Zero Reaction Limit for Hyperbolic Conservation Laws with Source Terms

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Dedicated to Jack Hale on the occasion of his 70th birthday

Abstract

In this paper we study the zero reaction limit of the hyperbolic conservation law with stiff source term

$$\partial_t u + \partial_x f(u) = \tfrac{1}{\epsilon} u(1-u^2)\,.$$

For the Cauchy problem to the above equation, we prove that as $\epsilon \to 0$, its solution converges to piecewise constant (± 1) solution, where the two constants are the two stable local equilibrium. The constants are separated by either shocks that travel with speed $\frac{1}{2}(f(1)-f(-1))$, as determined by the Rankine-Hugoniot jump condition, or a non-shock discontinuity that moves with speed $f'(0)$, where 0 being the unstable equilibrium. Our analytic tool is the method of generalized characteristics. Similar results for more general source term $\frac{1}{\epsilon}g(u)$, having finitely many simple zeros and satisfying $ug(u) < 0$ for large $|u|$, are also given.

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1. Introduction

In this paper we study the hyperbolic equations with stiff source term

$$\begin{aligned} &\partial_t u + \partial_x f(u) = -\frac{1}{\epsilon} W'(u)\,, \\ &u(x,0) = u_0(x)\,, \end{aligned} \tag{1.1}$$

where $W(u)$ is the double well potential, and $\epsilon > 0$ is the reaction time. This is the simplest model for reacting flows, where the source term, being the derivative of the typical double well potential, accounts for chemical reaction. A typical form of W is $W(u) = \frac{1}{4}(u^2-1)^2$, and in this case (1.1) becomes

$$\begin{aligned} &\partial_t u + \partial_x f(u) = \frac{1}{\epsilon} u(1-u^2), \\ &u(x,0) = u_0(x). \end{aligned} \tag{1.2}$$

Since most equations governing reacting flows or dynamics of phase transitions are combinations of inhomogeneous fluid dynamics equations and reaction-diffusion equations [AK, VK], equation (1.1) can serve as a prototype model to study issues involved in reacting flows. In a reacting flow, the typical scale of the reacting time ϵ is much smaller than the characteristic time scale of the fluid, which makes the source term in (1.2) stiff. The goal of the paper is to understand the limiting behavior of the solution to (1.2), as $\epsilon \to 0$.

First notice that the source term in (1.2) admits three local equilibria, namely, 0 and ± 1, with 0 being linearly unstable, while ± 1 linearly stable. Heuristically, as $\epsilon \to 0$, the solution of (1.2) should tend to the two stable local equilibria ± 1, thus the limiting solution becomes piecewise constant. In this paper, we will rigorously justify this heristics, and furthermore, investigate how the discontinuities that connect these constants propagate. Our analytic tool is the method of generalized characteristics [Daf].

Before stating our main results, we assume that the flux $f(u)$ is a convex function of u, i.e., $f''(u) > 0$, and the initial datum $u_0(x)$ satisfies

$$u_0(x) \in C^1(\mathbb{R};\mathbb{R}) \quad \text{has finitely many zeros } a_j,\ j = 1, 2, ..., n, \text{ with } u_0'(a_j) \neq 0\,. \tag{1.3}$$

From the classical theory of conservation laws [Kru], we know that there is a unique admissible solution of (1.2) in BV space for each fixed ϵ. When $f'' > 0$, a solution $u(x,t)$ of (1.2) in BV is called admissible if $u(x-,t) \geq u(x+,t)$ holds for all (x,t) in the domain of definition of u. The requirements $u \in C^1$ and $u_0' \neq 0$ is for the simplicity of presentation. It is nonessential. With these assumptions, we establish the following results:

Theorem 1.1. *Let the initial value* $u_0(x)$ *satisfy (1.3), and* u^ϵ *be the admissible solution of (1.2). Then the limit*

$$u(x,t) := \lim_{\epsilon \to 0} u^\epsilon(x,t)$$

exists for almost all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. *The function* $u(x,t)$ *is piecewise constant with the constants being* ± 1. *Constant pieces of* $u(x,t)$ *are separated by Lipschitz continuous curves* $x = z_j(t)$ *defined on* $[0, T_j)$, $j = 1, 2, ...n$. *Moreover, the following hold for these curves* $x = z_j(t)$, $j = 1, 2, ..., n$*:*

(i) $z_j(0) = a_j$.

(ii) If $\lim_{x \to a_j -} sign(u_0(x)) = 1$, *then* $z_j(t) = a_j + \frac{f(1) - f(-1)}{2} t$.

(iii) If $\lim_{x \to a_j -} sign(u_0(x)) = -1$, *then* $z_j(t) = a_j + f'(0)t$.

(iv) Curves $x = z_j(t)$ *do not intersect each other except at* $t = T_j$, *the end points of their domain of definition.*

(v) At $t = T_j < \infty$, *the curve* $x = z_j(t)$ *must intersect with another curve* $x = z_k(t)$.

Theorem 1.1 reveals that, as $\epsilon \to 0$, there are two types of discontinuities that will connect the constant equilibrium states ± 1. The first is the classical shock waves, as described in (ii), that propagate with the speed determined by the Rankine-Hugoniot jump condition and satisfy the entropy condition for the homogeneous equation $\partial_t u + \partial_x f(u) = 0$. Of particular interest is the new type discontinuity, called non-shock discontinuity throughout this paper, as described by (iii) in Theorem 1.1. This discontinuity violates the Rankine-Hugoniot jump condition and the entropy condition for the homogeneous equation. Despite this, our numerical experiment in section 4 still shows that this non-shock discontinuity is admissible by the viscosity regularization of (1.2):

$$u_t + f(u)_x = \frac{1}{\epsilon} u(1 - u^2) + \mu u_{xx}. \tag{1.4}$$

Our methods and results can be easily extended to problems with a more general source term

$$\partial_t u + \partial_x f(u) = \tfrac{1}{\epsilon} g(u)$$

where the source term $g(u)$ has finitely many simple equilibria.

The long-time behavior and attractors of hyperbolic conservation laws with source term that admits multiple equilibria similar to (1.2), but in the non-stiff regime $\epsilon = O(1)$, has been studied by several authors [FH1, FH2, Har, Lyb, Mac, MS, Sin1, Sin2, Sin3]. In particular, these earlier literatures focused on studying the long-time behavior of periodic traveling wave solutions that are admissible by the entropy condition. These solutions exhibit different behavior than those being studied here. In particular, the non-shock discontinuity, even though may be obtained as a limit of the wave connecting -1 to 1 as studied in these earlier works, is a new type of discontunity.

Since (1.2) is stiff, a practical numerical method for such problem would require an underresolved temporal discretization (time step Δt much bigger than the reacting time ϵ). Failing to do so in shock capturing methods induces to incorrect shock speed [BKT, CMR, LY]. In this paper we also report incorrect propagation speed for the non-shock discontinuity, if the numerical time step does not resolve ϵ.

This paper is organized as follows: In Section 2, we examine the basic asymptotic behavior of (1.2) as $\epsilon \to 0$. In Section 3, we shall prove Theorem 1.1 using the method of generalized characteristics. We also extend this theorem for the case of more general source term that exhibits multiple equilibria. In Section 4 we study numerically the viscosity regularization (1.4), and investigate the incorrect propagation speed generated by underresolved shock capturing methods.

Finally, we point out that the incorrect shock speed problem observed in the underresolved shock capturing method has been solved recently by Bao and Jin using the random projection method [BJ].

2. Asymptotic Behavior of the Solution for $\epsilon \to 0$

In this section we study the formal asymptotic behavior of (1.2) with $\epsilon \to 0$. A linear stability analysis indicates that 0 is an unstable equilibrium while ± 1 are the stable ones. An initial layer analysis shows that, for any initial data, the initial layer projects the positive part of the solution to 1 and the negative part of the solution to -1. We will then analyze the dynamics of the discontinuities that connect the two constant states ± 1. They are either a shock propagating with the speed determined by the Rankine-Hugoniot condition on the homogeneous equation

$$\partial_t u + \partial_x f(u) = 0 \tag{2.1}$$

or a non-shock discontinuity that propagates with the speed of rarefaction wave at the unstable local equilibrium 0, namely, $f'(0)$.

2.1. The Initial Layer Analysis

In order to study the behavior of the initial layer we introduce the stretching variable $\tau = t/\epsilon$. Let

$$u_I(\tau, x) = u(t/\epsilon, x)\,. \tag{2.2}$$

Under this new variable (1.2) becomes

$$\partial_\tau u_I + \epsilon \partial_x f(u_I) = u_I(1 - u_I^2)\,. \tag{2.3}$$

By omitting the $O(\epsilon)$ term one arrives at the simple ordinary differential equation (still use u_I)

$$\begin{cases} \partial_\tau u_I = u_I(1 - u_I^2)\,, \\ u_I(0, x) = u_0(x)\,. \end{cases} \tag{2.4}$$

The initial value problem to this ODE has a unique solution,

$$u_I(x, \tau) = \text{sgn}(u_0(x)) \frac{Ce^\tau}{\sqrt{1 + C^2 e^{2\tau}}}\,, \tag{2.5}$$

where C is the integration constant determined from the initial data, i.e., $C = |u_0(x)|/\sqrt{|1 - u_0(x)^2|}$. Since $\tau = t/\epsilon$, as $\epsilon \to 0$, the initial data will be driven to the two linearly stable local equilibria ± 1 exponentially fast, with the positive part of the initial data goes to 1 and the negative part to -1.

