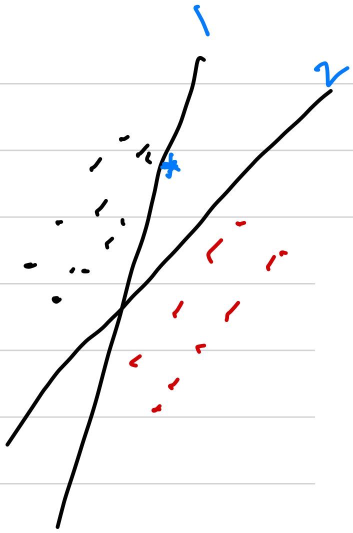
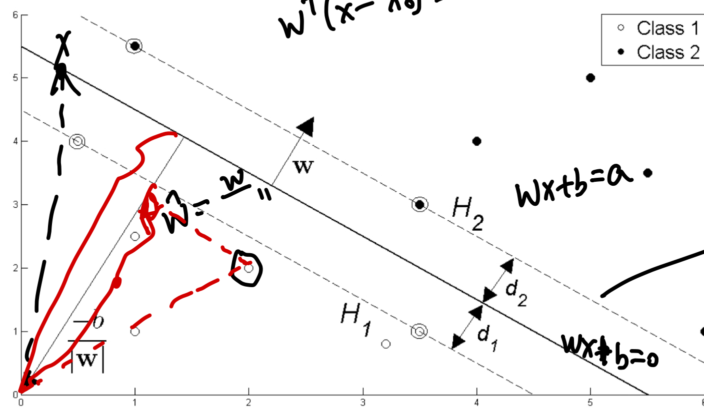


第八讲. 支持向量机. SVM

Support vector machine.

H $\quad w^T x + b = 0$

$w^T(x - x_0) = 0$



$\text{dist}(0, H_2)$

$= \frac{x w}{\|w\|} + \frac{b}{\|w\|}$

$= \frac{a - b + b}{\|w\|}$

$= \frac{a}{\|w\|}$

$x \in H \qquad x \cdot \hat{w} = \frac{wx}{\|w\|} = \frac{-b}{\|w\|}$

$x \notin H \qquad$ x 到 H 的距离

$$\text{dist}(x, H) = \frac{-b}{\|w\|} - \frac{wx}{\|w\|} = \frac{-(wx+b)}{\|w\|}$$

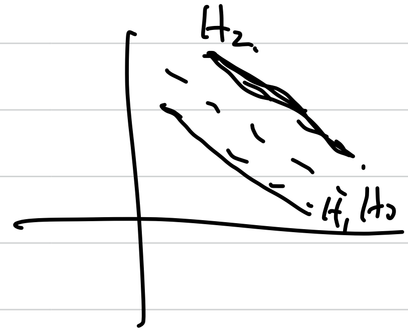
$y = w^T x + b.$

$\{(x_i, y_i)\}. \qquad x_i \in R^d. \qquad y_i \in \{-1, +1\}. \qquad i=1, \cdots n.$

$$\left.\begin{array}{ll} y_i = +1 & w^T x_i + b \geq 1 \\ y_i = -1 & w^T x_i + b \leq -1 \end{array}\right\} \Rightarrow.$$

