho &= 2.0, \quad j = 0, \qquad -1 < x < 0, \\ \rho &= 1.0, \quad j = 0, \qquad 0 < x < 1. \end{aligned} \tag{5.1}$$

As ϵ goes to zero this problem becomes a classical Riemann problem for the heat equation. That is the problem of two semi-infinite rods having temperature ρ_L and ρ_R brought into contact at $t = 0$. Numerical results for this problem has been recently obtained in [20].

We carry out the numerical experiments for different range of ϵ, and present the numerical results against the "exact" solution (obtained using $\Delta x = 0.001$) in Figure 5.1. We first use $\epsilon = 0.7, \Delta x = 0.01, \Delta t = 0.0025$, and the results are shown in Figure 5.1a,b. In Figure 5.1c,d, we use $\epsilon = 10^{-6}, \Delta x = 0.02, \Delta t = 0.0002$ ($\sim (\Delta x)^2 >> \epsilon^2$). In Figure 5.1e,f, we take $\epsilon = 0.1, \Delta x = 0.02, \Delta t = 0.001$. In all examples, the numerical solutions have excellent agreements with the "exact" solution in smooth regions. When there are discontinuities, the numerical solutions enjoy a high resolution. These tests indicate that the numerical scheme is able to handle different ranges of ϵ, from $O(1)$, the kinetic regime, to an extremely small value where the diffusive behavior is captured

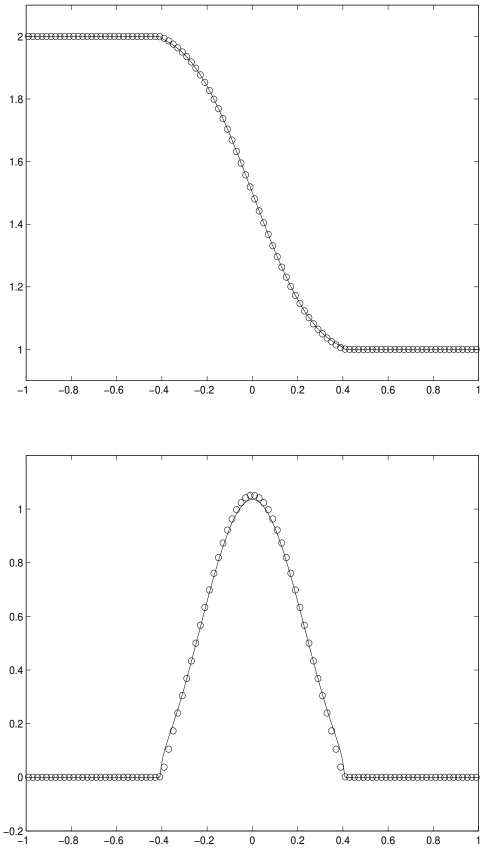

FIGURE 5.1E,F. *Numerical solution (circles) of the Goldstein-Taylor model with initial data (5.1). $\epsilon = 0.1, \Delta t = 0.001, \Delta x = 0.02, t = 0.04$. The mass density ρ (top) and the flow j (bottom) are compared against the "exact" solution represented by the solid lines.*

using very large (compared to ϵ^2) grid size and time steps.

Next we consider the nonlinear Carleman model with the same initial data as (5.1). We carry out similar numerical experiments, and the results are depicted in Figure 5.2. In Figure 5.2a,b, we use $\epsilon = 0.7, \Delta x = 0.01, \Delta t = 0.0025$. In Figure 5.2c,d, $\epsilon = 10^{-6}, \Delta x = 0.02, \Delta t = 0.0002$ ($\sim (\Delta x)^2 >> \epsilon^2$). In Figure 5.2e,f, $\epsilon = 0.1, \Delta x = 0.02, \Delta t = 0.001$. In all these examples, we also see similar performance with the first example.

6. Conclusion. In this paper we presented a class of asymptotic-preserving schemes for several kinetic models with multiple scales. This new class of schemes, compared to those derived in [11], are more robust and efficient since they are based on well-posed hyperbolic-parabolic systems. They possess numerical stability independent of the relaxation parameter, yet can be solved explicitly. Moreover, they are able to capture the correct diffusive behavior with underresolved numerical discretization. Numerical experiments show that these schemes perform well in all ranges of the relaxation parameter.

In the near future we will extend this idea to multidimensional problems.