2.2. Propagating Speed of the Discontinuities

As shown in the preceding section, beyond the initial layers, solution will become piecewise constant ± 1. In this section we will explore how the discontinuities that connect different constant states propagate. To serve this purpose it suffices to analyze the Riemann problem of (1.2) with the initial data

$$u(x, 0) = u_L \quad \text{if} \quad x < 0\,; \qquad u(x, 0) = u_R \quad \text{if} \quad x > 0\,; \tag{2.6}$$

where $|u_L| = |u_R| = 1, u_L + u_R = 0$.

First assume this initial condition gives an admissible shock [Lax] for the homogeneous equation (2.1), i.e., $u_L = 1, u_R = -1$. With these initial data, the source term in (1.2) becomes identically zero, thus the solution is exactly the same as the homogeneous equation (2.1), i.e., a shock that connects 1 with -1 and moves with the speed determined by the Rankine-Hugoniot jump condition

$$s = \frac{f(u_R) - f(u_L)}{u_R - u_L} = \tfrac{1}{2}(f(1) - f(-1))\,. \tag{2.7}$$

With this solution, the source term will remain vanished for all later time, allowing this shock to persist for all later time.

Now assume that the initial data (2.6) gives a rarefaction wave solution to (2.1), i.e., $u_L = -1, u_R = 1$. The fan-like solution in the rarefaction wave will generate nonequilibrium state between -1 and 1, which will trigger the reaction term. With the reaction on, the nonequilibrium state will again be projected into the equilibria ± 1, while the location of $u = 0$ remain unchanged during the reaction. Such a combination of the convection and the reaction term then yields a discontinuity that moves with the speed $f'(0)$. More specifically, for this initial condition, the entropy solution $u^\epsilon(x,t)$ to (1.2) can be easily expressed as

$$u^\epsilon(x,t) = \begin{cases} u_L, & x < f'(u_L)t \\ U(t/\epsilon; \Theta(t/\epsilon; x/\epsilon)), & f'(u_L)t < x < f'(u_R)t \\ u_R, & f'(u_R)t < x, \end{cases} \tag{2.8}$$

where $U(\tau;\theta)$ satisfies

$$\begin{cases} \dfrac{d}{d\tau}U = U(1-U^2), \\ U|_{\tau=0} = ((1+\theta)u_R + (1-\theta)u_L)/2, \quad \theta \in [-1,1] \end{cases} \tag{2.9}$$

and $\Theta(\tau;\xi)$ is the inverse of

$$\xi = X(\tau;\theta) := \int_0^\tau f'(U(\tau;\theta))\, d\tau, \quad f'' > 0 \tag{2.10}$$

with respect to θ for $-1 \le \theta \le 1$.

Solving (2.9) gives

$$U(\tau;\theta) = \frac{\theta}{\sqrt{\theta^2(1-e^{-2\tau}) + e^{-2\tau}}}. \tag{2.11}$$

From (2.11) we know that the solution $U(\tau;\theta)$ satisfies

$$U(\tau;\theta) = \begin{cases} -1 + o(1), & -1 \le \theta < 0 \\ 0, & \theta = 0 \\ 1 + o(1), & 0 < \theta \le 1, \end{cases} \tag{2.12}$$

and from (2.10) and (2.11) we have

$$X(\tau;\theta) = \begin{cases} f'(-1)\tau + o(1), & -1 \le \theta < 0 \\ f'(0)\tau, & \theta = 0 \\ f'(1)\tau + o(1), & 0 < \theta \le 1, \end{cases} \tag{2.13}$$

where $o(1) \to 0$ exponentially as $\tau \to +\infty$. Substituting (2.12) and (2.13) into (2.8) yields

$$u^\epsilon(x,t) = \begin{cases} -1, & x < f'(-1)t \\ -1 + o(1), & f'(-1)t \le x < f'(0)t \\ 0, & x = f'(0)t \\ 1 + o(1), & f'(0)t < x \le f'(1)t \\ 1, & f'(1)t < x, \end{cases} \tag{2.14}$$

where $o(1) \to 0$ exponentially as $\epsilon \to 0+$.

Therefore, the limit of $u^\epsilon(x,t)$ as $\epsilon \to 0+$ is

$$u(x,t) = \begin{cases} -1, & x < f'(0)t \\ 0, & x = f'(0)t \\ 1, & f'(0)t < x, \end{cases} \tag{2.15}$$

This is a discontinuous solution that does not satisfy the Rankine-Hugoniot jump condition, nor the entropy condition of (2.1). Thus it is a new type of discontinuity which differs from an expansion shock. We call it a non-shock discontinuity.

In next section we will rigorously prove these asymptotic results for a rather general Cauchy problem.

3. Convergence of $u^\epsilon(x,t)$ and the Structure of the Limit

In this section, we shall prove that the solution of (1.2) converges as $\epsilon \to 0+$. We shall also reveal the structure of the limit

$$u(x,t) := \lim_{\epsilon \to 0+} u^\epsilon(x,t). \tag{3.1}$$

The major tool used in this section is the method of generalized characteristics. We shall first review some results about the generalized characteristics in the following subsection.

3.1. Generalized Characteristics of (1.2)

A Lipschitzian curve $x = \xi(t)$ defined on an interval $[a,b]$ is called a characteristic curve associated to the solution $u(x,t)$ of (1.2) if, for almost all $t \in [a,b]$,

$$\frac{d\xi}{dt} \in [f'(u(\xi(t)+,t)), f'(u(\xi(t)-,t))]. \tag{3.1.1}$$

From [Fil], for any $(\bar{x},\bar{t}) \in \mathbb{R} \times (0,\infty)$, there exists at least one backward characteristic $\xi(t;\bar{x},\bar{t})$ defined on a maximal interval $(s,\bar{t}]$, $s \geq 0$, with $\xi(\bar{t};\bar{x},\bar{t}) = \bar{x}$. The set of all backward characteristics through $(\bar{x},\bar{t})$ form a funnel confined between the minimal and the maximal backward characteristics through $(\bar{x},\bar{t})$. We denote the minimal and maximal backward characteristics by $\xi_-(t;\bar{x},\bar{t})$ and $\xi_+(t;\bar{x},\bar{t})$ respectively. The following Lemmas 3.1.1, 3.1.2 and 3.1.3 are from [Daf]:

Lemma 3.1.1. *The extremal backward characteristic* $\xi_\pm(t;\bar{x},\bar{t})$ *associated with the solution* $u(x,t)$ *of (1.2) satisfies, for* $t \in (s,\bar{t}]$,

$$\begin{aligned} \frac{d\xi}{dt} &= f'(v(t)), \\ \frac{dv}{dt} &= \frac{1}{\epsilon} v(1-v^2)\,, \end{aligned} \tag{3.1.2}$$

with initial conditions

$$(\xi_-(\bar{t};\bar{x},\bar{t}), v(\bar{t})) = (\bar{x}, u(\bar{x}-,\bar{t})) \tag{3.1.3-}$$

and

$$(\xi_+(\bar{t};\bar{x},\bar{t}), v(\bar{t})) = (\bar{x}, u(\bar{x}+,\bar{t})) \tag{3.1.3+}$$

respectively. Furthermore, for both $\xi(t) := \xi_-(t)$ *and* $\xi(t) = \xi_+(t)$, *equations*

$$v(t) = u(\xi(t)-,t) = u(\xi(t)+,t) \tag{3.1.4}$$

hold for almost all $t \in (s,\bar{t}]$.

Lemma 3.1.2. *Any two extremal backward characteristics do not intersect.*

Lemma 3.1.3. *If the solution* $|u(x,t)| \le C$ *for some constant* $C > 0$, *then backward characteristics associated with* $u(x,t)$ *are defined on* $[0,\bar{t}]$.