$$y_i(w x_i + b) \geq 1$$

$$\text{dist}(H_1, H_0) = \text{dist}(H_2, H_0) = \frac{1}{\|w\|}$$



objective $\max \frac{1}{\|W\|} \Leftrightarrow \min \|W\|$

$\Leftrightarrow \min \frac{1}{2}\|W\|^2$

s.t $y_i(Wx_i+b)-1 \geq 0 \quad \forall i$

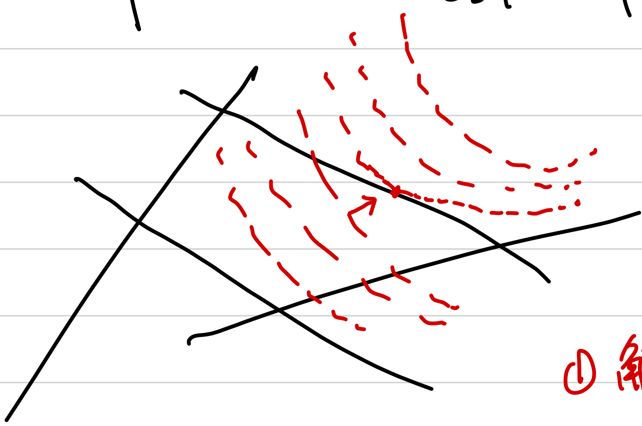
(w, b · 变量)

2. 优化问题

$\min \quad f_0(w) = \frac{1}{2}\|W\|_2^2$

s.t $\quad f_i(W,b) = 1 - y_i(Wx_i+b) \leq 0. \quad \forall i$

feasible set: $D_s = \{W, b \mid$ satisfies $f_i(w,b) \leq 0. \ \forall i\}$.



$p^* = \min_{w \in D_s} f_0(w,b)$

$f_0(w) = r.$

① 解唯一.

② 在边界上取极值点

Consider an equivalent problem

$\min f_0(w,b)$

s.t $\quad f_i(w,b) \leq 0$

$L(w,b,\alpha) = f_0(w) + \sum \alpha_i f_i(w,b) \qquad \alpha_i \geq 0$

$\theta = (w,b)$

$L(\theta, \alpha)$

$$g(\alpha) = \inf_{\theta} L(\theta, \alpha)$$

$p^* = f(\theta^*) \quad \theta^* \in D_s \qquad f_i(\theta^*) \le 0$, $\alpha_i \geqslant 0$

$$\geqslant f(\theta^*) + \sum \alpha_i f_i(\theta^*)$$

$$= L(\theta^*, \alpha) \geqslant g(\alpha).$$

$$g(\alpha) \le p^*$$

令 $d^* = \max_{\alpha} g(\alpha) \qquad d^* \overset{?}{=} p^*$

$$\max_x \inf_y \frac{x}{y} = 0.$$

$$\inf_y \max_x \frac{x}{y} = +\infty$$

对偶 $\quad d^* = \max_{\alpha} g(\alpha)$

$$= \max_{\alpha} \inf_{\theta} L(\theta, \alpha)$$

$$= \inf_{\theta} \max_{\alpha} L(\theta, \alpha)$$

<1> $\theta \notin D_s$. (not feasible) $\quad \exists i \quad f_i(w) > 0$

$$G(\theta) = \max_{\alpha} \; f_0(w) + \sum_i \alpha_i f_i(\theta)$$

let $\alpha_i \to +\infty \quad \tilde{\alpha} \quad \tilde{\alpha}_i = +\infty$

$$G(\theta) \to +\infty$$

$$\inf_{\theta \notin D_s} G(\theta) \longrightarrow +\infty$$

<2> $\theta \in D_s$. $\qquad f_i(\theta) \le 0 \quad \alpha_i \geqslant 0$.

$$\alpha_i f_i(\theta) \le 0.$$

(4

$\max_{\alpha} \sum \alpha_i f_i(\theta) = 0 \iff \boxed{\alpha_i f_i(\theta) = 0} \ \forall i$

$G(\theta) = f(\theta) < +\infty$

$\boxed{\text{KKT 条件}}$

互补松弛条件

③ 考虑 $\theta = \theta^*$. $p^* = f_0(\theta^*)$

$$L(\theta^*, \alpha) = f_0(\theta^*) + \underline{\sum \alpha_i f_i(\theta^*)}$$

$$p^* + \sum \alpha_i f_i(\theta) \le f_0(\theta) = p^*$$

$$\alpha_i f_i(\theta) = 0 \quad \forall i$$

$$\max_{x} \inf_{y} \frac{x}{y} = 0$$

$$\inf_{y} \max_{x} \frac{x}{y} = +\infty$$

$$
\begin{aligned}
d^* &= \max_{\alpha} \inf_{\theta} L(\theta, \alpha) \\
&= \inf_{\theta} \max_{\alpha} L(\theta, \alpha) \\
&= \inf_{\theta} \max_{\alpha} f_0(\theta) + \sum \alpha_i f_i(\theta) \qquad \alpha_i \ge 0,\ \theta \in D_s \\
&= \inf_{\theta \in D_s} f_0(\theta) \\
&= p^*.
\end{aligned}
$$