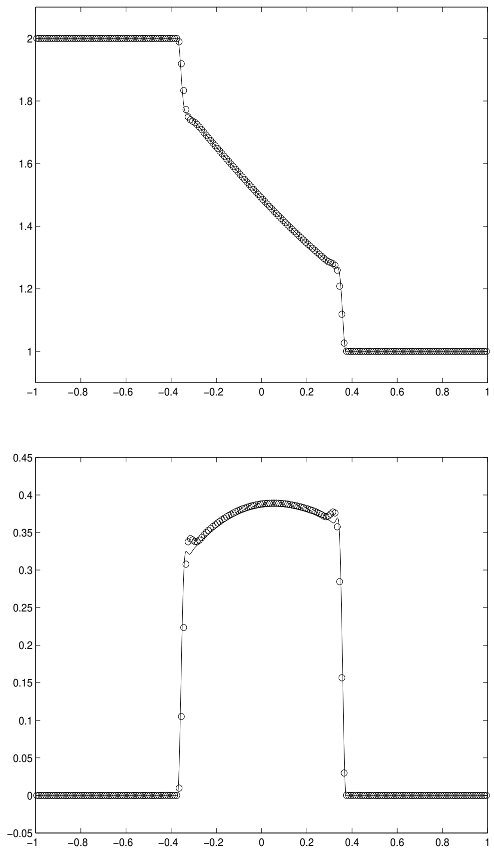

FIGURE 5.2A,B. *Numerical solution (circles) of the Carleman model with initial data (5.1).* $\epsilon = 0.7, \Delta t = 0.0025, \Delta x = 0.01, t = 0.25$. *The mass density ρ (top) and the flow j (bottom) are compared against the "exact" solution represented by the solid lines.*

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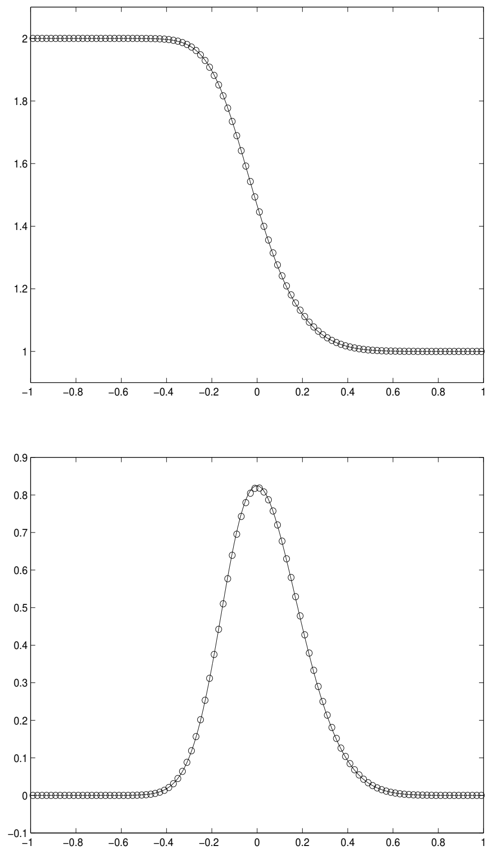

FIGURE 5.2C,D. *Numerical solution (circles) of the Carleman model with initial data (5.1).* $\epsilon = 10^{-6}, \Delta t = 0.0002, \Delta x = 0.02, t = 0.04$. *The mass density ρ (top) and the flow j (bottom) are compared against the "exact" solution represented by the solid lines.*

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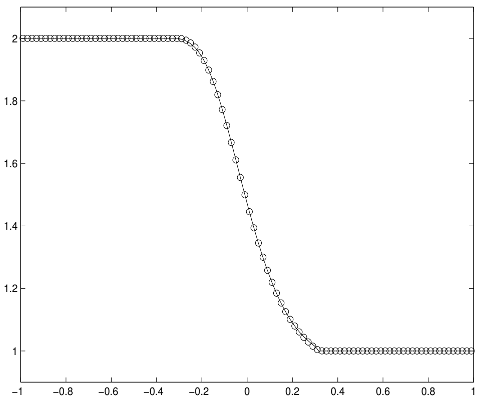

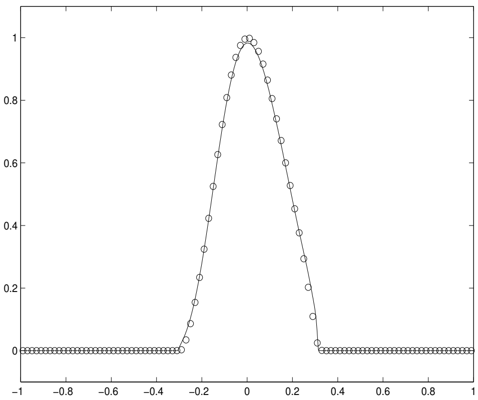

FIGURE 5.2E,F. *Numerical solution (circles) of the Carleman model with initial data (5.1).* $\epsilon = 0.1, \Delta t = 0.001, \Delta x = 0.02, t = 0.03$. *The mass density* ρ *(top) and the flow* j *(bottom) are compared against the "exact" solution represented by the solid lines.*

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