Lemma 3.1.4. *If* $f \in C^1(\mathbb{R};\mathbb{R})$, *then the solution of (1.2) satisfies*

$$||u(\cdot,t)||_{L^\infty(\mathbb{R})} \le M := \max(||u_0||_{L^\infty(\mathbb{R})}, 1), \quad \textit{for } t > 0. \tag{3.1.5}$$

Proof. According to [Kru], for any fixed ϵ, the solution $u(x,t)$ is the $\mu \to 0+$ limit of solutions $u(x,t;\mu)$ of

$$\begin{aligned} &\partial_t u + \partial_x f(u) = \frac{1}{\epsilon} u(1-u^2) + \mu \partial_x^2 u, \\ &u(x,0,\mu) = u_0(x). \end{aligned} \tag{3.1.6}$$

It is easy to see that the maximum and minimum principle hold in the region $u > 1$ and $u < -1$ respectively. Thus the solution $u(x,t;\mu)$ satisfies (3.1.5). Therefore the limit $u(x,t) = \lim_{\mu\to 0+} u(x,t;\mu)$ also satisfies (3.1.5). ∎

The following corollary immediately follows Lemma 3.1.3 and 3.1.4:

Corollary 3.1.5. *Backward characteristics through the point* $(\bar{x},\bar{t})$ *are defined on* $[0,\bar{t}]$.

Lemma 3.1.6. *From any point $(\bar{x}, \bar{t})$, $\bar{t} > 0$, there is a unique forward generalized characteristics $\zeta(t; \bar{x}, \bar{t})$ of (1.1) defined as*

$$\frac{d\zeta}{dt} = \begin{cases} f'(u(\zeta(t), t)), & \text{if } u(\zeta(t)-, t) = u(\zeta(t)+, t), \\ \frac{f(u(\zeta(t)-,t)) - f(u(\zeta(t)+,t))}{u(\zeta(t)-,t) - u(\zeta(t)+,t)}, & \text{if } u(\zeta(t)-, t) > u(\zeta(t)+, t), \end{cases} \tag{3.1.7}$$
$$\zeta(\bar{t}) = \bar{x}, t > \bar{t}.$$

Furthermore a minimal backward characteristics $x = \xi_-(t)$ of (1.1) cannot cross $x = \zeta(t)$ from the left as t decreases. Similarly, a maximal backward characteristics $x = \xi_+(t)$ of (1.1) cannot cross $x = \zeta(t)$ from the right as t decreases.

Proof. The existence and uniqueness of the forward generalized characteristics $\zeta(t; \bar{x}, \bar{t})$ is stated and proved in Theorem 3.4 of [Daf]. To see that $\xi_-(t)$ cannot cross $\zeta(t)$ from the left as t decreases, we note that if $\zeta(t) = \xi_-(t)$, it would hold that

$$\frac{d\zeta}{dt} = \begin{cases} f'(u(\zeta(t), t)), & \text{if } u(\zeta(t)-, t) = u(\zeta(t)+, t), \\ \frac{f(u(\zeta(t)-,t)) - f(u(\zeta(t)+,t))}{u(\zeta(t)-,t) - u(\zeta(t)+,t)}, & \text{if } u(\zeta(t)-, t) > u(\zeta(t)+, t), \end{cases}$$
$$\leq f'(u(\zeta(t)-, t)) = \frac{d\xi_-}{dt}. \tag{3.1.8}$$

The proof for the corresponding statement for $\xi_+(t)$ is similar. ∎

Lemma 3.1.7. *Let $\xi(t)$ be a genuine backward charateristics and $\zeta(t)$ be a forward characteristics of (1.1). If curves $x = \xi(t)$ and $x = \zeta(t)$ intersect at $t = T$, then the domain of definition of $\xi(t)$ is $[0, T]$.*

Proof. Let T_1 be the minimum of

$$\{t \in [0, \infty) \mid \zeta(t) = \xi(t). \tag{3.1.9}$$

It suffices to prove that T_1 is the upper end of the domain of definition of $\xi(t)$. To this end, we assume the contrary, i.e. that $\xi(t)$ is defined on $[0, T_2 > T_1]$. There are the following two cases:

Case I. $\xi(t) \neq \zeta(t)$ for $T_1 < T_2 < t < T_2 + \delta$ for some $T_2 > T_1$ and $\delta > 0$.

Let $\bar{t} \in [T_1, T_1 + \delta]$ and $\bar{x}$ be between $\xi(\bar{t})$ and $\zeta(\bar{t})$. The extremal backward characteristics $\xi_\pm(t; \bar{t}, \bar{x})$ must intersect $x = \zeta(t)$ since they cannot intersect $x = \xi(t)$ in view of Lemma 3.2.1. If $\zeta(\bar{t}) > \bar{x}$, then $x = \xi_-(t; \bar{t}, \bar{x})$ crosses $x = \zeta(t)$ from the left as t decreases. If $\zeta(\bar{t}) < \bar{x}$, then $x = \xi_+(t; \bar{t}, \bar{x})$ crosses $x = \zeta(t)$ from the right as t decreases. Either cases are prohibited by Lemma 3.1.6.

Case II. $\xi(t) = \zeta(t)$ for $T_1 < t < T_1 + \delta$ for some $\delta > 0$.

Since almost all points on the extremal backward characteristics are points of continuity of $u(x,t)$, we can select one of such points $(\bar{x},\bar{t})$ with $T_1 < \bar{t} < T_1 + \delta$. The minimal and maximal backward characteristics $\xi_-(t;\bar{x},\bar{t})$ and $\xi_+(t;\bar{x},\bar{t})$ are identical when $(\bar{x},\bar{t})$ is a point of continuity of u, see (3.1.2) and (3.1.3). Since forward characteristics passing $(\bar{x},\bar{t})$, being one of the generalized characteristics, is between $\xi_\pm(t;\bar{x},\bar{t})$ for $t < \bar{t}$, we have

$$\zeta(t) = \xi_\pm(t;\bar{x},\bar{t})$$

which violates the definition of T_1.

The contradictions in both cases complete the proof. ∎

3.2. Convergence of Solutions of (1.2) as $\epsilon \to 0+$ and the Structure of the Limit.

In this section, we shall prove that admissible solution of (1.2) converges as $\epsilon \to 0+$. By performing the transformation $x \mapsto x - f'(0)t$ in (1.2) if necessary, we can assume $f'(0) = 0$ in this section without loss of generality.

First, we investigate the structure of the solution $u^\epsilon(x,t)$ of (1.2).

Lemma 3.2.1. *Let $u(x,t)$ be the solution of (1.2) with initial value $u_0(x)$ satisfying assumption (1.3). Then at each fixed $t \geq 0$, there are points $z_1(t) < z_2(t) < \ldots\ldots < z_m(t)$, $m = m(t) \leq n$ being an integer, such that changes of sign$(u(x,t))$ occur and only occur when x crosses $x = z_j(t)$, $t < t_j$.*

Proof. This Lemma holds when $t = 0$ because the assumptions on $u_0(x)$, (1.3). Fix $\bar{t} > 0$. Consider the set

$$\mathcal{A}(\bar{t}) := \{\bar{x} \in \mathbb{R} \mid \lim_{x\to\bar{x}-} \operatorname{sign}(u(x,\bar{t})) = -\lim_{x\to\bar{x}+} \operatorname{sign}(u(x,\bar{t}))\}. \tag{3.2.1}$$

The system for the minimal backward characteristic is

$$\begin{aligned} &\frac{d\xi}{dt} = f'(v(t)), \\ &\frac{dv}{dt} = \frac{1}{\epsilon}v(1-v^2), \\ &(\xi(\bar{t};\bar{x},\bar{t}), v(\bar{t})) = (\bar{x}, u(\bar{x}-,\bar{t})). \end{aligned} \tag{3.2.2}$$

Along the curve $\xi(t) := \xi(t;\bar{x},\bar{t})$, we have $u(\xi(t)-,t) = u(\xi(t)+,t)$ for almost all $t \in [0,\bar{t})$ and $v(t)$ does not change sign. From (3.2.2) and assumption that u_0 has n zeroes, we see that there are no more than n zeroes for $u(\cdot,t)$ for any fixed t. Thus, $\lim_{x\to\bar{x}\pm} \operatorname{sign}(u(x,t)) \neq 0$ for all $\bar{x} \in \mathbb{R}$. It remains to prove that the number of points in $\mathcal{A}(\bar{t})$ is no more than n. To this end, we assume the contrary and arbitrarily select $n+1$ points in $\mathcal{A}(\bar{t})$: $x_1 < x_2 < .... < x_{n+1}$. According to the definition of $\mathcal{A}(\bar{t})$, one can choose $y_j^{\pm}$, $j = 1, 2, ..., n+1$ such that

$$x_j \in (y_j^-, y_j^+), \quad \text{sign}(u(y_j^-, \bar{t}) = -\text{sign}(u(y_j^+, \bar{t}) \neq 0 \tag{3.2.3}$$

and that intervals $[y_j^-, y_j^+]$, $j = 1, 2, ..., n+1$ are disjoint. Then, at $t = 0$, the intervals $[\xi(0, y_j^-, \bar{t}), \xi(0, y_j^+, \bar{t})]$ are also disjoint since extremal backward characteristics do not intersect each other. Furthermore, by (3.2.3) and the fact that $v(t)$ does not change sign, one has $\text{sign}(u_0(\xi(0, y_j^-, \bar{t})) = -\text{sign}(u(\xi(0, y_j^+, \bar{t})) \neq 0$ which implies that there is at least one zero point of u_0 in $[\xi(0, y_j^-, \bar{t}), \xi(0, y_j^+, \bar{t})]$ for each $j = 1, 2, ..., n+1$. However, there are only n zero points of u_0. This contradiction shows that the number of points in $\mathcal{A}(\bar{t})$ is at most n at any time $t \geq 0$. ∎

Remark: It is clear from Lemma 3.2.1 that there are $m(0) \leq n$ curves $z_1(t) < z_2(t) < .... < z_{m(0)}(t)$, where $z_j(t)$ is defined on $[0, T_j)$, $j = 1, 2, ..., m(0)$, so that between two adjacent curves, the sign of $u(x,t)$ is fixed. Furthermore, two curves $z_j(t)$ and $z_k(t)$ cannot intersect except at the end points of their domain of definition.