④ $\max_{\alpha} \inf_{\theta}$

$\min_{\theta} \max_{\alpha} L(\theta, \alpha) = f_0(\theta) + \sum \alpha_i f_i(\theta)$

s.t. $\alpha_i \ge 0 \ \forall i$

$\nabla_\theta L = 0$.

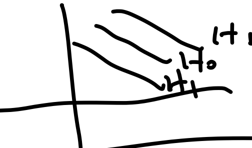
$$f_0(w) = \frac{1}{2}\|w\|^2.$$

$$f_i(\theta) = 1 - y_i(wx_i + b)$$

$$\frac{\partial L}{\partial w} = 0 \Rightarrow w = \sum_{i=1}^{n} \alpha_i y_i x_i$$

$$\frac{\partial L}{\partial b} = 0 \Rightarrow \sum \alpha_i y_i = 0$$

$$L = \frac{1}{2}\|w\|^2 + \sum \alpha_i (1 - y_i(wx_i + b))$$



讨论 极值性质

$$\alpha_i(1 - y_i(wx_i + b)) = 0$$

$$g(\alpha) = -\frac{1}{2} \sum_i \sum_j \alpha_i \alpha_j y_i y_j x_i x_j + \sum \alpha_i$$

令 $H_{ij} = y_i y_j x_i x_j$

$$\max_{\alpha} g(\alpha) \qquad \alpha \geq 0. \quad \sum \alpha_i y_i = 0.$$

$$\Leftrightarrow \text{Prob1} \quad \min_{\alpha} \frac{1}{2}\alpha^T H \alpha - 1^T \alpha \qquad \text{QP.}$$

$$\text{s.t.} \quad \alpha \geq 0$$

$$\sum \alpha_i y_i = 0$$

利用 QP solver Prob 1.

得 α. $\quad \alpha_i(1 - y_i(wx_i + b)) = 0.$

$\exists S.$ $\quad \forall i \in S \quad \alpha_i \neq 0.$

$$w = \sum_{i \in S} \alpha_i y_i x_i$$

$$y_i^2 (wx_i + b) = y_i$$

$$b = y_i - wx_i$$

$$= y_i - \sum_{j \in S} \alpha_j y_j x_j x_i$$

$$b = \frac{1}{N_S} \sum_{i \in S} \left(y_i - \sum_{j \in S} \alpha_j y_j x_j x_i\right)$$

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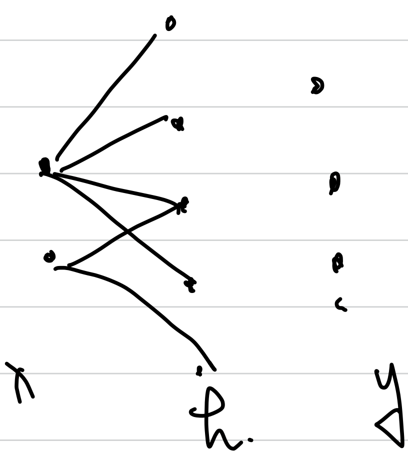
$$y = wx + b = \sum \alpha_i y_i x_i \cdot x - b$$

RKHS

Linear Model —— SVM → Kernel.

$x \to \phi(x)$ $\quad\quad$ $\phi(x) \to y$.

one-layer NN —— Multiple Layer NN



$x \to y$

$h(x) \to y$