Lemma 3.2.2. *Let $u_0(x)$ satisfy the assumption (1.3). Then $m(0) = n$ and $z_j(0) = a_j$, $j = 1, 2, ..., n$. Furthermore, the curves $z_j(t)$, $j = 1, 2, ..., n$ given in Lemma 3.2.1 are Lipschitzian with Lipschitzian constant $\leq \max_{|u| \leq M} |f'(u)|$. Moreover, if the end points of the domain of definition of $z_j(t)$, $T_j < \infty$, then there is another curve $z_{j'}(t)$ intersecting $z_j(t)$ at $t = T_j = T_{j'}$.*

Proof. Let $\bar{t}$ be any point in the domain of definition of $z_j(t)$. If $(z_j(\bar{t}), \bar{t})$ is a point of continuity of $u(x,t)$, then by definition of $z_j(t)$, it is necessary that $u(z_j(\bar{t}), \bar{t}) = 0$. From (3.2.2), it is clear that $u = 0$ along the backward characteristics $\xi(t, \bar{x}, \bar{t}) \equiv \bar{x}$. We claim that $z_j(t) = \bar{x}$ and hence is Lipschitzian. To this end, let $x_1 < \bar{x} < x_2$ and x_1 and x_2 are sufficiently close to $\bar{x}$ so that $\text{sign}(u(x_1-, \bar{t})) = -\text{sign}(u(x_2-, \bar{t})) \neq 0$. Along the minimal backward characteristics $\xi(t; x_1, \bar{t})$ and $\xi(t; x_2, \bar{t})$, the sign of $u(\xi(t; x_1, \bar{t}), t)$ and that of $u(\xi(t; x_2, \bar{t}), t)$ do not change. Therefore there is at least one $z_j(t)$ between $\xi(t; x_1, \bar{t})$ and $\xi(t; x_2, \bar{t})$. Since extremal backward characteristics do not intersect, the inequality $\xi(t; x_1, \bar{t}) < \xi(t, \bar{x}, \bar{t}) = \bar{x} < \xi(t; x_2, \bar{t})$ holds for all $0 \leq t \leq \bar{t}$. Furthermore, by the continuous dependence of solutions of ordinary differential equations on initial values,

$$\bar{x} = \lim_{x_1 \to \bar{x}-} \xi(t; x_1, \bar{t}) \leq z_j(t) \leq \lim_{x_2 \to \bar{x}+} \xi(t; x_2, \bar{t}) = \bar{x}. \tag{3.2.4}$$

The claim that $z_j(t) = \bar{x}$ is proven.

If $(z_j(\bar{t}), \bar{t})$ is a point of discontinuity of $u(x,t)$, then $u(\bar{x}-, \bar{t})$ and $u(\bar{x}+, \bar{t})$ are of opposite sign. Since $u(x,t)$ is the admissible solution of (1.2), it is clear that

$u(\bar{x}-,\bar{t}) > u(\bar{x}+,\bar{t})$. From (3.1.2) and (3.1.3), we see that the minimal and maximal backward characteristics through the point $(\bar{x},\bar{t})$ satisfy $\xi_-(t,\bar{x},\bar{t}) < \xi_+(t,\bar{x},\bar{t})$ and along these curves, $u(x,t)$ does not change sign. Thus,

$$\xi_-(t,\bar{x},\bar{t}) \le z_j(t) \le \xi_+(t,\bar{x},\bar{t}) \text{ for } 0 < t \le \bar{t} \tag{3.2.5}$$

with "=" holds only at $t = \bar{t}$. This implies that for $0 \le t < \bar{t}$

$$\left|\frac{z_j(\bar{t}) - z_j(t)}{\bar{t} - t}\right| \le \max_{t\in[0,\bar{t}]} \left|\frac{d\xi_\pm}{dt}\right| \le \max_{|u|\le M} |f'(u)|$$

where $M > 0$ is the constant givenm in Lemma 3.1.4. By the arbitrariness of $\bar{t} > 0$, $z_j(t)$ is uniformly Lipschitzian on its domain of definition.

To see that $m(0) = n$, one only needs to notice that when $t > 0$ is very small, at the midpoints of adjacent zeroes of u_0, $x_j = (a_j + a_{j+1})/2$, $j = 1,2,...,n-1$, and at points $x_0 << a_1$ and $x_n >> a_n$, the sign of $u(x_j,t)$ is the same as that of $u_0(x_j)$. This observation can be justified by the fact that sign of $u(x,t)$ does not change along extremal backward characteristics and and that the characteristic through (x_j,t) satisfy $a_j < \xi(0,x_j,t) < a_{j+1}$, $j = 0,1,2,...,n$ when $t > 0$ is small, where $a_0 := -\infty$ and $a_{n+1} := \infty$. Thus, for small $t > 0$, there are at least n points $z_1(t) < z_2(t) < ... < z_n(t)$ so that $\text{sign}(u(x,t))$ is fixed between $z_j(t)$ and $z_{j+1}(t)$. On the other hand, the last lemma shows that the number of these points is no more than n. Therefore, it is necessary that $m(0) = n$.

It remains to prove that $z_j(0) = a_j$, $j = 1,2,...,n$. To this end, we assume the contrary, i.e. for some $1 \le j_0 \le n$, $z_{j_0}(0)$ is not a zero of u_0. For definiteness, we assume that $u_0(z_{j_0}(0)) > 0$. Then for some small $\gamma > 0$, the value $u_0(x) > 0$ for all $|x - z_{j_0}(0)| \le \gamma$. Since the absolute value of the slope of backward characteristics is $\le \max_{|u|\le M} |f'(u)|$, there is a $\gamma_1 > 0$ such that $|\xi(0,\bar{x},\bar{t}) - z_{j_0}(0)| \le \gamma$ for small $\bar{t} > 0$ and $|\bar{x} - z_{j_0}(0)| \le \gamma_1$. This yields that $u(x,t) > 0$ for all

$$(x,t) \in \mathcal{B} := \{(x,t) \mid 0 \le t \le \bar{t},\ \xi(t, z_{j_0}(0) - \gamma_1, \bar{t}) \le x \le \xi(t, z_{j_0}(0) + \gamma_1, \bar{t})\}.$$

This contradicts the definition of $z_j(t)$ and that $(z_j(0), 0) \in \mathcal{B}$.

If there is no other $z_{j'}(t)$ intersecting $z_j(t)$ at the end point $t = T_j > 0$ of the domain of definition of $z_j(t)$, then there is a small number $\gamma_2 > 0$ such that $\text{sign}(u(x,t)) = 1$ for x in one of intervals $(z_j(T_j) - \gamma_2, z_j(T_j))$ and $(z_j(T_j), z_j(T_j) + \gamma_2)$ while $\text{sign}(u(x,t)) = -1$ for x in the other. Then by (3.2.8) and boundedness of slope of characteristics, the sign

$$\text{sign}(u(z_j(T_j) - \gamma_2/2, t)) = -\text{sign}(u(z_j(T_j) + \gamma_2/2, t)) \ne 0$$

for $T_j + \gamma_3 > t > T_j$ where $\gamma_3 > 0$ is some small number. Thus $z_j(t)$ can be extended to at least $[0, T_j + \gamma_3)$ which contradicts the definition of T_j. ∎

Lemma 3.2.3. *For any sequence* $\{\epsilon_n\}_{n=1}^{\infty}$ *with* $\epsilon_n \to 0+$ *as* $n \to \infty$, *there is a subsequence, also denoted by* $\{\epsilon_n\}$ *for simplicity, such that the limit*

$$u(\bar{x}, \bar{t}) = \lim_{\epsilon_n \to 0+} u^{\epsilon_n}(\bar{x}, \bar{t}) \tag{3.2.6}$$

exists for almost all $(\bar{x}, \bar{t}) \in \mathbb{R} \times \mathbb{R}^+$. *The range of* $u(\bar{x}\pm, \bar{t})$ *is* $\{-1, 1\}$. *Furthermore, there are uniform Lipschitzian curves* $z_1(t) < z_2(t) < ... < z_n(t)$ *defined* $[0, T_j)$, $j = 1, 2, ..., n$ *respectively such that for each fixed* $t > 0$, $u(x, t)$ *is constant for all* x *between two adjacent curves* $z_j(t)$.

Proof. Since, by Lemma 3.2.2, the curves $z_j^{\epsilon_n}(t)$ defined on $[0, T_j^{\epsilon_n}]$ are Lipschitzian uniformly in $\epsilon_n > 0$ and j, there is a subsequence of $\{\epsilon_n\}$, also denoted by $\{\epsilon_n\}$, such that

$$z_j^0(t) := \lim_{n \to \infty} z_j^{\epsilon_n}(t) \tag{3.2.7}$$

exits on $[0, T_j := \lim_{n \to \infty} T_j^{\epsilon_n}]$. By the definition of $z_j^{\epsilon_n}(t)$, for each fixed $t > 0$, $\lim_{n \to \infty} \text{sign}(u^{\epsilon_n}(x, t))$ is fixed for all x between two adjacent points among $z_j(t)$, $j = 1, 2, ..., n$. In above statement and in the rest of this proof, if t is out of the domain of definition of $z_j(t)$, we just ignore $z_j(t)$.

Fix $\bar{t} > 0$. Any point $\bar{x} \in \mathbb{R}$ must fall between some adjacent curves $x = z_j^0(\bar{t})$, $j = 0, 1, ..., n+1$, where $z_0^0(t) := -\infty$, $z_{n+1}^0(t) := \infty$. Let $z_j^0(\bar{t}) < z_{j'}^0(\bar{t})$ be two adjacent points at $t = \bar{t}$. From above discussion, the limit $\lim_{n \to \infty} \text{sign}(u^{\epsilon_n}(\bar{x}, \bar{t}))$ is a constant for all $z_j^0(\bar{t}) < \bar{x} < z_{j'}^0(\bar{t})$. For definiteness, we assume this constant is 1. i.e. for all $z_j^0(\bar{t}) < \bar{x} < z_{j'}^0(\bar{t})$,

$$u^{\epsilon_n}(\bar{x}, \bar{t}) > 0 \quad \text{for large } n.$$

The other case can be handled in the same way. The minimal backward characteristic $(\xi^{\epsilon_n}(t; \bar{x}, \bar{t})$ of (1.2) through the point $(\bar{x}, \bar{t})$ satisfies

$$\begin{aligned} &\frac{d\xi}{dt} = f'(v(t)), \\ &\frac{dv}{dt} = \frac{1}{\epsilon} v(1 - v^2), \\ &(\xi(\bar{t}), v(\bar{t})) = (\bar{x}, u^{\epsilon_n}(\bar{x}-, \bar{t})). \end{aligned} \tag{3.2.8}$$

According to Corollary 3.2.5, solution of (3.2.8) is defined on $[0, \bar{t}]$ and $v^{\epsilon_n}(t) = u^{\epsilon_n}(\xi^{\epsilon_n}(t; ...)-, t) = u^{\epsilon_n}(\xi^{\epsilon_n}(t; ...)+, t)$ for all most all $t \in [0, \bar{t}]$. If $u^{\epsilon_n}(\bar{x}, \bar{t}) = 1$, then $v^{\epsilon_n}(t) \equiv 1$. Hence the limit $u(\bar{x}, \bar{t}) = 1$ in this case. If $u^{\epsilon}(\bar{x}, \bar{t}) \neq 1$, then one can solve $(3.2.2)_2$ to obtain

$$\frac{u^{\epsilon_n}}{\sqrt{|1 - (u^{\epsilon_n})^2|}} = c_{\epsilon_n} e^{t/\epsilon_n}, \tag{3.2.9a}$$

where

$$c_{\epsilon_n} = \frac{u_0(\xi^{\epsilon_n}(0;\bar{x},\bar{t}))}{\sqrt{|1-[u_0(\xi^{\epsilon_n}(0;\bar{x},\bar{t}))]^2|}} > 0. \tag{3.2.9b}$$

Therefore,

$$u(\bar{x},\bar{t}) = \begin{cases} \lim_{\epsilon_n\to 0+} \frac{c_{\epsilon_n} e^{t/\epsilon_n}}{\sqrt{c_{\epsilon_n}^2 e^{2t/\epsilon_n}+1}}, & \text{if } c_{\epsilon_n} < 1, \\ \lim_{\epsilon_n\to 0+} \frac{c_{\epsilon_n} e^{t/\epsilon_n}}{\sqrt{c_{\epsilon_n}^2 e^{2t/\epsilon_n}-1}}, & \text{if } c_{\epsilon_n} > 1. \end{cases} \tag{3.2.10}$$

There are the following two possibilities for c_{ϵ_n}:

Case I.

$$\liminf_{\epsilon\to 0+} c_\epsilon > 0. \tag{3.2.11}$$

It is easy to see from (3.2.10) that $u(\bar{x},\bar{t}) = 1$ in this case.

Case II. Condition (3.2.11) fails. In other words, there is a subsequence of $\{\epsilon_n\}_{n=1}^\infty$, still denoted by $\{\epsilon_n\}$, such that

$$\lim_{n\to\infty} c_{\epsilon_n} = 0+. \tag{3.2.12}$$

Then equation (3.2.9) infers that

$$\lim_{n\to\infty} u_0(\xi^{\epsilon_n}(0,\bar{x},\bar{t})) = 0. \tag{3.2.13}$$

By the continuity of $u_0(x)$ and by further extracting a subsequence if necessary, we have

$$\xi^{\epsilon_n}(0,\bar{x},\bar{t}) \to a \tag{3.2.14}$$

with $u_0(a) = 0$. For different subsequences, the value a may be different. However, this does not affect our argument from (3.2.15 - 17) below because there are only finitely many possible values for a. Since $u^{\epsilon_n}(\bar{x}-,\bar{t}) > 0$, it is necessary in view of (3.2.8) that

$$a \le \bar{x}. \tag{3.2.15}$$

A straightforward calculation based on (3.2.8) yields that

$$\frac{\bar{x}-\xi^{\epsilon_n}(0,\bar{x},\bar{t})}{\epsilon_n} = \int_{u_0(\xi^{\epsilon_n}(0,\bar{x},\bar{t}))}^{u^{\epsilon_n}(\bar{x}-,\bar{t})} \frac{f'(u)du}{u(1-u^2)} \tag{3.2.16}$$

Note that $u=0$ is not a singular point for the intgral in (3.2.16) since $f'(0)=0$ and f' is differentiable. If $a < \bar{x}$, we can let $n\to\infty$ and apply (3.2.13-15) in (3.2.16) to obtain that

$$u(\bar{x},\bar{t}) := \lim_{n\to\infty} u^{\epsilon_n}(\bar{x}-,\bar{t}) = 1. \tag{3.2.17a}$$

If $a = \bar{x}$ in (3.1.15), then above arguments apply to points x near $\bar{x}$ to yield

$$u(\bar{x}-, \bar{t}) = u(\bar{x}+, \bar{t}) = 1. \tag{3.1.17b}$$

From above arguments, we see that the limit $u(x,t)$ is a piecewise constant function with constants being ± 1 which are separated by the Lipschitzian curves $z_j(t)$, $j = 1, 2, ..., n$. These curves intersect each other only at the end points of their domain of definition. ∎.

Theorem 3.2.4. *Let u_0 satisfy the assumption (1.3) and u^ϵ be the solution of (1.2). Then the limit*

$$u(\bar{x}, \bar{t}) = \lim_{\epsilon \to 0+} u^\epsilon(\bar{x}, \bar{t}) \tag{3.2.18}$$

exists for almost all $(\bar{x}, \bar{t}) \in \mathbb{R} \times \mathbb{R}^+$. The value of $u(\bar{x}\pm, \bar{t})$ is either 1, or -1. Furthermore, There are n curves $z_1^0(t) < z_2^0(t) < ... < z_n^0(t)$ of $u(x,t)$, defined on $[0, T_j)$, $j = 1, 2, ..., n$ respectively, such that:

(i) if $u(z_j^0(t)-, t) > u(z_j^0(t)+, t)$, then $x = z_j^0(t)$ satisfies the Rankine-Hugoniot condition

$$\frac{dz_j^0}{dt} = \frac{f(u(z_j^0(t)-, t)) - f(u(z_j^0(t)+, t))}{u(z_j^0(t)-, t) - u(z_j^0(t)+, t)} = \frac{f(1) - f(-1)}{2}. \tag{3.2.18}$$

(ii) if $u(z_j^0(t)-, t) < u(z_j^0(t)+, t)$, then the curve $z_j^0(t) = a_j + f'(0)t$ and hence the speed of the discontinuity is $f'(0)$.

Proof. By Lemma 3.2.3, there is a sequence $\{\epsilon_n\}$ such that

$$u(x,t) := \lim_{n \to \infty} u^{\epsilon_n}(x,t) \tag{3.2.19}$$

exists for almost all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. From Lemma 3.2.3, $u(x,t) = 1$ or -1. The connected components of $\{x \mid u(x\pm, t) = 1\}$ and $\{x \mid u(x\pm, t) = -1\}$ are intervals separated by $z_j(t)$, $j = 1, 2, ..., m(t)$ with $m(0) = n$. The curves $z_j(t)$, defined on $[0, T_j)$, $j = 1, 2, ..., n$ are uniformly Lipschitzian which do not intersect each other except at the end points $t = T_j$. Furthermore, the curves $z_j(t)$ satisfy $z_j(0) = a_j$, $j = 1, 2, ..., n$. Thus, the set of points of discontinuity of $u(x,t)$ is the union of these curves $z_j(t)$, $j = 1, 2, ..., n$. There are two possibilities at $x = z_j(t)$:

Case I. $u(z_j(t)-, t) > u(z_j(t)+, t)$.

In this case, it is necessary that $u(z_j(t)-, t) = -u(z_j(t)+, t) = 1$. Consider points $\bar{x} < z_j^{\epsilon_n}(\bar{t})$ and close enough to $z_j^{\epsilon_n}(\bar{t})$ at $t = \bar{t} < T_j$. Then $u^{\epsilon_n}(\bar{x}, \bar{t}) > 0$ holds. The minimal backward characteristic associated to u^{ϵ_n} defined by (3.2.10) at $t = 0$, $\xi^{\epsilon_n}(0,\bar{x},\bar{t})$, is confined to two possibilities according to Case I and II in the proof of Lemma 3.2.3: They are (i) $\liminf_{n\to\infty} u_0(\xi^{\epsilon_n}(0,\bar{x},\bar{t})) > 0$ and (ii) $\liminf_{n\to\infty} u_0(\xi^{\epsilon_n}(0,\bar{x},\bar{t})) = 0$. For (i),

$$\lim_{n\to\infty} \xi^{\epsilon_n}(0,\bar{x},\bar{t})) = \bar{x} - f'(1)\bar{t}. \tag{3.2.20}$$

For the same reason as stated after (3.2.14), we have for (ii) that

$$\lim_{n\to\infty} \xi^{\epsilon_n}(0,\bar{x},\bar{t})) = a_j \leq \bar{x} \tag{3.2.21}$$

for some $1 \leq j \leq n$. In both cases (i) and (ii),

$$a := \lim_{n\to\infty} \xi^{\epsilon_n}(0,\bar{x},\bar{t})) \leq \bar{x} \tag{3.2.22}$$

Subcase I(1): $a < \bar{x}$ in (3.2.22).

We can use (3.2.16) to obtain for large n that

$$\begin{aligned} &\frac{\bar{x} - \xi^{\epsilon_n}(0,\bar{x},\bar{t})}{\epsilon_n} = \int_{u_0(\xi^{\epsilon_n}(0,\bar{x},\bar{t}))}^{u^{\epsilon_n}(\bar{x}-,\bar{t})} \frac{f'(u)du}{u(1-u^2)} \\ &\leq \max_{u\in[-1,1]} \left(\frac{f'(u)}{u}\right) \int_0^{u^{\epsilon_n}(\bar{x}-,\bar{t})} \frac{du}{1-u} \\ &\leq C \ln|1 - u^{\epsilon_n}(\bar{x}-,\bar{t})| \end{aligned} \tag{3.2.23}$$

where $C := \max_{u\in[-1,1]}(f'(u)/u)$. From above,

$$|1 - u^{\epsilon_n}(\bar{x}-,\bar{t})| \leq \exp\left(-C_1 \frac{\bar{x} - \xi^{\epsilon_n}(0,\bar{x},\bar{t})}{\epsilon_n}\right) \tag{3.2.24}$$

where the constant C_1 only depends on f.

Subcase I(2): $a = \bar{x}$ in (3.2.22).

By (3.2.20), this case occurs only if $\liminf_{n\to\infty} u_0(\xi^{\epsilon_n}(0,\bar{x},\bar{t})) = 0$ and hence $a = a_j$ for some $j \in \{1,2,...,n\}$.

We claim that this subcase cannot happen for small enough $\epsilon_n > 0$. Indeed, if otherwise, we would have

$$\xi^{\epsilon_n}(t) := \xi^{\epsilon_n}(t;,\bar{x},\bar{t}) \to \bar{x} = a \tag{3.2.25}$$

from (3.2.22). For small $\delta > 0$, our arguments for Subcase I(1) proves that $0 \leq 1 - u^{\epsilon_n}(x,\bar{t}/2) < \delta$ uniformly for $x \in [a_{j-1}+\delta, a_j - \delta]$ when $\epsilon_n > 0$ is small enough. To establish above claim, it suffices to prove that the forward characteristics $\zeta^{\epsilon_n}(t;\bar{x}-$

$\delta, \bar{t}/2)$, $t > \bar{t}/2$ intersect $\xi^{\epsilon_n}(t) \approx a = \bar{x}$ at some $t < \bar{t}$, which is impossible in view of Lemma 3.1.7. To this end, we observe that before $x = \zeta^{\epsilon_n}(t; \bar{x} - \delta, \bar{t}/2)$ intersects $x = \xi^{\epsilon_n}(t)$, the estimate

$$u^{\epsilon_n}(x, \bar{t}) > 0 \text{ for } \bar{x} - \delta < x < \bar{x} \tag{3.2.26}$$

holds due to $\bar{x} < z^{\epsilon_n}(\bar{t})$. This implies

$$\frac{d\zeta^{\epsilon_n}(t; \bar{x} - \delta, \bar{t}/2)}{dt} > 0 \tag{3.2.27}$$

and hence

$$\zeta^{\epsilon_n}(t; \bar{x} - \delta, \bar{t}/2) > \bar{x} - \delta. \tag{3.2.28}$$

For any point $x_1 \in [(a_{j-1} + a_j)/2, \zeta^{\epsilon_n}(t_1; \bar{x} - \delta, \bar{t}/2)]$, we have $\xi(0; x_1, t_1) < \bar{x} - \delta$, since maximal backward characteristics cannot cross the forward characteristics $\zeta^{\epsilon_n}(t; \bar{x} - \delta, \bar{t}/2)$ from the left as t decreases. Then our arguments for Subcase I(1) also leads to

$$0 \leq 1 - u^{\epsilon_n}(\zeta^{\epsilon_n}(t; \bar{x} - \delta, \bar{t}/2)-, t) < \delta \tag{3.2.29}$$

when $\epsilon_n > 0$ is sufficiently small. This infers that

$$\frac{d\zeta^{\epsilon_n}(t; \bar{x} - \delta, \bar{t}/2)}{dt} \geq f(1 - \delta) - f(0) \tag{3.2.30}$$

before meeting $x = \xi(t; \bar{x}, \bar{t})$. Thus, the curve $x = \zeta^{\epsilon_n}(t; \bar{x} - \delta, \bar{t}/2)$ and $x = \xi^{\epsilon_n}(t)$ must intersect at some $t < \bar{t}$ when $\epsilon_n > 0$ is sufficiently small. This proves our claim.

Summerizing Subcase I(1) and (2), we see that

$$\frac{1}{\epsilon_n} u^{\epsilon_n}(\bar{x}-, \bar{t})(1 - (u^{\epsilon_n}(\bar{x}-, \bar{t}))^2) \to 0 \tag{3.2.31}$$

uniformly for $\bar{x} < z_j(t)$ and close to $z_j(t)$. Similarly one can prove that (3.2.31) also holds uniformly for $\bar{x} > z_j(t)$ and close to $z_j(t)$. Apply these estimates to the the weak form of (1.2)

$$\int_0^{T_j} \int_{\mathbb{R}} \left(-u\phi_t - f(u)\phi_x - \frac{1}{\epsilon_n} u^{\epsilon_n}(1 - (u^{\epsilon_n})^2)\phi \right) dxdt = 0$$

for test functions with compact support confined near $x = z_j(t)$, $0 < t < T_j$, one sees that the shock $x = z_j(t)$, $0 < t < T_j$, is a weak solution of $u_t + f(u)_x = 0$. Thus, the Rankine-Hugoniot condition

$$\frac{dz_j}{dt} = \frac{f(u(z_j(t)+,t)) - f(u(z_j(t)-,t))}{u(z_j(t)+,t) - u(z_j(t)-,t)} = \frac{f(1)-f(-1)}{2}$$

holds if $u(z_j(t)+,t) < u(z_j(t)-,t)$.

Case II. $u(z_j(t)-,t) < u(z_j(t)+,t)$.

In this case, one has $u(z_j(t)-,t) = -1 = -u(z_j(t)+,t)$. We claim that in this case, $z_j(\bar{t}) = a_j$ for some $1 \le j \le n$ and all $\bar{t} \in [0, T_j)$ under the assumption $f'(0) = 0$. To this end, we consider two points x_1 and x_2 sufficiently close to $z_j(\bar{t})$ and $x_1 < z_j(\bar{t}) < x_2$. By definition of $z_j(t)$, $\text{sign}(u^{\epsilon_n}(x_1-,\bar{t})) = 1 = -\text{sign}(u^{\epsilon_n}(x_2-,\bar{t}))$ for large n. From (3.2.10), the minimal backward characteristics through points $(x_1,\bar{t})$ and $(x_2,\bar{t})$ satisfy

$$\frac{d\xi^{\epsilon_n}(t,x_1,\bar{t})}{dt} < 0 < \frac{d\xi^{\epsilon_n}(t,x_2,\bar{t})}{dt}. \tag{3.2.32}$$

Since the sign of u^{ϵ_n} is constant along extremal backward characteristics, one has

$$\xi(t,x_1,\bar{t}) < z_j(t) < \xi(t,x_2,\bar{t}). \tag{3.2.33}$$

Now, let $x_1 \to z_j(t)-$ and $x_2 \to z_j(\bar{t})+$, estimates (3.2.32) and (3.2.33) imply that $z_j(t)$ is constant for $t \in [0,\bar{t}]$. It follows immediately from the arbitrariness of $\bar{t} \in [0,T_j)$ and $z_j(0) = a_j$ that $z_j(t) \equiv a_j$ for all t in its domain of definition.

From above analysis, we see that the the limit function $u(x,t)$ is completely determined by the curves $z_j(t)$, $j = 1,2,...,n$. Furthermore, these curves $z_j(t)$ are uniquely determined by the Rankine-Hugoniot condition with $z_j(0) = a_j$ or is equal to a constant a_j for some $1 \le j \le n$. In other words, no matter how the subsequence $\{\epsilon_n\}$ are chosen, the limit functions $u(x,t) = \lim_{n\to\infty} u^{\epsilon_n}(x-,t)$ are the same. This proves the convergence of u^ϵ as $\epsilon \to 0+$. ∎

Our above analysis already contains a complete picture of the structure of the $\epsilon \to 0+$ limit of $u^\epsilon(x,t)$, the solution of (1.2). We summarize the structure of limit function $u(x,t)$ as follows:

Corollary 3.2.5. *The $\epsilon \to 0+$ limit of $u^\epsilon(x,t)$, the solution of (1.2),*

$$u(x,t) := \lim_{\epsilon\to 0+} u^\epsilon(x,t)$$

is a piecewise constant function with constants being ± 1. The constant pieces of $u(x,t)$ are separated by Lipschitzian curves $x = z_j(t)$, $j = 1,2,...,n$ defined on $[0,T_j]$.

(i) $z_j(0) = a_j$.

(ii) If $\lim_{x\to a_j-}$ *sign*$(u_0(x)) = 1$, *then* $z_j(t) = a_j + \frac{f(1)-f(-1)}{2}t$.

(iii) If $\lim_{x\to a_j-}$ *sign*$(u_0(x)) = -1$, *then* $z_j(t) = a_j + f'(0)t$.

(iv) *Curves* $x = z_j(t)$ *do not intersect each other except at* $t = T_j$*, the end points of their domain of definition.*

(v) *At* $t = T_j < \infty$*, the curve* $x = z_j(t)$ *must intersect with another curve* $x = z_k(t)$*.*

3.3. General Source Terms.

The results in previous sections can be extended to a more general source term that possesses similar equilibrium structure and more general initial data. In particular, we extend the results to the following hyperbolic conservation law with source term

$$\partial_t u + \partial_x f(u) = \frac{1}{\epsilon} g(u) \tag{3.3.1}$$

where $g(u)$ has finitely many zeros, $b_1 < b_2 < \cdots < b_{2k+1}$, all simple, and there is an $M_0 > 0$ such that

$$ug(u) < 0 \quad \text{for} \quad |u| > M_0 \,. \tag{3.3.2}$$

Then the points $u = b_{2i+1}$, $i = 0, 1, ..., k$ are stable equilibria of (3.3.2) while $u = b_{2i}$, $i = 1, 2, ...k$ are unstable equilibria. The assumption (3.3.2) implies that the solution of (3.3.1) is bounded in L^∞ uniformly in $\epsilon > 0$ and t, see [FH2]. This uniform boundedness ensures that extremal backward characteristics through any point $(\bar{x}, \bar{t}) \in \mathbb{R} \times \mathbb{R}^+$ are defined on $[0, \bar{t}]$. Similar to our above analysis, we can derive the following Theorem.

Theorem 3.3.1. *Suppose the initial value* $u_0(x) \in C^1(\mathbb{R}; \mathbb{R})$ *has* $n < \infty$ *points* $x = a_1 < a_2 < \cdots < a_n$ *such that* $u_0(a_j) \in \{b_{2i} \; : \; i = 1, 2, ..., k\}$ *and* $u_0'(a_j) \neq 0$, $j = 1, 2, \cdots, n$. *Let* u^ϵ *be the admissible solution of (3.3.1) with the initial value* $u_0(x)$*. Then the limit*

$$u(x,t) := \lim_{\epsilon \to \infty} u^\epsilon(x,t)$$

exists for almost all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$*. The function* $u(x,t)$ *is piecewise constant with the constants being* $b_{2i+1} (i = 0, 1, \cdots, k)$*. Constant pieces of* $u(x,t)$ *are separated by Lipschitz continuous curves* $x = z_j(t)$ *defined on* $[0, T_j)$, $j = 1, 2, ...n$*. Moreover, the following hold for these curves* $x = z_j(t)$, $j = 1, 2, ..., n$*:*

(i) $z_j(0) = a_j$.

(ii) *If two curves* $x = z_j(t)$ *and* $x = z_{j'}(t)$ *intersect at* $t = t_0$*, then either both curves terminate at* $t = t_0$ *or* $z_j(t) = z_{j'}(t)$ *for* $t \geq t_0$.

(iii) *At* $t = T_j < \infty$*, the curve* $x = z_j(t)$ *must intersect with another curve* $x = z_{j'}(t)$ *and* $T_{j'} = T_j$.

(iv) *If* $u_0'(a_j) < 0$ *and* $u_0(a_j) = b_{2i}$ *for some* $1 \leq i \leq k$*, then* $u(z_j(t)-, t) = b_{2i+1} > u(z_j(t)+, t) = b_{2i-1}$ *for all* $0 < t < \gamma$ *and some small* $\gamma > 0$.

(v) If $u_0'(a_j) > 0$ *and* $u_0(a_j) = b_{2i}$ *for some* $1 \le i \le k$*, then* $u(z_j(t)-,t) = b_{2i-1} < u(z_j(t)+,t) = b_{2i+1}$ *for all* $0 < t < \gamma$ *and some small* $\gamma > 0$.

(vi) If $u(z_j(t)-,t) > u(z_j(t)+,t)$*, then the curve* $x = z_j(t)$ *satisfies the Rankine-Hugoniot condition*

$$\frac{dz_j}{dt} = \frac{f(u(z_j(t)-,t)) - f(u(z_j(t)+,t))}{u(z_j(t)-,t) - u(z_j(t)+,t)}.$$

(vii) If $u(z_j(t)-,t) < u(z_j(t)+,t)$*, then* $u(z_j(t)-,t) = b_{2i-1}$*,* $u(z_j(t)+,t) = b_{2i+1}$ *for some* $1 \le i \le k$*. Furthermore, the curve* $x = z_j(t)$ *is* $z_j(t) = a_j + f'(b_{2i})t$.

Remark: Once again, the statements (vi) and (vii) state that If $u(z_j(t)-,t) > u(z_j(t)+,t)$, then the curve $x = z_j(t)$ is an ordinary shock. If $u(z_j(t)-,t) < u(z_j(t)+,t)$, then $x = z_j(t)$ is a non-shock discontinuity. Moreover, the speed of a non-shock discontinuity must be $f'(b_{2i})$ and the values of u at the two sides of the discontinuity are $u(z_j(t)-,t) = b_{2i-1}$, $u(z_j(t)+,t) = b_{2i+1}$ for some $1 \le i \le k$.

Remark: The assumption $u_0 \in C^1$ is for the convenience of referring $u_0(a_j) \neq 0$ later. It is nonessential.

4. Other Relevant Issues

In this section we will discuss two relevant issues. First we would like to investigate the viscous regularization of (1.2):

$$\begin{aligned} &\partial_t u + \partial_x f(u) = \frac{1}{\epsilon} u(1-u^2) + \delta \partial_{xx} u \\ &u(x,0) = u_0(x). \end{aligned} \tag{4.1}$$

As studied in previous sections, in the inviscid case (1.2), if the initial data give a rarefaction for the homogeneous equation (2.1), the limit solution of (1.2) as $\epsilon \to 0$ gives a non-shock discontinuity that does not satisfy the Rankine-Hugoniot condition nor the entropy condition for (2.1). It is interesting to investigate how the viscosity affects the solution. As an example, in this section, we will always use $f(u) = u^2/2 + u$.

We use the initial data $u_0(x) = -1, x < 0.2; u_0(x) = 1, x > 0.2$, and then solve numerically the homogeneous equation (2.1), the inviscid equation (1.2) with $\epsilon = 0.01$, and the viscous problem (4.1) with $\epsilon = 0.01, \delta = 0.1$. The solutions at $t = 0.3$ are presented in Figure 4.1. It shows that although the viscosity coefficient $\delta >> \epsilon$, the solution of (4.1) with the given initial data is not a rarefaction as in the solution of (2.1), rather, the competition coming from the reaction term is very strong and one still has a layer that is much closer to the inviscid solution than to the rarefaction

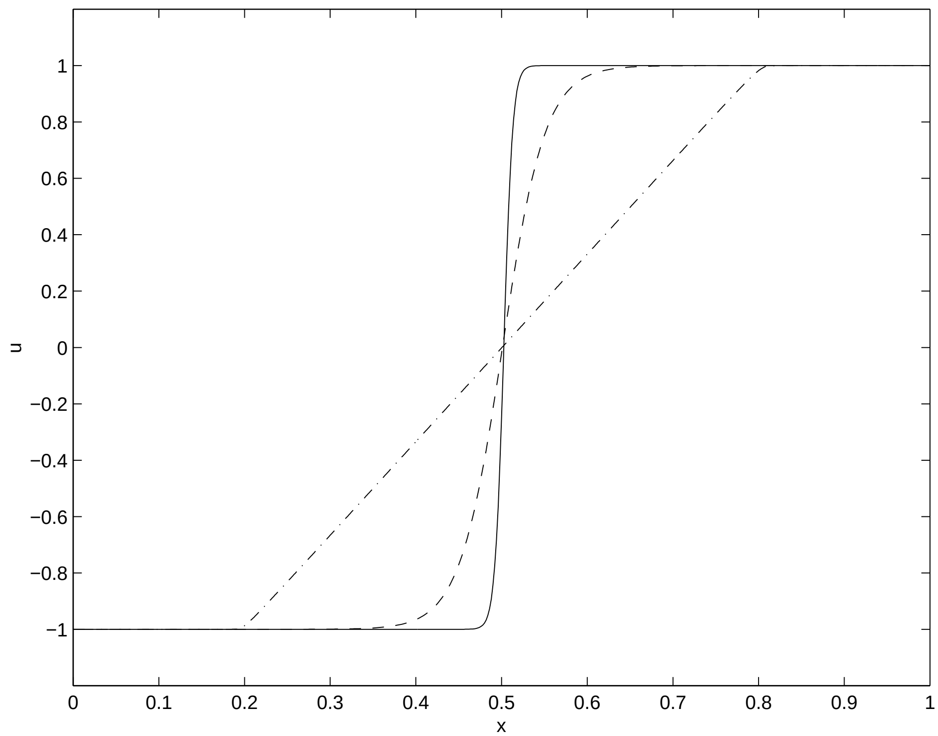

FIG.4.1. At $t = 0.3$. Solid: numerical solution of the inviscid equation (1.2) with $\epsilon = 0.01$; dashed line: numerical solution of the viscous equation (4.1) with $\epsilon = 0.01, \delta = 0.1$. Dot-dashed line: numerical solution of the homogeneous eqn (2.1).

solution. This experiment suggests that the non-shock discontinuity is admissible even by the viscosity criterion.

The second issue we will discuss is the behavior of numerical solutions for (1.2). Numerically solving a hyperbolic system with stiff source term is known to be challenging if one does not numerically resolve the small reaction time ϵ, i.e., if $\Delta t > \epsilon$. Physically ϵ is extremely small compared to other characteristic length of the problem, thus resolving ϵ numerically is impractically expensive. On the other hand, it is known that, for almost all shock capturing schemes, failing to resolve ϵ temporally will result in wrong shock speed [BKT, CMR, LY]. This is due to the numerical smearing of the shock, which induces artificial nonequilibrium across the shock that will ignite the reaction term that sends the nonequilibrium state into the incorrect equilibria. Here we use a numerical example to show that such a numerical phenomenon also occurs in the non-shock discontinuity.

As an example, we use a splitting method that treats the homogeneous convection and the stiff source in separated steps. Introduce the two split operators

$$\mathcal{S}_1: \quad \partial_t u + \partial_x f(u) = 0\,, \tag{4.1a}$$

$$\mathcal{S}_2: \quad \partial_t u = \frac{1}{\epsilon} u(1-u^2)\,. \tag{4.1b}$$

A simple time splitting method is

$$U^{n+1} = \mathcal{S}_2 \mathcal{S}_1(\Delta t) U^n \tag{4.2}$$

where $\mathcal{S}_1(\Delta t)$ stands for the exact or numerical solution of (4.1a) after one time step Δt, and $\mathcal{S}_2(\Delta t)$ is similarly defined. Numerically $\mathcal{S}_1$ can be solved by a standard modern shock capturing method, which $\mathcal{S}_2$ can be solved by a standard implicit ODE integrator, or even by an exact ODE solver since the solution of (4.1b) can be explicitly found as was done in section 2. As $\epsilon << \Delta t$, S_2 can be effectively replaced by a simpler projection operator

$$\mathcal{S}_3 : U^{n+1}(x) \begin{cases} 1 & \text{if} \quad U^n(x) > 0; \\ -1 & \text{if} \quad U^n(x) < 0. \end{cases} \tag{4.3}$$

The scheme now becomes simply

$$U^{n+1} = \mathcal{S}_3 \mathcal{S}_1(\Delta t) U^n\,. \tag{4.4}$$

As observed earlier, such a simple splitting gives a wrong shock speed. The shock will move either one grid point per time step, or does not move at all, depending on the structure of the numerical the shock profile (i.e., whether the smeared numerical shock point is positive or symmetric with respect to zero). Such a phenomenon is not restricted to the splitting method. In fact, it appears in essentially all shock **capturing** methods, split or unsplit [CMR, LY]. Here we just report that not only the shock speed is wrong, the non-shock discontinuity also has a wrong numerical speed. We solve (1.2) by method (4.4), with initial data $u_0(x) = 1$ for $0.2 < x < 0.4$ and $u_0(x) = -1$ otherwise. For $f(u) = u^2/2 + u$, this initial datum gives a shock and a non-shock discontinuity, both moving to the right with speed one. However, the numerical solution, as shown in Figure 4.2, gives zero speed for both discontinuities.

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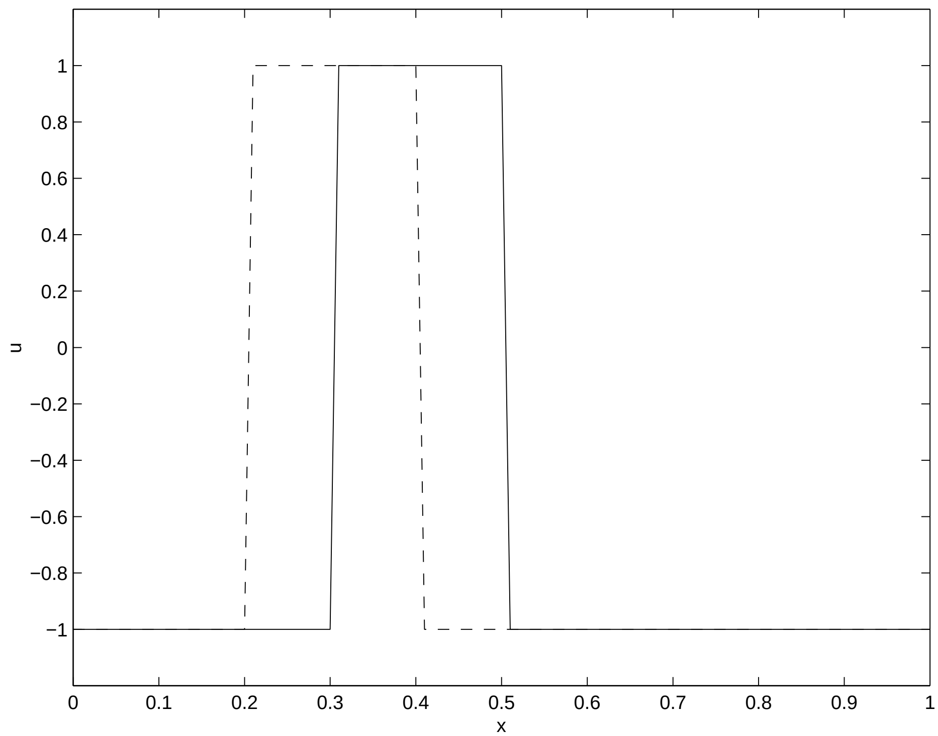

FIG.4.2. Spurious numerical discontinuity speed. Solid line: exact solution; dashed line: an underresolved numerical solution. $t = 0.1$.

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